Szkoła Doktorska Ekonometria Finansowa Blok 2

Michał Rubaszek
SGH Warsaw School of Economics

TOPICS

1. Risk of a univariate portfolio: GARCH model

2. Risk of a multivariate portfolio: MGARCH model

3. Risk of a multivariate portfolio: Copula

4. Backtesting

Meeting 1. Risk of a univariate portfolio: GARCH model

Plan for today

1. Downloading data from stooq.pl to R (stocks listed on WSE)

2. Descriptive stats: moments, ACF, density plot, QQ plot

3. Unconditional variance models for VaR/ES:
   - Parametric (normal/t-Student distribution)
   - Non-parametric (Historical simulation)

4. Conditional variance models for VaR/ES:
   - EWMA
   - GARCH

5. GARCH extensions (GJR-GARCH, EGARCH, GARCH-in-mean)

Michał Rubaszek, Financial Econometrics

Rates of return: reminder

Simple returns:

- ➢ Easy to calculate for a portfolio of assets: $R_p = \sum_{k=1}^{K} w_k R_k$
- ➢ Easy to communicate to non-statisticians
- ➢ Not symmetric nor additive...

Log returns:

- ➢ Symmetric and additive
- ➢ Easy to communicate to statisticians
- ➢ Difficult to calculate for a portfolio of assets: $r_p \neq \sum_{k=1}^{K} w_k r_k$

Even though the latter, we will work with log returns

Michał Rubaszek, Financial Econometrics

Financial series characteristics

1. No autocorrelation of returns (which is not equivalent to independence):

$$cor(r_t, r_{t-s}) = 0$$

2. Fat tails:

$$kurtosis(r_t) > 3$$

3. Volatility clustering:

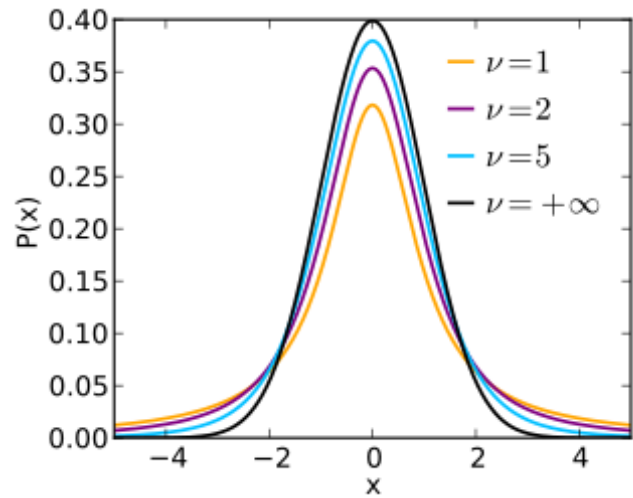$$cor(r_t^2, r_{t-s}^2) > 0$$

4. Leverage effect (sometimes):

$$cor(r_t^2, r_{t-s}) < 0$$

Michał Rubaszek, Financial Econometrics

Fat tails: t-Student distribution

t-Student distribution t_v:
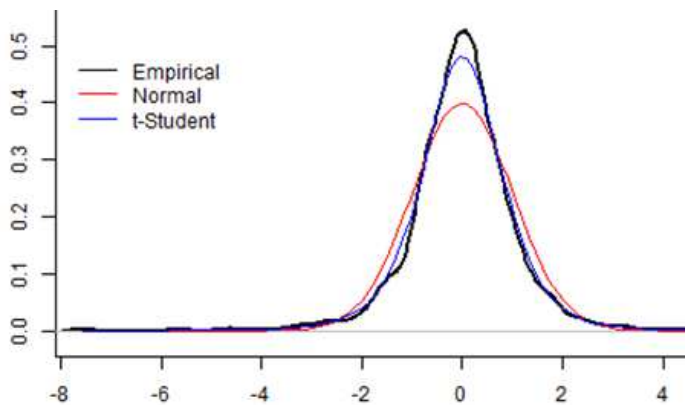
- for $v = \infty$ equivalent to $N(0,1)$
- for $v \leq 2$ no variance
  (which is equal to $\frac{v}{v-2}$)
- Kurtosis: $K = 3 + \frac{6}{v-4}$
- For stocks usually $v \sim 5$
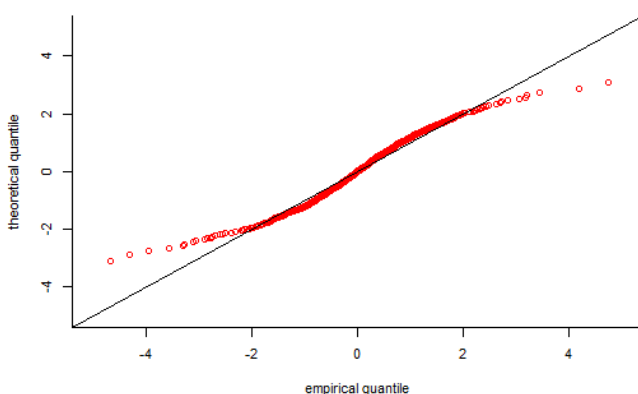- One can check with
  QQ plot / density plot

5% critical values for two-tailed t-Student t_v

v	1	2	3	5	10	50	∞
t*	-12.71	-4.30	-3.18	-2.57	-2.23	-2.01	-1.96

7

Michał Rubaszek, Financial Econometrics

Fat tails: illustration

```
|N     |  250.063|
|mu    |    7.534|
|sig   |   32.134|
|min   |  -15.352|
|max   |   12.607|
|skew  |   -0.197|
|kurt  |    7.577|
|JB    |  2199.87|
```

8

Michał Rubaszek, Financial Econometrics

Risk is not equivalent to variance!!!

Three series with $E(r) = 0$ and $Var(r) = 1$ (see Danielson, 2012)

Michał Rubaszek, Financial Econometrics
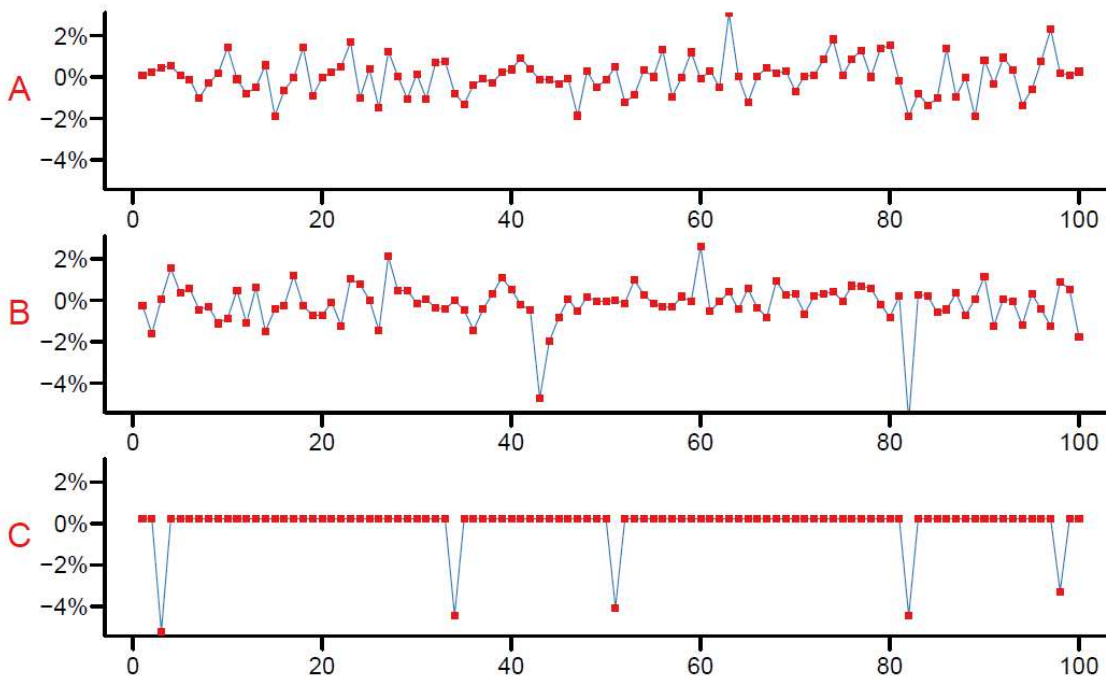
VaR and ES: definitions

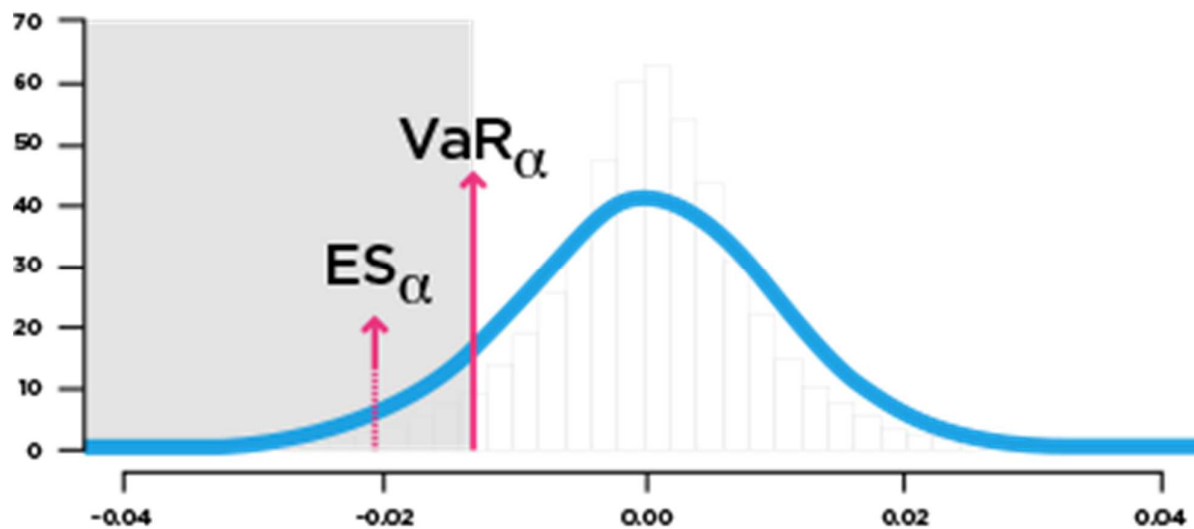**Value at Risk (VaR) for tolerance level *p*:**

$$p = \int_{-\infty}^{VaR_p} f(r)dr$$

$$P(r \leq VaR_p) = p$$

Expected shortfall (ES) = Conditional VaR (CVaR):

$$ES_p = E(r|r \leq VaR_p)$$

$$ES_p = \frac{1}{p}\int_{-\infty}^{VaR_p} rf(r)dr$$

Michał Rubaszek, Financial Econometrics

VaR and ES: calculation stages

SGH

1. Setting tolerance level p
2. Setting horizon H
3. Choosing estimation sample period $1:T$
4. Choosing a model
5. VaR/ES computation (for period $T+1$)
  + Model validation

Basel ii/iii: VaR as a risk measure

Basel iv: plans to change into ES

Quantitative standards Basel II

a. 99th percentile V*aR* must be computed on a daily basis

b. In calculating VaR the minimum *"holding period"* will be ten trading days. Banks may use VaR numbers calculated according to shorter holding periods scaled up to ten days by the square root of time

c. The choice of sample period for calculating VaR is constrained to a minimum length of one year.

d. banks will be free to use models based, for example, on variance-covariance matrices, historical simulations, or Monte Carlo simulations

e. The multiplication factor will be set by individual supervisory authorities on the basis of their assessment of the quality of the bank's risk management system, subject to an absolute minimum of 3. Banks will be required to add to this factor a "plus" directly related to the ex-post performance of the model, thereby introducing a built in positive incentive to maintain the predictive quality of the model. The plus will range from 0 to 1 based on the outcome of so-called "backtesting."

Michał Rubaszek, Financial Econometrics

A. Parametric / non-parametric models

B. Analytical formula / Monte-Carlo simulations

C. Conditional / unconditional volatility

Michał Rubaszek, Financial Econometrics

Parametric models: normal distribution

Analytical formula for $r \sim N(\mu, \sigma^2)$:

$$VaR_p = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p = \mu - \sigma \frac{\phi(\Phi^{-1}(p))}{p}$$

where ϕ and Φ are normal distribution pdf and cdf.

Numerical integral formula

$$ES_p = \mu + \sigma \frac{\int_0^p \Phi^{-1}(s)ds}{p}$$

Tables for $r \sim N(0,1)$:

p	0.5	0.1	0.05	0.025	0.01	0.001
VaR	0	1.282	1.645	1.960	2.326	3.090
ES	0.798	1.755	2.063	2.338	2.665	3.367

Parametric models: t-Student distribution

Formula for VaR = quantile p:

$$VaR_p = \mu + \sigma T_v^{-1}(p) \sqrt{\frac{v-2}{v}}$$

where T_v is the cdf of t-Student with v degrees of freedom

Numerical integral formula for ES

$$ES_p = \mu + \sigma \frac{\int_0^p T_v^{-1}(s)ds \sqrt{\frac{v-2}{v}}}{p}$$

Notes:

- The variance of $X \sim t_v$: $Var(X) = \frac{v}{v-2}$
- In R functions relate to t_v (e.g. `rt`) or scaled t_v (e.g. `rdist` in `rugarch`)

Non-parametric model: historical simulation

- We assume that the distribution of returns is well approximated by past/historical returns

- We sort past T returns from the lowest to highest: $rs_1 \leq rs_2 \ldots \leq rs_T$ and calculate VaR as p^{th} quantile. For $m = mod(pT)$:

$$VaR_p = rs_m$$

- ES is equal to the average of the worst returns lower than VaR

$$ES_p = \frac{1}{m} \sum_1^m rs_i$$

VaR and ES for further horizons

- To measure risk of investment for horizons $H > 1$ we need to approximate the distribution of:

$$y_H = \sum_{h=1}^{H} r_h$$

- Two kind of methods:
    - analytical (square root of time, SQRT)
    - mumerical (Monte Carlo, bootstraping)

Square root of time method

Let us assume that $r_t \sim N(\mu, \sigma^2)$ and r_t are *IID*. Then:

$$y_H = \sum_{h=1}^{H} r_h \sim N(H\mu, H\sigma^2)$$

In this case:

$$VaR_H = H\mu + \sqrt{H} \times \sigma \Phi^{-1}(p)$$

$$ES_H = H\mu - \sqrt{H} \times \sigma \frac{\phi(\Phi^{-1}(p))}{p}$$

For $\mu = 0$ this simplifies to:

$$VaR_H = \sqrt{H} \times VaR \quad \text{and} \quad ES_H = \sqrt{H} \times ES$$

This is why we call this method **square root of time**

Note: this method applied only for IID returns with normal distribution

---

Monte Carlo simulations

- Let us assume that returns are t-Student
  (or any other distr. for which we don't know analytical formula for the sum of vars.)
  In this case we resort to Monte Carlo simulations

- MC steps to calculate VaR/ES for any horizon H:
  1. Draw a path $r_1, r_2, \ldots, r_H$ of returns over horizon H and calculate $y_H = \sum_{h=1}^{H} r_h$
  2. Repeat step 1 "N" Times. Save $y_H^{(n)}$ for $n = 1, 2, \ldots, N$
  3. Sort cumulated returns $ys_H^{(1)} \leq ys_H^{(2)} \leq \ldots$
  4. Set $M = mod(pN)$
  5. Use formulas :

$$VaR_H = ys_H^{(M)}$$

$$ES_H = \frac{1}{M} \sum_{1}^{M} ys_H^{(i)}$$

Bootstrap

- When we use historical simulation method, an equivalent to MC simulations is Bootstrap
- Bootstrap steps to calculate VaR/ES for any horizon H:
  1. Draw H times with replacement from sample $r_{1:T}$. Use draws $r_1, r_2, \ldots, r_H$ to calculate $y_H = \sum_{h=1}^{H} r_h$
  2. Repeat step 1 "N" Times. Save $y_H^{(n)}$ for $n = 1, 2, \ldots, N$
  3. Sort cumulated returns $ys_H^{(1)} \leq ys_H^{(2)} \leq \ldots$
  4. Set $M = mod(pN)$
  5. Use formulas :

$$VaR_H = ys_H^{(M)}$$

$$ES_H = \frac{1}{M} \sum_{1}^{M} ys_H^{(i)}$$

Michał Rubaszek, Financial Econometrics

Exercises

Exercise 1.1.

The rate of return of a portfolio is t-Student distributed, where the number of degrees of freedom is equal to 5 (critical values are provided in table below). Moreover, it is known that the expected rate of return is 5% and standard deviation is 20%.

p	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
t-Student	-3.36	-2.76	-2.42	-2.19	-2.02	-1.87	-1.75	-1.65	-1.56	-1.48
scaled t-Student	-2.61	-2.14	-1.88	-1.70	-1.56	-1.45	-1.36	-1.28	-1.21	-1.14

a. Select the tolerance level p
b. Calculate VaR with pen and paper for $H = 1$ and $H = 4$ (with SQRT)
c. Calculate VaR and ES with R (for $H = 1$ and $H = 4$)
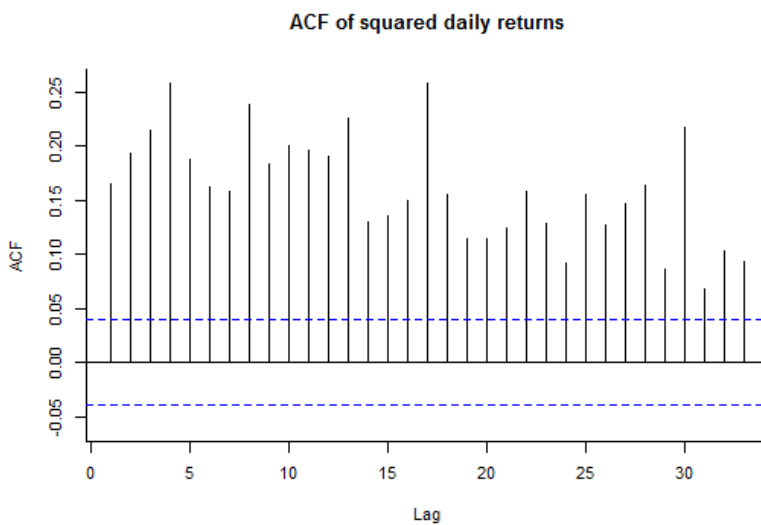d. Compare the results from points b and c

Is SQT justified?

Exercise 1.2.

The rate of return has an IID uniform distribution $r \sim U(-0.05; 0.05)$.
a. Calculate VaR and ES for $p = 0.05$ or 0.10
b. Can you find distribution for horizon $H = 2$?
c. Calculate VaR and ES for $p = 0.05$ or 0.10 for horizon $H = 2$. Compare the results with SQRT.

Michał Rubaszek, Financial Econometrics

Volatility clustering

Financial series characteristics: $cor(r_t^2, r_{t-s}^2) > 0$

ACF of squared daily returns

This implies that conditional variance is not constant in time:

$$\sigma_t^2 = E(r_t^2 \mid r_{t-1}^2, r_{t-2}^2, \ldots) \neq \sigma^2$$

Michał Rubaszek, Financial Econometrics

Volatility clustering, EWMA

- **Moving Average (MA):**

$$\sigma_t^2 = \frac{1}{S} \sum_{s=1}^{S} (r_{t-s} - \mu)^2$$

- **Exponentially Weighted Moving Average (EWMA),**
  A model proposed by JP Morgan in 1993, known also as RiskMetrics

$$\sigma_t^2 = \sum_{s=1}^{\infty} \lambda^s r_{t-s}^2$$

- Simplified formula for EWMA model, i.e. IGARCH(1,1):

$$\sigma_t^2 = (1 - \lambda) r_{t-1}^2 + \lambda \sigma_{t-1}^2$$

- JP Morgan calibrate $\lambda = 0.94$ for daily observations

Michał Rubaszek, Financial Econometrics

Volatility clustering, GARCH(1,1)

- **EWMA …**

$$\sigma_t^2 = (1 - \lambda)r_{t-1}^2 + \lambda\sigma_{t-1}^2$$

- **… as a specific version of GARCH(1,1):**

$$r_t = \mu + \epsilon_t, \ \epsilon_t \sim D(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

$$\omega > 0, \alpha, \beta \geq 0.$$

- **EWMA restrictions:**

$$\mu = 0$$
$$\omega = 0$$
$$\alpha = 1 - \lambda$$
$$\beta = \lambda$$

Volatility clustering, GARCH(1,1)

- **MA …**

$$\sigma_t^2 = \frac{1}{S}\sum_{s=1}^{S}(r_{t-s} - \mu)^2$$

- **… as a specific version of GARCH(S,0):**

$$r_t = \mu + \epsilon_t, \ \epsilon_t \sim D(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \alpha_1\epsilon_{t-1}^2 + \cdots + \alpha_S\epsilon_{t-S}^2$$

$$\omega > 0, \alpha, \beta \geq 0.$$

- **MA restrictions:**

$$\omega = 0$$
$$\alpha_s = 1/S \text{ for } s=1,2,\ldots S$$

Volatility clustering, GARCH(1,1)

- **GARCH(1,1):**

$$r_t = \mu + \epsilon_t, \qquad\qquad \epsilon_t \sim D(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$
$$\omega > 0, \alpha, \beta \geq 0.$$

- **Other notation:**

$$\sigma_t^2 = (1 - \alpha - \beta)\,\bar{\sigma}^2 + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

where

$$\bar{\sigma}^2 = \frac{\omega}{1 - (\alpha + \beta)}$$

is the equilibrium value of the variance.

- If $\alpha + \beta < 1$ then the variance is mean reverting (stationary model).
  For EWMA $\alpha + \beta = 1$: Integrated GARCH, IGARCH model

Michał Rubaszek, Financial Econometrics

GARCH: estimation

- The joint probability of all observations:

$$p(y_1, y_2, \ldots, y_T) = p(y_1|\Omega_0) \times p(y_2|\Omega_1) \times \cdots \times p(y_T|\Omega_{T-1})$$

where Ω_t is information set available till moment t

- If we assume that:

$$\epsilon_t|\Omega_{t-1} \sim N(0, \sigma_t^2)$$

then the likelihood is:

$$\mathcal{L}(\theta|y_{1:T}) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left(-\frac{(y_t - \mu_t)^2}{2\sigma_t^2}\right)$$

where θ is the vector of model parameters

Michał Rubaszek, Financial Econometrics

GARCH: estimation

- In many cases the conditional distribution of returns is also characterised by excess kurtosis or skewness.

- In this case we can assume that conditional distribution has t-Student distribution or skewed t-Student distribution.

- For t-Student distribution the likelihood is:

$$\mathcal{L}(\theta|y_{1:T}) = \prod_{t=1}^{T} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)}\sigma_t} \left(1 + \frac{(y_t - \mu_t)^2}{(v-2)\sigma_t^2}\right)^{-\left(\frac{v+1}{2}\right)}$$

- For skewed t-Student distribution, see rugarch vignette (link, page 19)

Michał Rubaszek, Financial Econometrics

GARCH(P,Q): specification selection

Specification selection stages:
1. Select the specification for levels (μ_t), usually a constant
2. Select the specification for the variance (σ_t^2), usually GARCH(1,1)
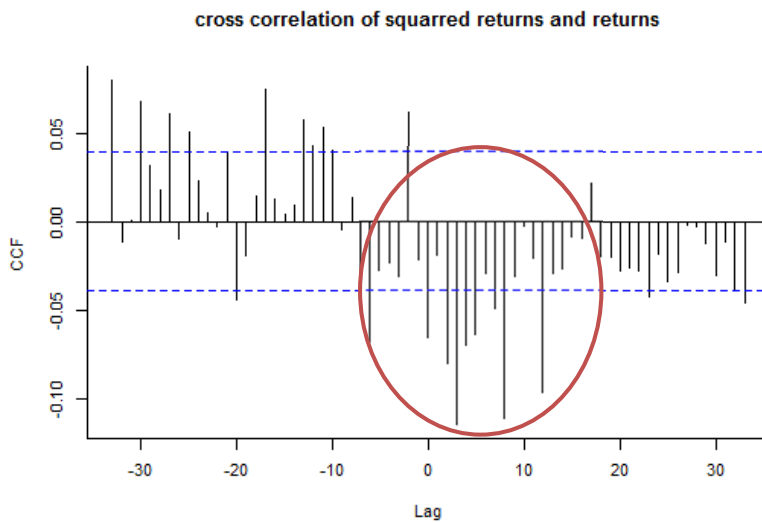3. Decide on the conditional distribution, usually t-Student

Criteria:
A. No autocorrelation for levels and squares of standarized residuals

$$u_t = \epsilon_t / \sigma_t$$

B. Minimization of information criteria (AIC, BIC, HQ)

Michał Rubaszek, Financial Econometrics

Leverage effect

One of financial series characteristics: $cor(r_t^2, r_{t-s}) < 0$

cross correlation of squared returns and returns

This implies that conditional variance depends on the sign of past returns

---

Asymetric GARCH models
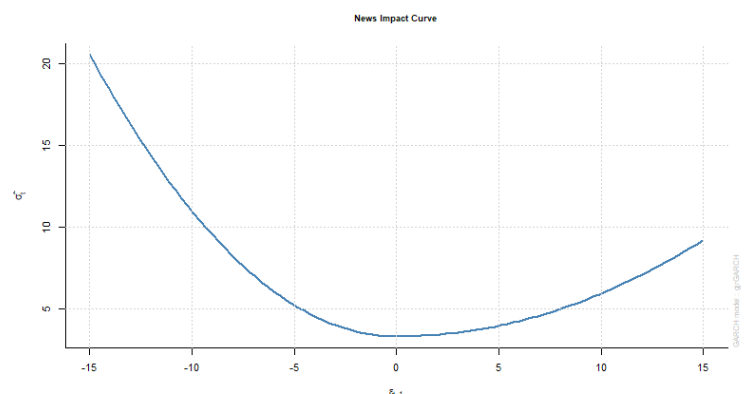
GJR-GARCH(1,1) by Glosten-Jagannathan-Runkle (1993):

$$r_t = \mu + \epsilon_t,$$
$$\sigma_t^2 = \omega + (\alpha + \gamma I(\epsilon_{t-1} < 0))\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$
$$\omega > 0, \alpha, \beta, \gamma \geq 0.$$

where:

$$I(x) = \begin{cases} 1 \text{ if } x \text{ is TRUE} \\ 0 \text{ if } x \text{ is FALSE} \end{cases}$$

so that:

News Impact Curve

$$\sigma_t^2 = \begin{cases} \omega + (\alpha + \gamma)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \text{ for } \epsilon_{t-1} < 0 \\ \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \quad\quad \text{ for } \epsilon_{t-1} \geq 0 \end{cases}$$

Asymetric GARCH models

E(xponential)GARCH(1,1) by Nelson (1991):

$$r_t = \mu + \epsilon_t,$$
$$\ln(\sigma_t^2) = \omega + \alpha u_{t-1} + \gamma |u_{t-1}| + \beta \ln(\sigma_{t-1}^2)$$

where $u_t = \epsilon_t / \sigma_t$ is a standarized error term

As a result:

$$\ln(\sigma_t^2) = \begin{cases} \omega + (\alpha - \gamma)u_{t-1} + \beta \ln(\sigma_{t-1}^2) \text{ for } \epsilon_{t-1} < 0 \\ \omega + (\alpha + \gamma)u_{t-1} + \beta \ln(\sigma_{t-1}^2) \text{ for } \epsilon_{t-1} \geq 0 \end{cases}$$

GARCH in Mean

- If investors are risk averse then expected return of risky (volatile) assets should be higher than the rate of return of stable assets (e.g. eturn on SP500 was on average 5% higher than from 3M TB)

- GARCH-M (GARCH in Mean, Engle, Lilien i Ronbins, 1987) :

$$r_t = \mu + \delta \sigma_t + \epsilon_t,$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

- Alternative specifications

$$r_t = \mu + \delta \sigma_t^2 + \epsilon_t$$
$$r_t = \mu + \delta \ln \sigma_t + \epsilon_t$$

Forecasting volatility with GARCH(1,1)

Variance forecast from GARCH model:

$$r_t = \mu + \epsilon_t, \qquad \epsilon_t \sim D(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

Given information set Ω_T, i.e. $\epsilon_{1:T}$ and $\sigma_{1:T}^2$, we can compute that:

$$\sigma_{T+1|T}^2 = \omega + \alpha \epsilon_T^2 + \beta \sigma_T^2$$

For futher horizons we need to notice that:

$$E(\epsilon_{T+h}^2) = \sigma_{T+h|T}^2$$

Hence:

$$\sigma_{T+2|T}^2 = \omega + (\alpha + \beta) \sigma_{T+1|T}^2$$

Notice that for $\alpha + \beta < 1$ the forecast converges towards:

$$\lim_{H \to \infty} \sigma_{T+H|T}^2 = \frac{\omega}{1 - (\alpha + \beta)}$$

Simulating future returns form a GARCH

Steps to simulate a single path of returns over horizon H from GARCH model:

$$r_t = \mu + \epsilon_t, \qquad \epsilon_t \sim D(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

1. Given information set Ω_T calculate σ_{T+1}^2
2. Draw ϵ_{T+1} from distribution $D(0, \sigma_{T+1}^2)$
3. Calculate r_{T+1}
4. Conditional on the draw for ϵ_{T+1} calculate σ_{T+2}^2

… continue until you have the path for $r_1, r_2, \dots, r_H$

Calculating VaR/ES with GARCH models

MC steps to calculate VaR/ES for any horizon H from GARCH model:

1. Simulate a path $r_1, r_2, \ldots, r_H$ of returns over horizon H and cumulate $y_H = \sum_{h=1}^{H} r_h$
2. Repeat step 1 "N" Times. Save $y_H^{(n)}$ for $n = 1, 2, \ldots, N$
3. Sort cumulated returns $ys_H^{(1)} \le ys_H^{(2)} \le \ldots$
4. Set $M = mod(pN)$
5. Use formulas :

$$VaR_H = ys_H^{(M)}$$

$$ES_H = \frac{1}{M} \sum_{1}^{M} ys_H^{(i)}$$

Exercises

Exercise 1.3.

Let r_t be weekly log-return (expressed as %) for a portfolio. The estimatates of the GARCH(1,1) model as as follows:

$r_t = 0.08 + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_t^2)$
$\sigma_t^2 = 0.025 + 0.10\epsilon_{t-1}^2 + 0.80\sigma_{t-1}^2$

a. What is the average annual rate of return (assume that a year is 52 weeks)?
b. Calculate the unconditional variance (and standard deviation) for weekly data
c. Knowing that $\epsilon_T^2 = 0.15$ and $\sigma_T^2 = 0.4$ calculate the forecast $\sigma_{T+1|T}^2$
d. Select the tolerance level p and calculate VaR and ES using the values from table below

p	0.5	0.1	0.05	0.025	0.01	0.001
VaR	0	1.282	1.645	1.960	2.326	3.090
ES	0.798	1.755	2.063	2.338	2.665	3.367

Exercise 1.4.

Build an equally weighted protfolio consisting of two stocks quoted on WSE. You can download the data with the command:

```
data      <- read.csv("http://web.sgh.waw.pl/~mrubas/EFII/Dane/wig.csv", row.names = "X")
```

a. Make a graph of historical time series. If the history is shorter than 5 years, select other stocks
b. Select the tolerance level p
c. Calculate VaR/ES for horizons $H = 1$ and $H = 10$ using parametric models (normal, t-Student); historical simulation; EWMA; GARCH(1,1) model
d. Fill in the table below

	VaR		ES	
	H=1	H=10	H=1	H=10
Normal				
t-Student				
Historical simulation				
EWMA				
GARCH				

Exercise 1.5.

Build an equally weighted protfolio consisting of two stocks quoted on WSE. You can download the data with the command:

```
data      <- read.csv("http://web.sgh.waw.pl/~mrubas/EFII/Dane/wig.csv", row.names = "X")
```

Construct the best GARCH model:

a. Specify lags P and Q of GARCH(P,Q) as well as the error term distribution with the BIC criterion
b. Check for the autocorrelation of standardized residuals
c. Check for the leverage effect (GJR-GARCH / EGARCH)
d. Check for in-Mean effect
e. Calculate VaR/ES for horizons $H = 1$ and $H = 10$
f. Compare the results from point e to GARCH(1,1) – see Exercise 1.4
g. Calculate the forecast for standard deviation σ_t at horizon $H = 1000$ and compare it to sample standard deviation for portfolio returns. Are the differences sizeable?

Meeting 2. Risk of a multivariate portfolio: MGARCH models

MGARCH: general specification

Let y_t be a vector of returns for individual assets entering the investment portfolio.
For the joint distribution, let us assume that:

$$y_t = \mu_t + \epsilon_t$$

$$Cov(\epsilon_t) = H_t$$

where:

$y_t = (y_{1t}, y_{2t}, \dots, y_{Nt})'$ is the vector of returns

$\mu_t = (\mu_{1t}, \mu_{2t}, \dots, \mu_{Nt})'$ is the conditional mean

$\epsilon_t = (\epsilon_{1t}, \epsilon_{2t}, \dots, \epsilon_{Nt})'$ is the error term

$H_t = [h_{ij}]_{N \times N}$ is the conditional covariance matrix

In MGARCH model we model the dynamics of H_t as a function of:

- past values of the covariance matrix H_{t-q} for $q = 1, 2, \dots, Q$
- realization of the error term $\epsilon_{t-p}\epsilon'_{t-p}$ for $p = 1, 2, \dots, P$

Michał Rubaszek, Financial Econometrics

MGARCH: classification

MGARCH models can be classified depending on the specification of the dynamics for the covariance matrix H_t into (see Bauwens et al. 2006, JAE):

1. Direct generalizations of the univariate GARCH
    (VEC GARCH or BEKK)

2. Linear combinations of univariate GARCH model
    (GO-GARCH)

3. Nonlinear combinations of univariate GARCH models
    (DCC-GARCH)

VEC-GARCH

VEC-GARCH(1,1) proposed by Bollerslev, Engle and Wooldridge (1988)

$$y_t = \mu_t + \epsilon_t, \epsilon_t \sim N(0, H_t)$$
$$h_t = \omega + Ae_{t-1} + Bh_{t-1}$$
$$h_t = vech(H_t)$$
$$e_t = vech(\epsilon_t \epsilon_t')$$

where $vech(\cdot)$ denotes the operator that stacks the lower triangular portion of a $N \times N$ matrix as a $\frac{(N+1)N}{2} \times 1$ vector.

Problems:

- Large number of parameters: A and B are $\frac{(N+1)N}{2} \times \frac{(N+1)N}{2}$ matrices
- Difficulties in ensuring that H_t is positive definite

VEC-GARCH

Bivariate example of VEC GARCH:

$$h_t = \omega + Ae_{t-1} + Bh_{t-1}$$

$$\begin{bmatrix} h_{11,t} \\ h_{22,t} \\ h_{12,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{12} \end{bmatrix} + \begin{bmatrix} \alpha_{11,1} & \alpha_{11,2} & \alpha_{11,3} \\ \alpha_{22,1} & \alpha_{22,2} & \alpha_{22,3} \\ \alpha_{12,1} & \alpha_{12,2} & \alpha_{12,3} \end{bmatrix} \begin{bmatrix} e_{11,t-1} \\ e_{22,t-1} \\ e_{12,t-1} \end{bmatrix} + \begin{bmatrix} \beta_{11,1} & \beta_{11,2} & \beta_{11,3} \\ \beta_{22,1} & \beta_{22,2} & \beta_{22,3} \\ \beta_{12,1} & \beta_{12,2} & \beta_{12,3} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{12,t-1} \end{bmatrix}$$

where $e_{ij} = \epsilon_i \times \epsilon_j$

DVEC-GARCH

To limit the number of parameters in VEC-GARCH model, Bollerslev et al (1988) proposed its restriction version, in which matrices A and B from
$$h_t = \omega + Ae_{t-1} + Bh_{t-1}$$
are assumed to be diagonal, so that equation changes into:

$$h_{ij,t} = \omega_{ij} + \alpha_{ij}\epsilon_{i,t-1}\epsilon_{j,t-1} + \beta_{ij}h_{ij,t-1}$$

Even though the number of parameters decreases, the problem of ensuring that H_t is positive definite remains

Important: multivariate EWMA from Riskmetrics is calibrated DVEC-GARCH:

$$h_{ij,t} = (1 - \lambda)\epsilon_{i,t-1}\epsilon_{j,t-1} + \lambda h_{ij,t-1}$$

Bivariate example of DVEC GARCH:

$$h_t = \omega + Ae_{t-1} + Bh_{t-1}$$

$$\begin{bmatrix} h_{11,t} \\ h_{22,t} \\ h_{12,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{12} \end{bmatrix} + \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{12} \end{bmatrix} \begin{bmatrix} e_{11,t-1} \\ e_{22,t-1} \\ e_{12,t-1} \end{bmatrix} + \begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & \beta_{12} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{12,t-1} \end{bmatrix}$$

where $e_{ij} = \epsilon_i \times \epsilon_j$

Engle and Kroner (1995) proposed BEKK-GARCH model, in which H_t is always positive definite

$$y_t = \mu_t + \epsilon_t, \epsilon_t \sim N(0, H_t)$$
$$H_t = \Omega\Omega' + A\epsilon_{t-1}\epsilon'_{t-1}A' + BH_{t-1}B'$$

where Ω is lower triangular matrix, whereas A and B are $N \times N$ matrices.

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} & 0 \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{12} \\ 0 & \omega_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} e_{11,t-1} & e_{21,t-1} \\ e_{12,t-1} & e_{22,t-1} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} h_{11,t-1} & h_{21,t-1} \\ h_{12,t-1} & h_{22,t-1} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix}$$

where $e_{ij} = \epsilon_i \times \epsilon_j$

Note: Due to a large number of params BEKK model is rarely used when $N > 3$

Exercise 2.1.

The BEKK GARCH model describing the dynamics of a bivariate vector $r = (r_1, r_2)'$:

$$r_t \sim N(0, H_t)$$

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0.3 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.0 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} e_{11,t-1} & e_{12,t-1} \\ e_{12,t-1} & e_{22,t-1} \end{bmatrix} \begin{bmatrix} 0.2 & 0.1 \\ 0.0 & 0.2 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.0 \\ 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} h_{11,t-1} & h_{12,t-1} \\ h_{12,t-1} & h_{22,t-1} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.0 & 0.6 \end{bmatrix}$$

a. Assuming that $\epsilon_{1,T} = 0$; $\epsilon_{2,T} = 0$; $h_{11,T} = 3$; $h_{12,T} = 1$; $h_{22,T} = 5$; make a forecast for H_{T+1}
b. Calculate the variance σ_{T+1}^2 of a portfolio with weights $w = (0.5, 0.5)'$
c. Calculate the $VaR_{5\%}$ of a portfolio with weights $w = (0.5, 0.5)'$ knowing that $\Phi^{-1}(0.05) = -1.64$
d. Repeat points b and c for $w = (0.25, 0.75)'$

Factor-GARCH

SGH

Engle, Ng and Rothschild (1990) proposed a factor specification of MGARCH model, in which the dynamics of H_t is described by K factors:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)$$

$$\epsilon_t = \Lambda f_t + \eta_t, \quad \eta_t \sim N(0, \Gamma), \qquad f_t \sim N(0, D_t),$$

$$\Gamma = \text{diag}(\gamma_1^2, \gamma_2^2, ..., \gamma_N^2), \quad D_t = \text{diag}(d_{1t}, d_{2t}, ..., d_{Kt})$$

$$H_t = \Lambda D_t \Lambda' + \Gamma$$

$$d_{kt} = \omega_k + \alpha_k f_{k,t-1}^2 + \beta_k d_{k,t-1}$$

Note: This specification allows to transform the problem of finding the dynamics for multidimentional matrix H_t into a problem of finding the dynamics of K univariate processes

Factor-GARCH

Bivariate Factor-GARCH(1,1,2) with no idiosyncratic term

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)$$
$$\epsilon_t = \Lambda f_t, \quad f_t \sim N(0, D_t),$$

$$d_{1t} = \omega_1 + \alpha_1 f_{1,t-1}^2 + \beta_1 d_{1,t-1}$$
$$d_{2t} = \omega_2 + \alpha_2 f_{2,t-1}^2 + \beta_2 d_{2,t-1}$$

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} d_{1,t} & 0 \\ 0 & d_{2,t} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{bmatrix}$$

Michał Rubaszek, Financial Econometrics

Exercises

Exercise 2.2.

On the basis of the below relationship:

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} d_{1,t} & 0 \\ 0 & d_{2,t} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{bmatrix}$$

write the formula for the value of conditional correlation in Factor-GARCH model as a function of d_{1t} and d_{2t}. Is the dynamic for the calculated formula transparent?

Note that: $\rho_{12,t} = \dfrac{h_{12,t}}{\sqrt{h_{11,t}}\sqrt{h_{22,t}}}$

Michał Rubaszek, Financial Econometrics

GO-GARCH (Generalized Orthogonal)

Van der Weide (1990) proposed a specific verion of Factor-GARCH model, based on spectral decomposition of population cov. matrix, combined with rotation:

$$H \quad = PLP^{-1}, P - \text{eigenvectors matrix}, L - \text{eigenvalues matrix}$$
$$\Lambda \quad = PL^{0.5}U$$
$$UU' = I, \qquad U - \text{orthonormal, rotation matrix}$$

GO-GARCH model:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)$$
$$\epsilon_t = \Lambda f_t, \qquad f_t \sim N(0, D_t),$$
$$D_t = \text{diag}(d_{1t}, d_{2t}, \ldots, d_{Nt})$$
$$H_t = \Lambda D_t \Lambda'$$
$$d_{it} = \omega_i + \alpha_i f_{i,t-1}^2 + \beta_i d_{i,t-1}$$

Michał Rubaszek, Financial Econometrics

CCC-GARCH (Constant Conditional Correlation)

- Bollerslev (2002) proposed the CCC GARCH of the following form:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)$$
$$H_t = D_t P D_t'$$
$$D_t = \text{diag}(\sqrt{d_{1t}}, \sqrt{d_{2t}}, \ldots, \sqrt{d_{Nt}})$$
$$d_{it} = \omega_i + \alpha_i \epsilon_{i,t-1}^2 + \beta_i d_{i,t-1} \text{ for } i = 1, 2, \ldots, N$$

where P is the unconditional correlation matrix

- E(xtended)CCC-GARCH by Jeantheau (1998)

$$d_{it} = \omega_i + \sum_{j=1}^{N} \alpha_{ij} \epsilon_{j,t-1}^2 + \sum_{j=1}^{N} \beta_{ij} d_{j,t-1} \text{ for } i = 1, 2, \ldots, N$$

Michał Rubaszek, Financial Econometrics

Engle (2002) proposed the DCC GARCH of the following form:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)$$
$$H_t = D_t P_t D_t'$$
$$D_t = \text{diag}(\sqrt{d_{1t}}, \sqrt{d_{2t}}, \ldots, \sqrt{d_{Nt}})$$
$$d_{it} = \omega_i + \alpha_i \epsilon_{i,t-1}^2 + \beta_i d_{i,t-1} \text{ for } i = 1,2, \ldots, N$$

where P_t is the conditional correlation matrix with the law of motion:

$$z_t = D_t^{-1} \epsilon_t$$
$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha z_{t-1} z_{t-1}' + \beta Q_{t-1}$$
$$P_t = (Q_t \odot I)^{-0.5} Q_t (Q_t \odot I)^{-0.5}$$

where $\odot$ is the Hadamard operator (element-by-element multiplication)

Michał Rubaszek, Financial Econometrics

Exercises   **SGH**

Exercise 2.3.

The CCC GARCH model describing the dynamics of a bivariate vector $r = (r_1, r_2)'$ is:

$$r_t = \begin{bmatrix} 0.8 \\ 1.2 \end{bmatrix} + \epsilon_t; \quad \epsilon_t \sim N(0, H_t)$$
$$H_t = D_t \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} D_t'$$
$$\begin{bmatrix} d_{1t} \\ d_{2t} \end{bmatrix} = \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix} + \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix} \begin{bmatrix} \epsilon_{1t-1}^2 \\ \epsilon_{2t-1}^2 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.0 \\ 0.0 & 0.7 \end{bmatrix} \begin{bmatrix} d_{1,t-1} \\ d_{2,t-1} \end{bmatrix}$$

a.   Assuming that $\epsilon_{1T}^2 = 23$; $\epsilon_{2T}^2 = 16$; $d_{1T} = 16$; $d_{2T} = 9$; make a forecast for H_{T+1}

b.   Calculate the parameters of the distribution of returns a portfolio with weights $w = (0.25, 0.75)'$ for the period $T + 1$

c.   Calculate the $VaR_{2.5\%}$ of a portfolio with weights $w = (0.25, 0.75)'$ knowing that $\Phi^{-1}(0.025) = -1.96$

d.   Calculate the equilibrium value of H_t?

Michał Rubaszek, Financial Econometrics

Exercise 2.4.

For a protfolio consisting of two stocks quoted on WSE (you can download the data with)

```
data      <- read.csv("http://web.sgh.waw.pl/~mrubas/EFII/Dane/wig.csv", row.names = "X")
```

   a.   Make a graph of historical time series. If the history is shorter than 5 years, select other stocks
   b.   Select the tolerance level p
   c.   Estimate GO-GARCH(1,1) and DCC-GARCH(1,1) model. Which is better fitted to the data?
   d.   Make a graph of conditional std. dev. for two vars. from both models (one chart per. variable)
   e.   Make o graph of conditional correlation dynamics from both models (one chart)
   f.   Make a graph of conditional std. dev. of a portfolio from both models (one chart)
   g.   Calculate VaR/ES for horizons $H = 1$ and $H = 10$ and fill in the table below

	VaR		ES	
	H=1	H=10	H=1	H=10
GO-GARCH				
DCC-GARCH				

SGH

Meeting 3. Risk of a multivariate portfolio: Copulas

Multivariate normal distribution

- In many applications it is convenient to assume that multivariate returns have multivariate normal distribution:

$$r \sim N(\mu, \Sigma)$$

- For a portfolio of assets with weights w the rate of return:

$$r_p = w'r \sim N(\mu_p, \sigma_p^2)$$

where $\mu_p = w'\mu$ and $\sigma_p^2 = w'\Sigma w$

Multivariate normal distribution

- Multivariate normal distribution implies that the relationship between variables Y and X is linear

$$Y = a + bX + \epsilon, \qquad b = \frac{cov(Y, X)}{var(X)}, \qquad a = \bar{Y} - b\bar{X}$$

- In other words, the relationship is always the same and does not depend on the scale of change …

- … but at financial markets dependences tend to be stronger during crashes that in normal times, which leads to risk undervaluation!!!

Non-linear dependencies: illustration

The scale of nonlinearity of dependencies between two variables can be illustrated by comparing a scatter plot of realized and simulated series

---
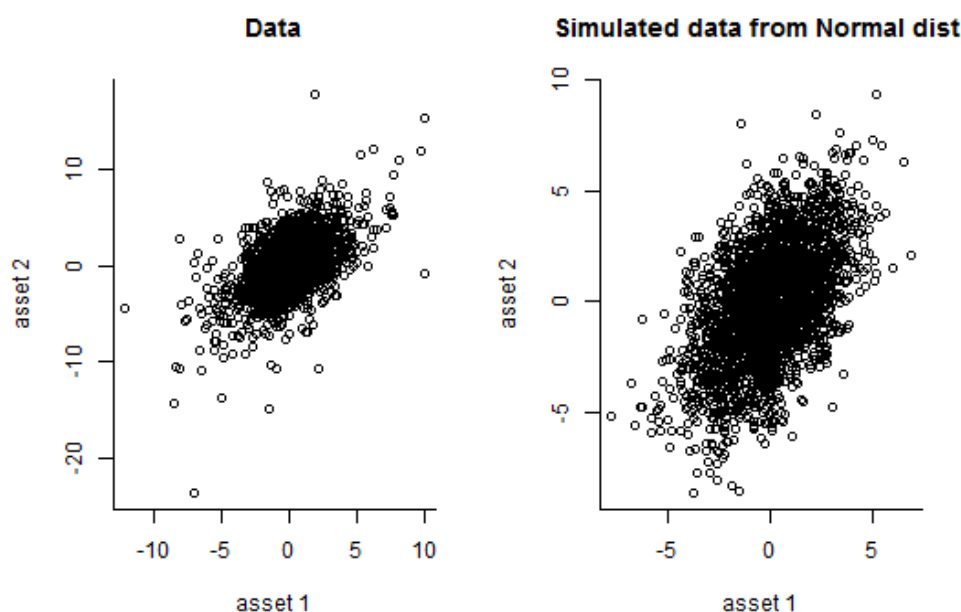
Copula function: a general idea

- Copula as a convenient method to model non-linear dependencies between variables X and Y (we will discuss only a bivariate case)

- A general idea is to decompose a (sophisticated) joint distribution of X and Y into:
    - Univariate marginal distributions for X and Y
    - Copula function, which combines both marginal distributions

Copula function

SKLAR'S THEOREM:

For:

$H(X, Y)$ multivariate/bivariate joint cdf

$F(X)$ and $G(Y)$ univariate marginal cdf

there exists a copula $C()$ for which:

$$H(X, Y) = C\big(F(X), G(Y)\big)$$

If F and G are continous, then C is unique

Copula function: notation

$f(X); g(Y)$:   pdf of marginal distributions

$U = F(X); V = G(Y)$:   cdf of marginal distributions

$h(X, Y)$:   pdf of joint distribution

$H(X, Y)$:   cdf of joint distribution

$C(U, V)$:   copula function

$$H(X, Y) = C(F(X), G(Y)) = C(U, V)$$

$$h(X, Y) = f(X)g(Y)C(F(X), G(Y))$$

To draw from the joint distribution, we need to decide on:

- the shape of marginal distributions $F(X)$ and $G(Y)$
- the shape of copula function $C(U, V)$

Most popular copulas

Empirical copula

- Let $\hat{F}$ and $\hat{G}$ be the empirical distribution functions for x_t and y_t:
$$\hat{F}(x) = \#(x_t < x)/T \text{ and } \hat{G}(y) = \#(y_t < y)/T$$

  and $u_t = \hat{F}(x_t)$ and $v_t = \hat{G}(y_t)$, where $t = 1, 2, \ldots, T$

- Definition of empirical copula:

$$C(u,v) = \frac{\#(u_t \leq u \ \wedge v_t \leq v)}{T}$$

- (discrete) probability density of empirical copula is:

$$c(u,v) = \frac{\#(u_t = u \ \wedge v_t = v)}{T}$$

where $\#(z)$ is the number of observations t that fulfill condition z

Exercise 3.1.

For a sample of observations:

t	1	2	3	4	5
x	1	2	3	4	6
y	7	5	1	3	4

calculate:

a. The values of u_t and v_t for $t = 1,2,3,4,5$
b. Empirical copula
c. Density of empirical copula

Exercise 3.2.

Let (X, Y) be the random variables describing the outcome of rolling two dices.

a. What is the marginal pdf/cdf for X and Y?
b. What is the joint pdf for (X, Y)
c. What is the density copula for (X, Y)
d. Roll two dices 10 times to create your sample for (x, y) and calculate the empirical copula / density of empirical copula. Use function `x <- sample(1:6,10,replace=TRUE)`

Michał Rubaszek, Financial Econometrics

Elliptic copulas

SGH

Normal copula:

$$C(U, V) = \Phi_\Sigma(\Phi^{-1}(U), \Phi^{-1}(V))$$

where Φ/Φ_Σ is univariate/multivariate normal cdf and Σ is the covariance matrix

t-Student copula:

$$C(U, V) = T_{v,\Sigma}(T_v^{-1}(U), T_v^{-1}(V))$$

where $T_v/T_{v,\Sigma}$ is univariate/multivariate t-Student cdf with v degrees of freedom

Michał Rubaszek, Financial Econometrics

Elliptic copulas

Normal (rho=0.5)
Contour for h(X,Y)

t-Student (df=3, rho=0.5)
Contour for h(X,Y)

Scatter plot

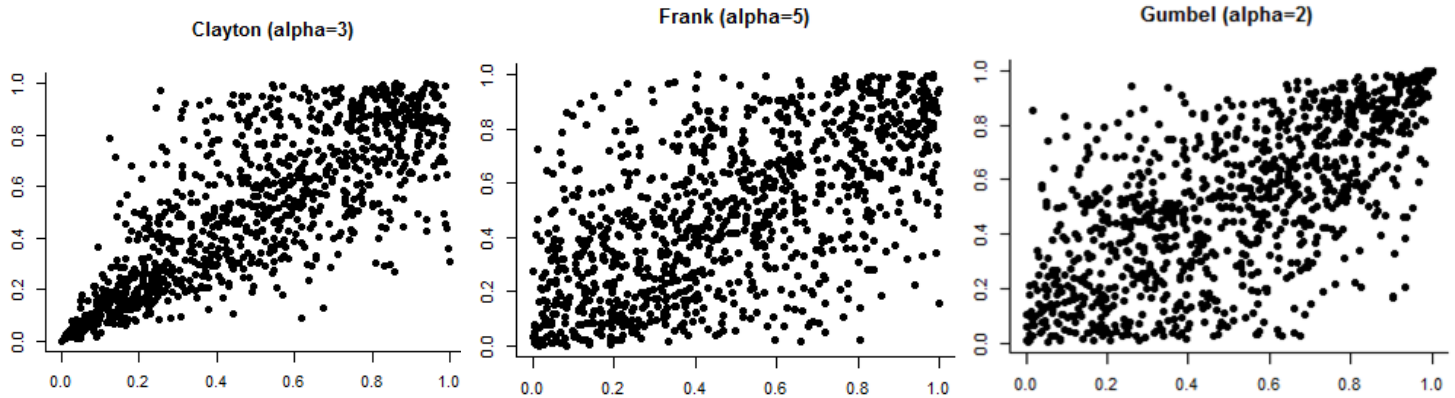Scatter plot

Michał Rubaszek, Financial Econometrics
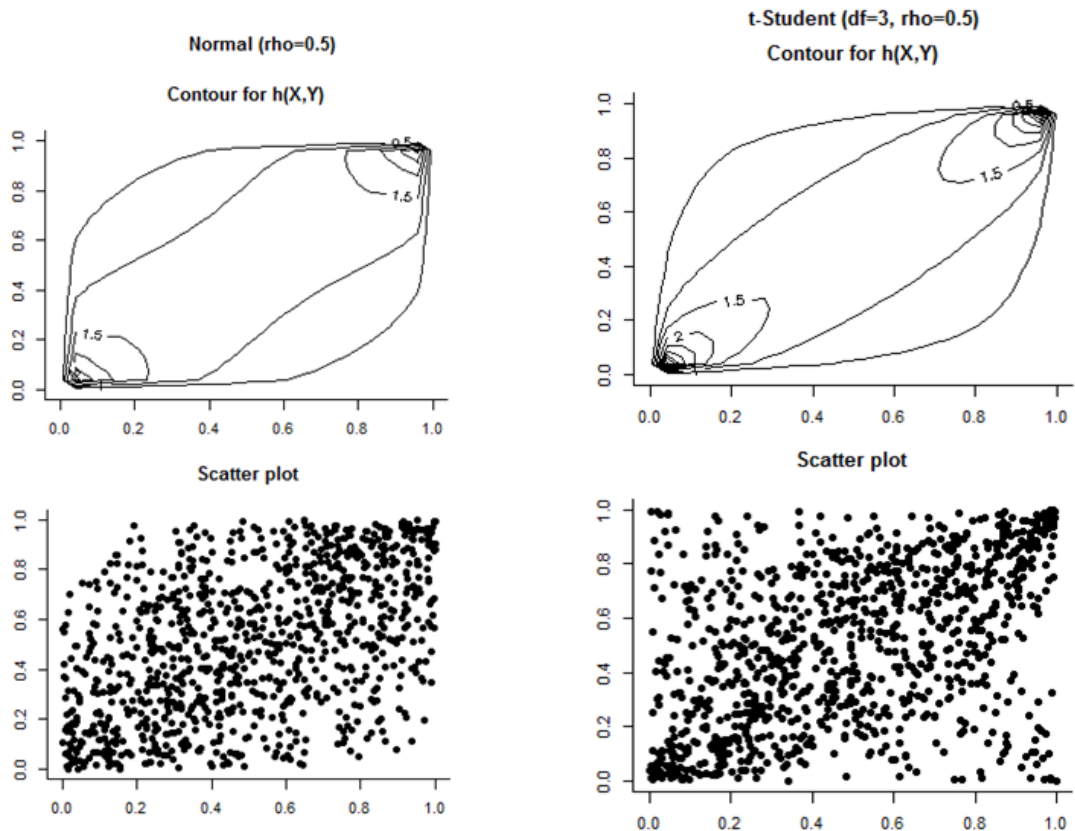
Archimedean copulas

The formula for copula:

$$C(U,V) = \varphi^{-1}(\varphi(U) + \varphi(V))$$

where φ is the generator

copula	$C(u,v)$	$\varphi(t)$	$\varphi^{-1}(t)$
Clayton	$C(u,v) = \left(\max\left\{u^{-\theta} + v^{-\theta} - 1; 0\right\}\right)^{-1/\theta}$	$\varphi(t) = (1+t)^{-1/\theta}$	$\varphi^{-1}(t) = t^{-\theta} - 1$
Gumbel	$C(u,v) = \mathrm{e}^{-\left((-\log(u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}}$	$\varphi(t) = \mathrm{e}^{\left(-t^{1/\theta}\right)}$	$\varphi^{-1}(t) = (-\log(t))^{\theta}$
Frank	$C(u,v) = \frac{-1}{\theta}\log\left(1 + \frac{(\mathrm{e}^{-\theta u}-1)(\mathrm{e}^{-\theta v}-1)}{\mathrm{e}^{-\theta}-1}\right)$	$\varphi(t) = \frac{-1}{\theta}\log(\mathrm{e}^{-t}(\mathrm{e}^{-\theta}-1)+1)$	$\varphi^{-1}(t) = -\log\left(\frac{\mathrm{e}^{-\theta t}-1}{\mathrm{e}^{-\theta}-1}\right)$

Michał Rubaszek, Financial Econometrics

Archimedean copulas

Clayton (alpha=3) | Frank (alpha=5) | Gumbel (alpha=2)

Michał Rubaszek, Financial Econometrics

Tail dependence

- Copulas are useful tool for panic/crisis scenarios, in which the dependencies between assets tend to be stronger

- A useful measure for these scenarios is downward:

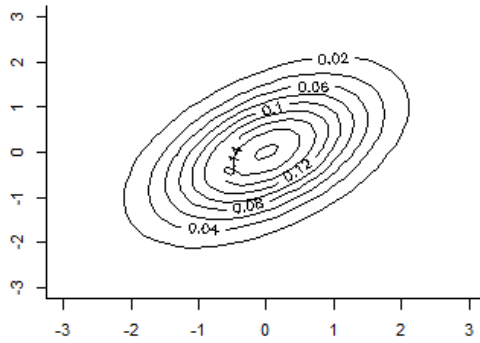$$\lambda^L = \lim_{q \to 0} \frac{C(q, q)}{q}$$

  or upward tail dependence:

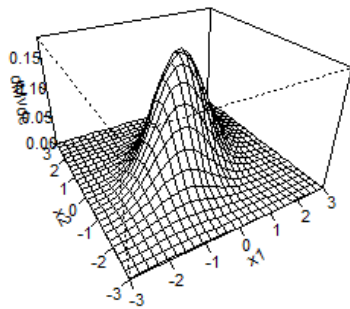$$\lambda^U = \lim_{q \to 0} \frac{1 - C(1 - q, 1 - q)}{q}$$

- The values for copulas:
  - Normal / t-Student: $\lambda^L = 0$ $\lambda^U = 0$ if $\rho < 1$
  - Clayton: $\lambda^L = 2^{-\frac{1}{\theta}}$ $\lambda^U = 0$
  - Gumbel: $\lambda^L = 0$ $\lambda^U = 2 - 2^{\frac{1}{\theta}}$
  - Frank: $\lambda^L = 0$ $\lambda^U = 0$

Michał Rubaszek, Financial Econometrics

Joint distribution from copulas

Marginal: normal/normal, Copula: normal (rho=0.5)
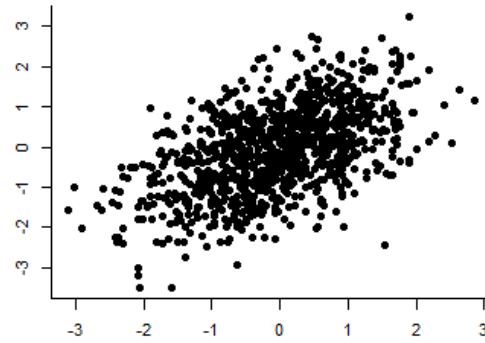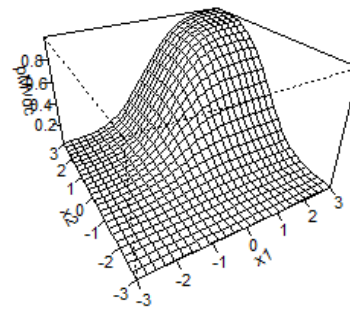
Contour for h(X,Y) — Scatter plot

density: h(X,Y) — cumulative density: H(X,Y)

Michał Rubaszek, Financial Econometrics

Joint distribution from copulas

Marginal: t/t (df=3), Copula: normal (rho=0.5)

Contour for h(X,Y) — Scatter plot

density: h(X,Y) — cumulative density: H(X,Y)

Michał Rubaszek, Financial Econometrics
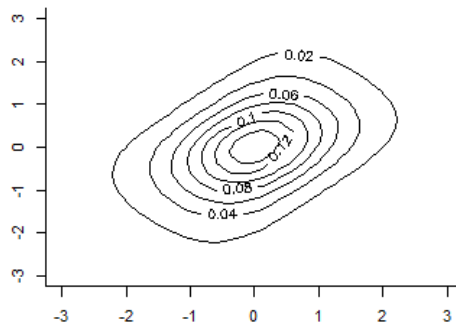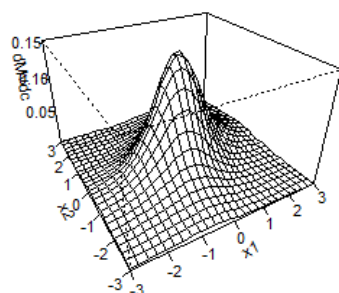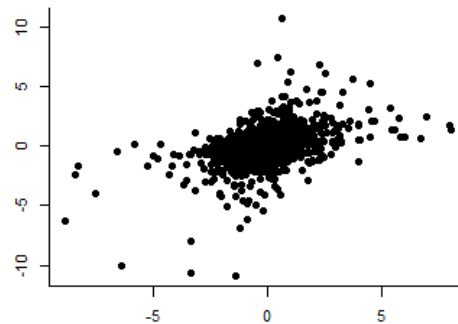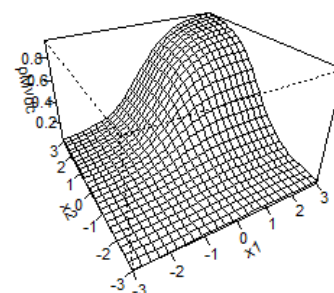
Marginal: gamma/gamma (2,1), Copula: normal (rho=0.5)

Michał Rubaszek, Financial Econometrics

SGH

Fitting to the data

Kendall tau correlation
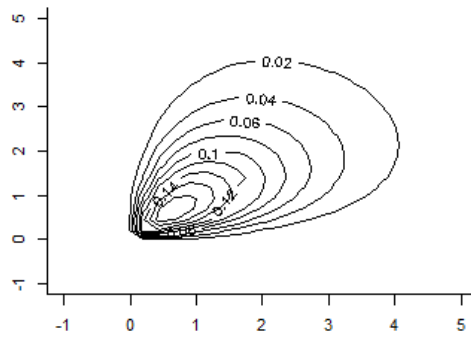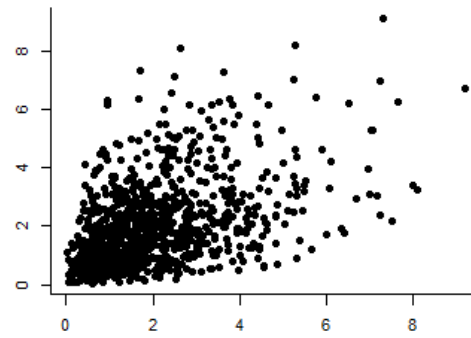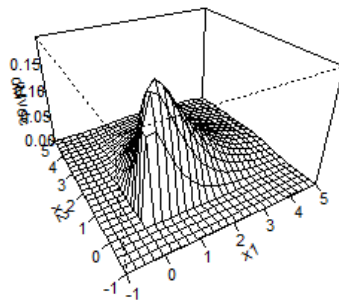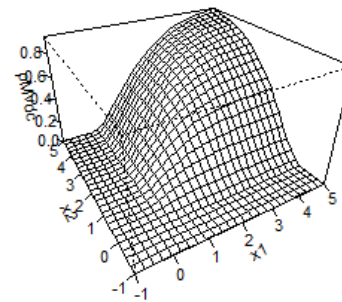
Definition:

$$\tau = P\{(x_i - x_j)(y_i - y_j) > 0\} - P\{(x_i - x_j)(y_i - y_j) < 0\}$$

Sample estimate:

$$\hat{\tau} = \frac{P - Q}{N(N-1)/2}$$

P – number of concordant pairs: $\quad N\{(x_i - x_j)(y_i - y_j) > 0\}$

Q – number of discordant pairs: $\quad N\{(x_i - x_j)(y_i - y_j) < 0\}$

Fitting copula to the data: method of moments

For copula $C(U, V|\theta)$ we are looking parameter θ for which:

$$\tau_\alpha = 4\int \int C(U,V|\theta) dC(U,V|\theta) - 1$$

is closest to $\hat{\tau}$

Formulas:

Normal copula / t-Student: $\quad \rho = \sin(\frac{\pi}{2}\tau)$

Clayton copula: $\quad \theta = 2\tau(2-\tau)^{-1}, \;\; \theta > 0, \tau > 0$

Gumbel copula: $\quad \theta = (1-\tau)^{-1} \quad, \;\; \theta \geq 0, \tau \geq 0$

Frank copula: $\quad \tau = 1 - \frac{4}{\theta}\left[1 - \int_0^1 \frac{\ln(1-t^\theta)(1-t^\theta)}{t^{\theta-1}} dt\right]$

Exercise 3.3.

For a sample of observations:

t	1	2	3	4	5
x	1	2	3	4	6
y	3	5	1	6	7

calculate:

a. Kandal tau
b. Estimate of the parameter for normal copula
c. Estimate of the parameter for Clayton copula
d. Estimate of the parameter for Gumbel copula

Michał Rubaszek, Financial Econometrics

Fitting copula to the data: maximum likelihood   **SGH**

One-step procedure (full ML)

We are looking for parameter θ which maximizes

$$l(\theta|X,Y) = \sum_{t=1}^{T} \log c[F(x_t|\theta), G(y_t|\theta)] + \log f(x_t|\theta) + \log g(y_t|\theta)$$

Two-step procedure

Step 1: Estimate marginal distribution parameters

Step 2: Estimate copula function parameters

Michał Rubaszek, Financial Econometrics

1. Draw (u, v) from $C(U, V)$
2. Calculate $x = F^{-1}(u)$ and $y = G^{-1}(v)$
3. Calculate the simulate rate of return of the portfolio $r = w_x x + w_y y$
4. Repeat steps 1-3 H times to simulate a path $r_1, r_2, \dots, r_H$ of returns over horizon H and calculate $y_H = \sum_{h=1}^{H} r_h$
5. Repeat steps 1-4 N times. Save $y_H^{(n)}$ for $n = 1, 2, \dots, N$
6. Sort cumulated returns $ys_H^{(1)} \leq ys_H^{(2)} \leq \dots$
7. Set $M = mod(pN)$
8. Use formulas :

$$VaR_H = ys_H^{(M)}$$

$$ES_H = \frac{1}{M} \sum_{1}^{M} ys_H^{(i)}$$

Michał Rubaszek, Financial Econometrics

Exercises

SGH

Exercise 3.4.

For a protfolio consisting of two stocks quoted on WSE (you can download the data with)

```
data      <- read.csv("http://web.sgh.waw.pl/~mrubas/EFII/Dane/wig.csv", row.names = "X")
```

a. Make a graph of historical time series. If the history is shorter than 5 years, select other stocks
b. Select the tolerance level p
c. Estimate 5 copulas listed in table below. Which is the best fitted to the data?
d. Calculate VaR/ES for horizons $H = 1$ and $H = 10$
e.   Fill in the table below

|	VaR		ES	
	H=1	H=10	H=1	H=10
Normal				
t-Student				
Clayton				
Gumbel				
Frank				

Michał Rubaszek, Financial Econometrics

# Meeting 4. Backtesting

---

What is backtesting

- "backtesting" in finance = "out-of-sample evaluation" in economics
- backtesting allows to assess model performance if it was used in the past
- for VaR we compare the share of VaR exceedances to tolerance level
- for ES we check if the scale of exceedances is correctly calibrated
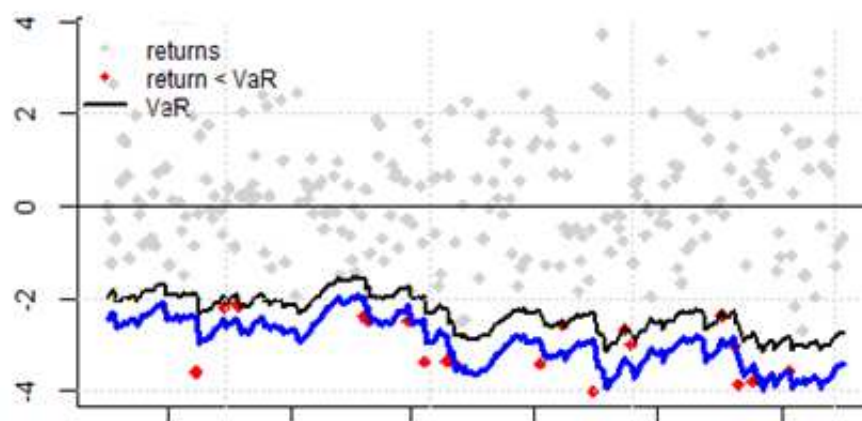
1. Set observation for the start backtesting T^* (usually $T - 250$)
2. Use data until period $t = T^*$ to calculate VaR for period $t + 1$: $VaR_{t+1|t}$
3. Compare $VaR_{t+1|t}$ to realization r_{t+1} to assess if VaR was exceeded
4. Repeat steps 2 and 3 for $t = T^* + 1, T^* + 2, \ldots, T - 1$
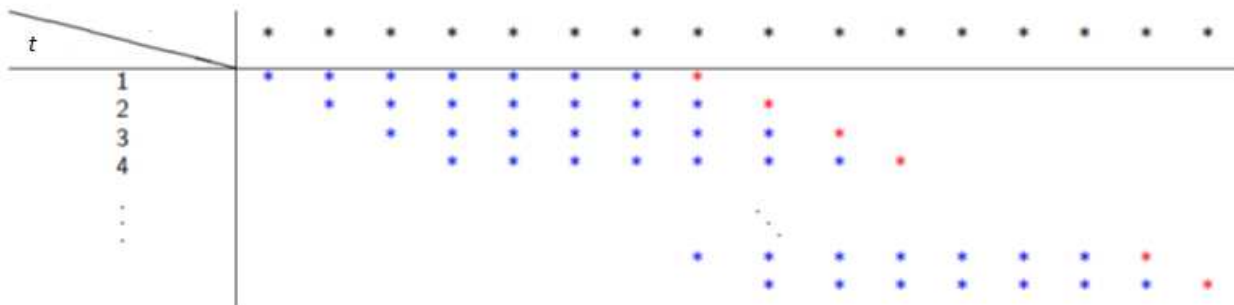
Note: We used a similar procedure for out-of-sample forecast evaluation in Block 1

VaR exceedances

SGH

- Using the series $VaR_{t|t-1}$ and r_t for $t = T^* + 1, T^* + 2, \ldots, T$ we can construct the series of exceedances

$$e_t = \begin{cases} 1 \text{ if } r_t \leq VaR_{t|t-1} \\ 0 \text{ if } r_t > VaR_{t|t-1} \end{cases}$$

- And calculate the number of exceedances (n_1) / no exceedances (n_0)

$$n_1 = \sum_{t=T^*+1}^{T} e_t$$
$$n_0 = n - n_1$$

where $n = T - T^*$ is the number of observations with which we evaluate VaR

Distribution for the number of VaR exceedances

SGH

- How many exceedances should we expect?
- For a well specified model e_t should be IID with:

$$e_t \sim B(1, p)$$
$$n_1 \sim B(n, p)$$

where $B(n, p)$ is binomial distribution with n trials and probability p

Distribution of n_1 for $n = 250$

n_1	$p = 5\%$ pdf	$p = 5\%$ cdf	$p = 1\%$ pdf	$p = 1\%$ cdf
0	0.0	0.0	8.1	8.1
1	0.0	0.0	20.5	28.6
2	0.0	0.0	25.7	54.3
3	0.1	0.1	21.5	75.8
4	0.3	0.5	13.4	89.2
5	0.9	1.3	6.7	95.9
6	1.8	3.1	2.7	98.6
8	5.4	11.9	0.3	99.9
10	9.6	29.1	0.0	100.0
12	11.6	51.8	0.0	100.0
14	10.0	72.9	0.0	100.0
16	6.4	87.5	0.0	100.0
18	3.1	95.3	0.0	100.0
20	1.2	98.5	0.0	100.0

Michał Rubaszek, Financial Econometrics

Basel Committee „Traffic lights" approach

SGH

Zone	Exceedances	Plus Factor k	cdf
🟩	0	0.00	8.11
🟩	1	0.00	28.58
🟩	2	0.00	54.32
🟩	3	0.00	75.81
🟩	4	0.00	89.22
🟨	5	0.40	95.88
🟨	6	0.50	98.63
🟨	7	0.65	99.60
🟨	8	0.75	99.89
🟨	9	0.85	99.97
🟥	10+	1.00	99.99

Quantitative standards Basel II

e. The multiplication factor will be set by individual supervisory authorities on the basis of their assessment of the quality of the bank's risk management system, subject to an absolute minimum of 3. Banks will be required to add to this factor a "plus" directly related to the ex-post performance of the model, thereby introducing a built in positive incentive to maintain the predictive quality of the model. The plus will range from 0 to 1 based on the outcome of so-called "backtesting."

Michał Rubaszek, Financial Econometrics

Exercise 4.1.

A VaR_p model was evaluated with a backtest using n observations. Let $\pi = n_1/n$ be the share of VaR exceedances, where n_1 is the number of VaR exceedances.

Calculate the 95% interval (left tailed and centered) for n_1 and π using `dbinom/pbinom/qbinom` functions in `R`, assuming that the VaR model is well specified and that:

a. $n = 250, p = 1\%$
b. $n = 250, p = 5\%$
c. $n = 100, p = 5\%$
d. $n = 100, p = 5\%$

Discuss the results

▪ For a model $e_t = \alpha + v_t$ we test for **unconditional coverage:**

$$H_0: \alpha = p$$

▪ For a model $e_t = \alpha + \rho e_{t-1} + v_t$ we test for **independence:**

$$H_0: \rho = 0$$

and **unconditional coverage:**

$$H_0: \alpha = p \wedge \rho = 0$$

▪ Why shouldn't we use LS regression to test the above hypotheses?

Kupiec test: unconditional coverage

- Let e_t are $IID\ B(\alpha)$ so that the likelihood of α given n_1 exceedeances in sample $n = n_0 + n_1$ is:

$$\mathcal{L}(\alpha|n_0, n_1) = \binom{n_0 + n_1}{n_1} \alpha^{n_1}(1 - \alpha)^{n_0}$$

- The formula for ML estimator:

$$\pi = \hat{\alpha} = n_1/n$$

- We can test the null of **unconditional coverage (Kupiec) test**:

$$H_0: \alpha = p$$

By calculating the likelihood ratio:

$$LR_{UC} = \frac{\mathcal{L}(p|n_0, n_1)}{\mathcal{L}(\pi|n_0, n_1)} = \frac{p^{n_1}(1 - p)^{n_0}}{\pi^{n_1}(1 - \pi)^{n_0}}$$

and the [likelihood ratio test](#) statistic:

$$-2\ln(LR_{UC}) \sim \chi^2(1)$$

Christoffersen test: independence

- Let's assume that the distribution of e_t depends on history:

$$e_t \sim \begin{cases} B(0, \alpha_0) \text{ if } e_{t-1} = 0 \\ B(0, \alpha_1) \text{ if } e_{t-1} = 1 \end{cases}$$

- The likelihood for $i \in \{0,1\}$ is:

$$\mathcal{L}_i(\alpha_i|n_{i0}, n_{i1}) = \binom{n_{i0} + n_{i1}}{n_{i1}} \alpha_i^{n_{i1}}(1 - \alpha_i)^{n_{i0}}$$

where $n_{ij} = \#(e_{t-1} = i \wedge e_t = j)$.

- ML estimator of α_i:

$$\pi_i = \hat{\alpha}_i = n_{i1}/(n_{i0} + n_{i1})$$

Christoffersen test: independence

- The null of **independence Christofersen test**:

$$H_0: \alpha_0 = \alpha_1 = \alpha$$

- Under the null the ML estimate for a single probability is :

$$\pi = \hat{\alpha} = (n_{01}+n_{11})/(n_{00}+n_{01}+n_{10}+n_{11})$$

- The likelihood ratio is:

$$LR_{IND} = \frac{\mathcal{L}_0(\pi|n_{00},n_{01}) \times \mathcal{L}_1(\pi|n_{10},n_{11})}{\mathcal{L}_0(\pi_0|n_{00},n_{01}) \times \mathcal{L}_1(\pi_1|n_{10},n_{11})} = \frac{\pi^{(n_{01}+n_{11})}(1-\pi)^{(n_{00}+n_{10})}}{\pi_0^{n_{01}}(1-\pi_0)^{n_{00}} \times \pi_1^{n_{11}}(1-\pi_1)^{n_{10}}}$$

with the likelihood ratio test statistic:

$$-2\ln(LR_{IND}) \sim \chi^2(1)$$

Michał Rubaszek, Financial Econometrics

Christoffersen test: conditional coverage

- **Conditional coverage Christofersen test** is a joint test of of unconditional coverage and independence

- For the null of the test:

$$H_0: \alpha_0 = \alpha_1 = p$$

the likelihood ratio is:

$$LR_{CC} = \frac{\mathcal{L}_0(p|n_{00},n_{01}) \times \mathcal{L}_1(p|n_{10},n_{11})}{\mathcal{L}_0(\pi_0|n_{00},n_{01}) \times \mathcal{L}_1(\pi_1|n_{10},n_{11})} = \frac{p^{(n_{01}+n_{11})}(1-p)^{(n_{00}+n_{10})}}{\pi_0^{n_{01}}(1-\pi_0)^{n_{00}} \times \pi_1^{n_{11}}(1-\pi_1)^{n_{10}}}$$

$$LR_{CC} = LR_{IND} \times LR_{UC}$$

with the likelihood ratio test statistic:

$$-2\ln(LR_{CC}) \sim \chi^2(2)$$

Michał Rubaszek, Financial Econometrics

Backtesting - illustration

- Number of observations: $n = 2500$
- Expected number of exceedances: $np = 125$
- Realized number of exceedances: $n_1 = 124$
- Kupiec UC test decision: H_0
- Christofersen CC test decision H_1
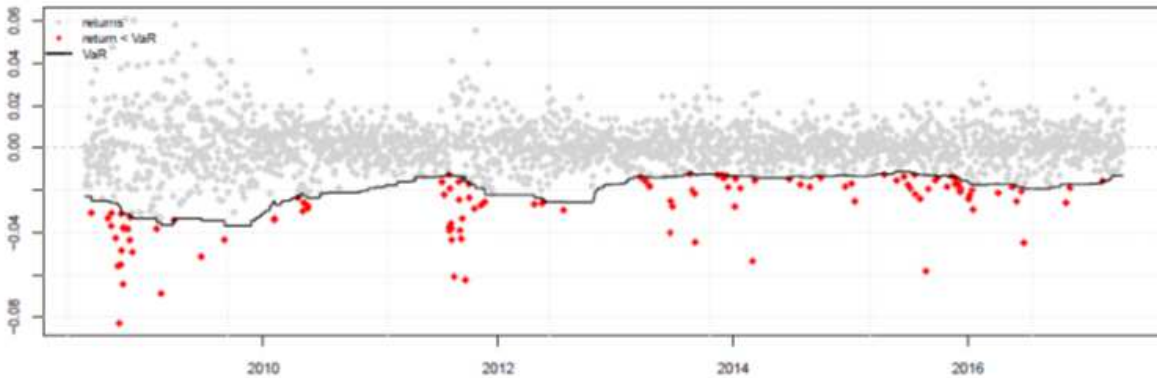
How should we improve the VaR model?

Backtesting ES: McNeila and Freya test

- Let $\tau = t_1, t_2, \ldots, t_{n_1}$ be the periods of VaR exceedances ($r_\tau < VaR_{\tau|\tau-1}$).
  Given the definition of ES:

$$ES_{\tau|\tau-1} = E(r_\tau | r_\tau < VaR_{\tau|\tau-1})$$

for a well specified ES model the variable:

$$z_\tau = \frac{r_\tau - ES_{\tau|\tau-1}}{\sigma_{\tau|\tau-1}}$$

should have $D(0,1)$, where $\sigma_{\tau|\tau-1}$ is the conditional standard deviation

- The null of **McNeil and Frey test**:

$$H_0: E(z_t) = 0$$

can be thereby verified with the standard t test (or bootstrapped version):

$$t = \frac{\bar{z}}{\hat{\sigma}_z/\sqrt{n_1}} \sim t_{v=n_1-1}$$

- **PIT – probability Integral Transform (already discussed in Block 1)**

$$PIT_{t|t-1} = \int_{-\infty}^{r_t} p_{t|t-1}(u)du$$

where $p_{t|t-1}()$ is forecasted pdf for returns.

- For a well calibrated model the $PIT_{t|t-1}$ should be $IID\ U(0,1)$, hence $z_t = \Phi^{-1}(PIT_{t|t-1})$ should be IID $N(0,1)$

- We can check it by estimating a model:

$$z_t = \alpha + \rho z_{t-1} + \epsilon_t$$

and verifying the null of the Berkowitz test:

$$H_0: \alpha = 0 \wedge \rho = 0$$

Exercises

SGH

Exercise 4.2.

Build a portfolio consisting of two stocks quoted on WSE (you can download the data with)

```
data  <- read.csv("http://web.sgh.waw.pl/~mrubas/EFII/Dane/wig.csv", row.names = "X")
```

Backtest risk models for the portfolio by making the following steps:

A. Make a graph of historical time series.

B. Select tolerance level ($p = 5\%$ or $p = 1\%$) and evaluation sample ($n = 250$)

C. Calculate the share of VaR exceedances for univariate models (nomal, HS, EWMA)

D. Backtest univariate models with Kupiec / Christofersen / McNnail-Frey tests

E. [Difficult] Try to perform points C and D for multivariate normal and compare with univariate normal. Are the results the same?

F. [Difficult] Try to perform points C and D for more sophisticated methods (GRACH, MGARCH, Copula)