Fertility Choice and Semi-Endogenous Growth: Where Becker Meets Jones*

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Abstract

We introduce fertility choice into an R&D-based semi-endogenous growth model so that the economy's long-run growth rate is again fully endogenously determined. The ultimate growth engine is located in the population equation of the model ("people reproduce in proportion to their number"), and R&D carries sustained population growth forward to GDP growth. We indicate the problems stemming from the fact that in the considered class of models, population size ought to enter the utility functional multiplicatively. In particular, we show that second order optimality conditions need not hold and flow utility is required to be positive (*levels* of utility matter). A simplified "Barro–Becker–Jones" model which we put forward, reconciles these problems, yields an asymptotic long-run fertility rate along an asymptotic BGP, and is open to further generalizations.

Keywords: fertility choice, semi-endogenous growth, R&D, long-run dynamics, second order conditions.
JEL Classification: J13, O41.

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*This is a **corrected version** of the article originally published in *Topics in Macroeconomics*, 6(2), Art. 10, 2006. Unfortunately, several errors in algebra have obstructed the presentation of the basic model in Section 3 of the published text. These errors have been corrected here. All further errors remain my responsibility. Current address: Warsaw School of Economics, al. Niepodległości 162, 02-554 Warszawa, Poland; e-mail: jakub.growiec@sgh.waw.pl.

1 Introduction

One of the most often discussed features of R&D-based models of semi-endogenous growth is that they imply a strong link between long-run economic growth and population growth. This link has been questioned for many reasons (e.g. "people become skillful researchers by education rather than birth" – Strulik, 2005), but the semi-endogenous theory remains one of the most prominent contemporary growth theories nevertheless. However, ever since the Jones' (1995) pathbreaking article, authors of semi-endogenous growth models have been customarily assuming population growth to be exponential and *exo*genous. On the one hand, the linear population equation of form $\dot{N} = nN$ is a natural assumption – as opposed to other linearity assumptions put forward in literature which are much more difficult to interpret – since "it is a biological fact of nature, that people reproduce in proportion to their number" (Jones, 2003), and policy and the economy are believed not to affect the population growth rate n much. On the other hand, it effectively pushes the *endo*genous growth mechanism out of these models.

Moreover, the assumption of sustained exogenous population growth may soon be at odds with evidence. Modern demographic trends, notably the Second Demographic Transition, already present in all developed countries (see van de Kaa, 1997), put the exponential population growth assumption into severe doubt.

Several ideas on how population growth can be endogenized are already present in the literature. Becker was probably the first to doubt the Malthusian (1798) claim, that "the passion between the sexes has appeared in every age to be so nearly the same, that it may always be considered, in algebraic language as a given quantity", and his ideas have influenced many economic theories (see e.g. Becker, 1981; and more notably, the growth theory of Barro and Becker, 1989) – semi-endogenous growth theories as well.[1]

The main point raised in this paper is that semi-endogenous growth models with the "naturally linear" population equation $\dot{N} = nN$ and endogenous fertility have the potential to generate sustained endogenous growth without

[1] Another question is whether perpetual population growth is, in fact, a desirable outcome. Mainstream semi-endogenous growth theories take into account neither finiteness of Earth nor the fact that production of goods depends on various kinds of natural resources, *exhaustible* resources in particular. If one believes that finiteness of Earth will ultimately put a limit to population growth (see Pimentel et al., 1999, for a survey on this "interdisciplinary" strand of literature), she will probably be not too pleased with the prediction of the semi-endogenous theory that growth in per capita wealth will also cease. Here, however, we only point at this problem, and do not consider it any further.

unmotivated knife-edge assumptions, as long as explicit limits such as finiteness of Earth or dependence of production on exhaustible resources are assumed out.

The article starts with a survey of the multiple ways to set up R&D-based growth models with endogenous fertility. For clarity of the discussion, we reduce to the necessary minimum the components of economic frameworks which do not influence their long-run dynamics. We present the decisive assumptions that various authors make, compute the weights attached by their optimizing agents to consecutive generations, and indicate the problems with second order conditions.

It is argued here that there exists a particular feature of growth models with endogenous fertility which has not received the necessary attention yet: population size enters the utility functional multiplicatively. Thus, the *level* of flow utility enters the first order conditions (not just *marginal* utility), its sign matters, and the second order optimality conditions are no longer automatically satisfied. Hence, numerous authors preferred to get rid of this uneasy feature of endongenous-fertility models by writing the (logarithm of) population size as an additive component in the utility functional (Barro and Sala-i-Martin, 1995, Chapter 9; Jones, 2003; Connolly and Peretto, 2003)[2] or by considering a static optimization problem instead (Jones, 2001). By providing an analysis of the case where population size enters the utility functional multiplicatively, we fill a gap in the literature.

In the quest to reconcile all the problems outlined above, we put forward a bare-bones model with infinite-horizon ("dynastic") optimization, endogenous fertility choice, R&D-based semi-endogenous growth, and population size entering the utility functional multiplicatively. The dynasty's optimization problem where consumption and fertility are the only sources of utility (the Barro–Becker approach), has been given the most attention. A detailed discussion of first order, second order and auxilliary conditions, as well as the long-run dynamics of the model, is the most important contribution of this paper. In particular, we identify and characterize the model's asymptotic balanced growth path (analogous to the one in Jones, 2001). We also find that the restriction on the value of the intertemporal elasticity of substitution (IES hereafter) in consumption is crucial for the long-run outcome of semi-endogenous growth models with endogenous fertility.

The main results obtained in this article are expected to hold also if one ex-

[2]Existence of long-run growth in these three models relies upon the weakly motivated knife-edge assumption that the intertemporal elasticity of substitution (IES) in consumption be exactly unity (the logarithmic case).

pands our basic model so that it allows for physical or human capital accumulation, endogenous labor allocation, imperfect competition in the production sector, and the like. The long-run behavior of R&D-based semi-endogenous growth models where population growth is exogenous, is typically qualitatively different to the behavior of models with endogenous fertility. In addition, more stringent conditions have to be imposed.

In section 2, we present the specific features of growth models with endogenous fertility which are not shared by models with exogenous fertility. In section 3, we lay out and solve the basic model. In section 4, we discuss the requirements for the IES in consumption, imposed by this model as well as by the alternative ones put forward in literature. Section 5 concludes.

2 Exogenous vs endogenous fertility in growth models

2.1 Writing down the utility functional

Prior to reading any involved literature, one would expect the mathematical treatment of demographics in economic growth models to be the same regardless of whether fertility is exogenous or endogenous: after all, it is just the population growth rate n that is endogenized. However, it is not the case. Even more curiously, the difference strikes usually already in the first equation: the objective functional of the model.

The usual growth model with infinite-horizon optimization and *exo*genous fertility would have population size entering the objective functional *multiplicatively*: the social planner or representative household would maximize $\mathcal{U}_0 = \int_0^\infty N_t u(c_t) e^{-\rho t} dt$, where $N_t = N_0 e^{nt}$ denotes the population size at time t, n is the exogenous population growth rate, c_t denotes per capita consumption at time t, and u is the flow utility function. One can then rewrite the utility functional $\mathcal{U}_0$ as $\mathcal{U}_0 = N_0 \int_0^\infty u(c_t) e^{-(\rho-n)t} dt$, from which it follows immediately that population growth provides a drag on the discount rate ρ. The *effective* discount rate is computed as $\rho - n$: the difference between the pure time preference rate and the population growth rate.

If fertility is endogenous, the objective functional gets changed, for technical reasons rather than conceptual (as we shall see shortly). In particular, Barro and Becker (1989) replace N_t in $\mathcal{U}_0$ with N_t^θ, with $\theta \in (0,1)$, thereby allowing for "incomplete altruism" – larger generations' ("vintages") utility is embedded in $\mathcal{U}_0$ with a decreasing marginal share. The effective discount rate becomes $\rho - \theta n$. Although the Barro and Becker's assumption is relatively easily interpretable, one ought to know that its introduction is also necessary to assure that the second order conditions for optimality hold. This formulation will be our preferred one throughout the paper. We admit, however, that there is a number of drawbacks of this approach. We shall discuss one of them in the next subsection 2.2.

There are strands of literature which deal with endogenous fertility differently than Barro and Becker do. In particular, it is widespread[3] to use the following trick: to redefine the objective functional $\mathcal{U}_0$ as $\mathcal{U}_0 = \int_0^\infty [u(c_t, b_t) + \epsilon v(N_t)]e^{-\rho t}dt$, where b_t is the per capita fertility rate (having children may be a source of utility). In this specification, population size enters $\mathcal{U}_0$ *additively*. We note that the above expression can be split into a sum of two integrals, $\mathcal{U}_0 = \int_0^\infty u(c_t, b_t)e^{-\rho t}dt + \epsilon \int_0^\infty v(N_t)e^{-\rho t}dt$.

Two facts should be mentioned here. First, the impact of births on agents' utility through an increasing population size is neglected. Thus, the channel of intertemporal substitution via the effective discount rate is switched off: the effective discount rate remains just ρ. One still treats population size as a normal good, but no longer considers it to be related to the marginal utility of consumption. Second, all involved models rely upon the knife-edge assumption that v is a logarithmic function: $v(N_t) = \ln N_t$ – the IES in population size is exactly unity.

Another way to simplify matters is to disregard *any* impact of births on population size whatsoever. One disposes then of all dynamic tradeoffs associated with the evolution of endogenously determined population size over time. This approach was taken in the previous, discussion-paper version of this article (Growiec, 2006). Similarily, Jones (2001) prefers to exchange the original dynamic setup for a static one where by definition, no dynamic tradeoffs attached to fertility decisions are present.

2.2 Weighting the generations

Let us now write down the general form of the objective functional as

$$\mathcal{U}_0 = \int_0^\infty N_t \omega_t u(c_t, b_t) e^{-\rho t} dt. \tag{1}$$

By ω_t, we denote the *weight function* of generation t. We shall simply purge all differences between alternative setups into ω_t.

In the "complete altruism" (or "egalitarian") case, characteristic for models with exogenous fertility, the weight function is $\omega_t \equiv 1$. The aggregate level

[3] See Barro and Sala-i-Martin (1995), Chapter 9, Jones (2003), and Connolly and Peretto (2003).

of utility is just the sum of individuals' utility levels. We take this result as benchmark.

In the Barro and Becker's (1989) "incomplete altruism" case, $\omega_t = N_t^{\theta-1}$. Thus, individuals in larger generations are systematically less taken care of than individuals in smaller generations. This implies in particular, that if population is increasing over time, later generations are less taken care of than earlier ones.[4]

In the case where population size enters $\mathcal{U}_0$ additively, it can be shown that $\omega_t = \frac{1}{N_t}\left(1 + \frac{v(N_t)}{u(c_t,b_t)}\right)$ and thus the weight of each generation depends on: (i) its level of per capita consumption, (ii) its size, and (iii) its fertility rate. Moreover, it does so in a rather unintuitive and complex way. In particular, it matters if the levels of $u(c_t, b_t)$ and $v(N_t)$ – flow utilities – are positive or negative. We see that the "degree of altruism" in this case exhibits no systematic pattern,[5] which is a serious drawback of this methodology in comparison to the Barro and Becker's one.

2.3 Special features of models with endogenous fertility

Incorporating endongenous fertility in economic growth models with infinite-horizon optimization requires us to think about several of its characteristics, which would be of less significance if fertility were to remain exogenous. The list of such characteristics goes as follows.

1. The natural specification of the population equation of motion is the linear one: $\dot{N}_t = n_t N_t = (b_t - d_t)N_t$, where b_t is the birth rate and d_t is the death rate. This provides a potential for fully endogenous sustained growth (as opposed to other endogenous growth models where the linearity assumption is *ad hoc* rather than natural: see the detailed discussion in Jones, 2003).

2. The fact that population size enters the dynasty's utility functional multiplicatively, $\mathcal{U}_0 = \int_0^\infty N_t^\theta u(c_t, b_t) e^{-\rho t} dt$, implies that the flow utility needs to be always *positive*: $\frac{\partial \mathcal{U}_0}{\partial N_t} = \theta N_t^{\theta-1} u(c_t, b_t) e^{-\rho t} > 0$ only if $u(c_t, b_t) > 0$. Otherwise, utility (and in general, life) would be a painful

[4]Please note that we do not insist that population size be increasing over time. It may be well constant or decreasing.

[5]For instance, it is obtained that $\omega_{c,t} \equiv \frac{\partial \omega_t}{\partial c_t} = -\frac{v(N_t)u'(c_t)}{N_t u^2(c_t,b_t)}$ and thus, $\omega_{c,t} \lesseqgtr 0$ if and only if $v(N_t) \gtreqless 0$. It is a mystery why the *direction* of the impact of per capita consumption on the weight attached to a generation should depend on whether flow utility from population size is positive or negative.

burden rather than a blessing and the benevolent dynastic head would not impose such burden on her descendants. In other words, if flow utility is negative, it always pays to lower fertility: the utility functional is then closer to zero and thus greater.[6]

3. Another consequence of the fact that population size enters the utility functional multiplicatively is that levels of flow utility enter the first order and second order conditions. The utility function loses one of its "textbook" properties: its ordinal character. The utility function becomes a cardinal measure instead; the agent's decisions cease to be invariant to affine transformations of her utility function. Adding a constant to it and multiplying it by a positive constant changes the first order conditions.

4. Population growth provides a drag on the effective discount rate. Thus, a problem of endogenous fertility may be understood as a problem of endogenous discounting (see e.g. Becker and Mulligan, 1997; Das, 2003 on this issue).

5. There are substantial costs of childrearing, ranging from financial to time (opportunity) costs. In particular, time devoted to bringing up children has to be deducted from the total time spent on productive activities (in our case, production and R&D).

6. Population growth rate adds to the depreciation rate of accumulated stocks (per capita capital, per capita stock of exhaustible resources, etc.), thus creating the "effective" depreciation rate. This is due to the fact that total stocks have to distributed among an increasing number of people.

7. Individuals may (but may not) receive utility from having children. In most fertility models discussed in literature, the birth rate b_t does enter the flow utility function $u(c_t, b_t)$.

We deal with most of these points when solving our basic model in section 3.

2.4 Problems with second order conditions

Having population size as a multiplicative factor in the utility functional can cause problems with second order conditions. To explain this claim, let us first

[6] I am grateful to Charles Jones for this point. The same result applies to models with health status which affects the survival probability (Hall and Jones, 2007).

write down the following "generic" optimization problem:

$$\max_{N_t, b_t, \ldots} \int_0^\infty N_t^\theta u(c_t, b_t, \ldots) e^{-\rho t} dt \qquad \text{s.t.} \qquad \dot{N}_t = (b_t - d_t) N_t, \ldots \tag{2}$$

Three dots denote further variables and further equations which can be freely added to the optimization problem under the condition that the population size N_t does not enter any of these equations (in particular, it implies that the production function ought to have constant returns to scale). The resulting Hamiltonian reads:

$$\mathcal{H}(N_t, b_t, \ldots) = N_t^\theta u(c_t, b_t, \ldots) e^{-\rho t} + \Lambda_t (b_t - d_t) N_t + \ldots \tag{3}$$

We write N_t as the first argument of the Hamiltonian for convenience – the term $\frac{\partial^2 \mathcal{H}}{\partial N_t^2}$ becomes then the first minor of its Hessian. We shall maintain this order throughout the paper.

Now, let us remind ourselves that for the FOCs to describe a maximum of the Hamiltonian, it is necessary that its Hessian is non-positive definite. This implies that there are no positive elements on the main diagonal.

However, it is quickly verified that

$$\frac{\partial^2 \mathcal{H}}{\partial N_t^2} = \theta(\theta - 1) N_t^{\theta - 2} u(c_t, b_t, \ldots) e^{-\rho t}, \tag{4}$$

and thus $\frac{\partial^2 \mathcal{H}}{\partial N^2}$ is positive if $\theta \in (0, 1)$ and the flow utility $u(c_t, b_t, \ldots)$ is negative. For the sake of optimality, we must rule this case out. Therefore, *flow utility must be positive in models with endogenous fertility.*

To emphasize the importance of this result, let us now restrict our attention to the class of CRRA utility functions, which is probably the one most often used by economists. The simplest CRRA functions are defined as $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. If $\gamma > 1$, then flow utility is always negative and thus cannot be used in models with endogenous fertility. However, $\gamma > 1$ implies that the IES in consumption is less than unity – a result endorsed by a multiplicity of empirical works (see the review in the section 4). Accordingly, the same negativity result holds for generalized CRRA utility functions of form $u(c_t, b_t) = \mu_0 + \mu_1 \frac{c_t^{1-\gamma}}{1-\gamma} + \mu_2 \frac{b_t^{1-\eta}}{1-\eta}$ where $\gamma, \eta > 1$, $\mu_0 \leq 0$, and $\mu_1, \mu_2 > 0$. Even adding an arbitrarily large positive constant $\mu_0 > 0$ to flow utility cannot help dispose of the negativity problem for small values of c_t, since $\lim_{c \to 0} u(c, b) = -\infty$ for *any* μ_0. We note that for CRRA utility functions, u can be everywhere positive only if $\gamma < 1$.

3 The basic model

We shall now analyze a simplified R&D-based semi-endogenous growth model with population size entering the utility functional multiplicatively. We shall keep in mind the remarks of above subsections, in particular of section 2.3 where we have listed the special features of such models.

3.1 Setup

3.1.1 Demographics

To give the demographics of the model an explicit treatment, we shall make use of a continuous-time overlapping-generations setup with indeterministic lifespan. For mathematical simplicity, we assume that N_t – population at time $t \geq 0$ – is in fact not the integer number of individuals, but rather the measure of an interval, populated by a continuum of agents. Thus, although the lifespan of each individual is random, the Law of Large Numbers enables us to treat the death rate at each instant of time as deterministic. For each individual, we shall introduce a survival function $m : \mathbb{R}_+ \to [0,1]$ such that $m(0) = 1, \lim_{t\to\infty} m(t) = 0$, and m is decreasing. The total number of births at time t is denoted $B_t \equiv b_t N_t$, with b_t being the birth rate. The size of generation t at time $z \geq t$ is equal to

$$S_{z,t} = B_t m(z-t), \tag{5}$$

and the total population at time t is

$$N_t = \int_0^t B_z m(t-z) dz. \tag{6}$$

The population growth rate can be calculated as

$$n_t = \frac{\dot{N}_t}{N_t} = \frac{B_t m(0)}{N_t} + \frac{\int_0^t B_z m'(t-z)dz}{N_t} \equiv \underbrace{b_t}_{\text{birth rate}} - \underbrace{d_t}_{\text{death rate}} . \tag{7}$$

Instead of maintaining the general form of the survival function m throughout the paper, we shall simplify the analysis by limiting ourselves to the "perpetual youth" case. Namely, we shall take the exponential function m implying a constant probability of death d at all ages $x \geq 0$, conditional on having reached the age x. This assumption reads:

$$m(x) = e^{-dx} \quad \Rightarrow \quad d_x \equiv d, \quad \text{where } d > 0. \tag{8}$$

It is possible for some individuals to live forever, although such probability can well be neglected. For simplicity, we also neglect the impact of technological progress and increasing per capita wealth on the survival function m.[7]

3.1.2 Production and R&D

The production function of the single consumption good is assumed to be Cobb-Douglas with constant returns to scale in labor L, and increasing returns to scale, once the technology level A is included as well, which reflects the concept of non-rivalry of ideas (see Jones, 2005, for a discussion):

$$Y_t = A_t^{\sigma} L_t = A_t^{\sigma}(1 - \beta(b_t))N_t, \qquad \sigma > 0, \tag{9}$$

where $\beta(b_t)$ describes the time cost of childrearing. For reasons which we shall describe later in more detail, we assume that the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, increasing and convex ($\beta' > 0$ and $\beta'' > 0$), and such that $\beta(0) = 0$. We are going to assume out all other forms of childrearing costs.

By taking (9) as our production function, we assume out physical capital accumulation. This assumption is made to ease the exposition – further derivations, focused on the endogenous fertility component, are more transparent without capital – but is not necessary. Indeed, one would make this model more realistic by considering capital accumulation as well.

Ideas are accumulated according to the Jones' (1995) R&D equation

$$\dot{A}_t = \nu L_t^{\lambda} A_t^{\phi}, \qquad 0 < \lambda < 1, \quad 0 < \phi < 1, \quad \nu > 0. \tag{10}$$

We assume that the spillovers in idea production are positive (the "standing-on-shoulders" effect) but not sufficiently strong for fully-endogenous R&D-driven growth ($\phi < 1$).

The above Cobb-Douglas assumptions are standard for the associated literature (see Jones, 2005, for a justification). At the same time, they greatly facilitate obtaining balanced growth. In this paper, we think of this property as desirable, because we shall emphasize *different* (population-side, not production-side) barriers to endogenous balanced growth.

It is assumed that the whole working population is employed both in R&D and in the production sector. To keep things as simple as possible, we do not consider the allocation of labor between these two sectors explicitly. In our setup, people receive remuneration for their production work but not for

[7] Provided that we rule out the possibility that the expected lifespan be growing without bound, this assumption does not change our results qualitatively.

their research. Thus, R&D is considered here an inevitable side-effect of production, rather than a distinct sector of the economy.[8] Despite the fact that this model may be interpreted a model of "learning by doing" in the Arrow's (1962) tradition (or better: "inventing by doing"), all the long-run results we are discussing would have clearly gone through if we had endogenized labor allocation and allowed for an explicit treatment of the R&D outlay. We abstract from these issues only to simplify exposition.

3.1.3 Households

The representative agent maximizes discounted utility of her dynasty, born at time 0. Her altruism is *imperfect*: the representative agent does not explicitly take into account the fact that some members of the dynasty are born, and some die at each instant of time, but she systematically attaches smaller weight to bigger generations. Utility is derived from consumption and the number of children (as in Barro and Becker, 1989):

$$\max_{\{b_t\}_{t=0}^{\infty}} \int_0^{\infty} N_t^{\theta} u(y_t, b_t) e^{-\rho t} dt, \qquad \rho > 0, \tag{11}$$

subject to the population equation of motion $\dot{N}_t = (b_t - d)N_t$ and the per capita production function $y_t = A_t^{\sigma}(1 - \beta(b_t))$. Since no savings are allowed, all production is immediately consumed and thus $c_t = y_t$ for all t.

Moreover, we shall assume that the flow utility function is of the argument-separable CRRA form with a non-negative constant:

$$u(y_t, b_t) = \mu_0 + \mu_1 \frac{y_t^{1-\gamma}}{1-\gamma} + \mu_2 \frac{b_t^{1-\eta}}{1-\eta}, \tag{12}$$

where $\mu_0 \geq 0$, $\mu_1, \mu_2 > 0$, and $\gamma, \eta \in (0, 1)$.[9] This closes the setup of our basic model.

[8] This is an admittedly heroic assumption. However, it does not change the results qualitatively, because in the long run, the ratio of researchers is expected to approach a constant. We assure it by trivially setting it to a constant – i.e. endowing each individual with a constant amount of time for production work and for research. Then, we say that the long-run research/production effort ratio is already included in ν and A_0.

[9] The assumption that $\gamma, \eta \in (0, 1)$ assures that flow utility is always positive. In fact, we could have been less stringent here. It would suffice if u were positive just for the values of variables actually realized along the time path of the economy. One would then have to be very careful with the choice of initial conditions and parameter values, though. We acknowledge that the case with $\gamma > 1$ and (despite that) $u > 0$ is a cumbersome but potentially very useful case to analyze. We leave it for future work.

3.2 The asymptotic balanced growth path

Before we start actually solving the model, we shall describe the necessary properties of the balanced growth path (BGP), or alternatively, the asymptotic BGP. These will be helpful in the subsequent derivations.

We shall define the balanced growth path as a sequence of time paths $\{A_t, N_t, y_t\}_{t=0}^{\infty}$, along which all economic variables grow at constant non-negative rates, possibly zero. It implies that the birth rate $\{b_t\}_{t=0}^{\infty}$ must be constant along the BGP.

Similarily, we shall say that the model approaches an *asymptotic* BGP if the growth rates of variables approach constant values. Please note that this happens only as the levels of A_t, N_t, y_t diverge to infinity. In such case no proper BGP exists – there is no exponential solution to the dynamical system at hand.

We shall claim that in our model no BGP exists, but under an appropriate parameter configuration, an asymptotic BGP is converged to as time goes to infinity. In the following subsections, we shall derive the conditions under which the birth rate approaches a constant $\bar{b}$ implying that the population growth rate also approaches a constant $\bar{n} = \bar{b} - d$. Consequently, the growth rate of knowledge stock and the economic growth rate approach constants as well.

From the R&D equation (10), it is obtained that along an asymptotic BGP, necessarily

$$\frac{\dot{A}_t}{A_t} = \frac{\lambda \bar{n}}{1-\phi}, \tag{13}$$

where $\bar{n} = \bar{b} - d$ is the endogenously determined long-run population growth rate.

This implies that the economic growth rate approaches:

$$g \equiv \frac{\dot{y}_t}{y_t} = \sigma \frac{\dot{A}_t}{A_t} = \frac{\sigma \lambda \bar{n}}{1-\phi}. \tag{14}$$

Please note that although the childrearing cost $\beta(b_t)$ imposes a significant level effect on production and R&D output, no long-run growth effects should be expected.

3.3 Optimization

To solve the households' optimization problem, we set up the Hamiltonian:

$$\mathcal{H}(N_t, b_t) = N_t^{\theta} u(y_t, b_t) e^{-\rho t} + \Lambda_t (b_t - d) N_t, \tag{15}$$

where Λ_t is the shadow price of population size. b_t is the only control variable in this setup, and N_t is the only state variable. Thanks to our prior simplifying assumptions, the accumulation of knowledge A_t is considered external to the household and no savings decision is allowed.

When dealing with transversality and summability conditions, we shall utilize the properties of the asymptotic BGP, derived in the previous subsection. For a time being, we assume that the asymptotic BGP is indeed approached; in the next subsection we shall prove it, and thus prove that the transversality and summability conditions hold. But of course, we shall begin with the FOCs.

3.3.1 First order conditions

The first derivatives of the Hamiltonian read:

$$\frac{\partial \mathcal{H}}{\partial N_t} = \theta N_t^{\theta-1} u(y_t, b_t) e^{-\rho t} + \Lambda_t (b_t - d) = -\dot{\Lambda}_t, \tag{16}$$

$$\frac{\partial \mathcal{H}}{\partial b_t} = N_t^{\theta} [u_b(y_t, b_t) - u_y(y_t, b_t) A_t^{\sigma} \beta'(b_t)] e^{-\rho t} + \Lambda_t N_t = 0. \tag{17}$$

Substituting the suitable expression for the shadow price Λ_t from the second FOC into the first FOC ($\Lambda_t = -N_t^{\theta-1}[u_b(y_t, b_t) - u_y(y_t, b_t) A_t^{\sigma} \beta'(b_t)] e^{-\rho t}$), using additive separability of u as apparent in (12), and rearranging yields the following equation of motion of the birth rate:[10]

$$\dot{b}_t \underbrace{\Big(u_{bb} + u_{yy} A_t^{2\sigma} (\beta'(b_t))^2 - u_y A_t^{\sigma} \beta''(b_t)\Big)}_{\text{Left-hand side } \equiv \Psi} = \underbrace{\theta u - \Big(\theta(b_t - d) - \rho\Big) u_b +}_{\text{Right-hand side } \equiv \Phi}$$

$$\underbrace{+ u_y A_t^{\sigma} \beta'(b_t) \Big(\theta(b_t - d) - \rho + \sigma \frac{\dot{A}}{A}\Big) + u_{yy} A_t^{2\sigma} \sigma \frac{\dot{A}}{A} (1 - \beta(b_t)) \beta'(b_t)}_{\text{Right-hand side } \equiv \Phi}. \tag{18}$$

We shall describe the properties of this rather complicated expression in subsection 3.4. We shall denote its left-hand side (apart from $\dot{b}_t$) as Ψ and the right-hand side as Φ.

3.3.2 Transversality condition

The transversality condition requires that the shadow value of population size tends to zero as time approaches infinity. For the condition $\lim_{t\to\infty} \Lambda_t N_t = 0$ to

[10] By u_x, we denote the derivative of u with respect to x. We omit the arguments of u and its derivatives for convenience.

hold, it suffices that from some time on, the growth rate of $\Lambda_t N_t$ is uniformly negative. Solving for this growth rate and taking an asymptotic balanced growth path approximation (which implies that the *positive* growth rate of population is $\bar{n}$, the economy grows at a rate g, and the marginal utility of consumption declines at a rate γg), we can write the transversality condition as:

$$\frac{\dot{\Lambda}_t}{\Lambda_t} + \frac{\dot{N}_t}{N_t} = \theta\bar{n} + (1-\gamma)g - \rho = \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)\bar{n} - \rho < 0. \tag{19}$$

In other words, the discount rate should be large enough compared to the population growth rate. Condition (19) is identical to the restriction of a positive effective discount rate in exogenous fertility models.

3.3.3 Summability condition

The summability condition requires that the integral (11) converges. To satisfy this requirement, it suffices that along the asymptotic BGP, the growth rate of the product $N_t^\theta y^{1-\gamma} e^{-\rho t}$ is negative. This implies that in our setup, the transversality condition and the summability condition coincide.

3.3.4 Second order conditions

Let us now proceed to the second order (sufficiency) conditions which would guarantee that the FOCs describe an actual maximum of the Hamiltonian. We are going to be conscientious here since we have already shown that in models where endogenous fertility enters the utility functional multiplicatively, satisfaction of second order conditions is not automatic. This step is typically omitted in the literature, though.

It turns out that assuming $u > 0$ is necessary for optimality, but by no means sufficient. The following proposition holds.

Proposition 1 *In the vicinity of the asymptotic BGP, the first order condition* (18) *describes a maximum of the Hamiltonian* (15) *if*

$$\frac{\beta''(\bar{b})(1-\beta(\bar{b}))}{(\beta'(\bar{b}))^2} > \frac{1-\gamma-\theta}{\theta}, \tag{20}$$

where $\bar{b}$ denotes the asymptotic steady-state birth rate.

Proof. See Appendix. ■

3.4 Existence of the solution and dynamics

We shall now characterize the conditions which have to be satisfied for the birth rate b_t to approach a constant $\bar{b}$ as time goes to infinity, thus yielding an asymptotic BGP. First, we shall derive the steady-state birth rate $\bar{b}$ (in most cases it will be unique, but we will not rule out the possibility of multiple solutions). The next step will be to analyze the dynamics of the model around its asymptotic BGP.

The existence issue must be addressed first. The following propositions hold.

Proposition 2 *The steady-state birth rate $\bar{b}$ is defined as a solution to the implicit equation:*

$$\frac{\theta}{1-\gamma}\left(\frac{1-\beta(\bar{b})}{\beta'(\bar{b})}\right) = \rho - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\bar{b}-d). \tag{21}$$

Proof. See Appendix. ■

Proposition 3 *A solution to the implicit equation (21) is guaranteed to exist if*

$$\beta'(0) < \frac{\theta}{(1-\gamma)\left(\rho + \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)d\right)} \tag{22}$$

and

$$\rho > \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\beta^{-1}(1) - d), \tag{23}$$

or if the signs in (22) and (23) are simultaneously reversed.

Proof. See Appendix. ■

Equation (21) in proposition 2 is central to this paper: by defining the endogenous asymptotic steady-state birth rate, it also pins down the long-run growth rate of the economy.

Please note that the parameters $\eta, \mu_0, \mu_1, \mu_2$ have disappeared in the course of taking limits. They play their roles during the transition, but not along the asymptotic BGP.

Imposing an upper bound on $\beta'(0)$ such as in (22) and a lower bound on ρ as in (23) closes the existence issue. The roots of (21) can be evaluated numerically, and we shall do so in the following subsection. Moreover, the sufficiency conditions (22)–(23) can be substantially weakened: it is enough to show that for some $b \in [0, \beta^{-1}(1)]$, the left-hand side of (21) is greater than the right-hand side, and for some other b, it is smaller, and the intermediate value theorem argument used in the proof of proposition 3 still goes through.

Let us pass on to the question of stability. The following proposition holds.

Proposition 4 *The steady-state birth rate $\bar{b}$, implicitly defined in (21), is asymptotically stable if in the vicinity of $\bar{b}$,*

$$\frac{\theta}{1-\gamma}\left(\frac{1-\beta(\bar{b})}{\beta'(\bar{b})}\right) < \rho - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\bar{b} - d) \tag{24}$$

for $b < \bar{b}$, and the inequality (24) is reversed for $b > \bar{b}$. Otherwise, the steady-state birth rate is unstable.

If the steady-state birth rate is unstable but $1 - \theta - \gamma > 0$, then no transitional adjustments in b_t are possible and the birth rate is always set so as to assure $\dot{b}_t = 0$. Otherwise, a solution that has $b_t \to \beta^{-1}(1)$ may also be chosen by the representative household (depending on initial conditions).

Proof. See Appendix. ■

Summarizing, the following conditions should be checked before one could be sure that a given $\bar{b}$ is the asymptotic steady-state birth rate delivered by our model, giving rise to an asymptotic BGP:

1. The necessary condition for optimality combined with the asymptotic steady-state requirement, summarized by the implicit equation (21).

2. The transversality/summability condition, given by (19).

3. The sufficient condition for optimality, given by (20).

Local dynamics around the asymptotic BGP may then be studied using proposition 4. In this respect, it must be noted that if the asymptotic steady-state birth rate proves to be unstable, then in many cases, there will be no transitional adjustments in b_t.[11] The logic is the following. The birth rate b_t is a control variable so it can always be set such that $\dot{b}_t = 0$; setting it at a different value might lead to an eventual violation of the transversality condition or of the second order optimality condition (as apparent in the proof of proposition 4). One has to bear in mind, however, that no proper BGP exists here. Thus, if the solution that has $b_t \to \beta^{-1}(1)$ is ruled out, then it must be the case that $\dot{b}_t = 0$ at all times, so that $\Phi(b_t) = 0$ at all times, but since $u_b/u \to 0$ and

[11] Such instability results may also be found in endogenous-fertility growth models of Barro and Sala-i-Martin (1995), Chapter 9, and Connolly and Peretto (2003).

$\frac{u}{y^{1-\gamma}} \to \frac{\mu_1}{1-\gamma}$ only gradually, then also b_t will approach $\bar{b}$ only gradually. In the end, there will be some transition dynamics here even though the asymptotic birth rate is unstable and divergence is ruled out. These will not be driven by purposeful dynamic adjustments in b_t, though.

Let us also comment on the following: we have been using asymptotic steady-state approximations when deriving all above inequalities. Thus, we have been implicitly assuming that the initial population size and the initial knowledge stock are large enough for the approximations to be valid. Using the continuity argument, one could claim that what is quite important here is the *margins* by which our approximate inequality conditions are satisfied. The narrower is the margin, the larger has to be the initial population size as well as the initial knowledge stock (which are our only stock variables) in order for a given condition in its precise form to be satisfied as well.

3.5 The case of a linear childrearing cost function

The case of a linear childrearing cost function is a particularly tractable one. Indeed, if such functional form is assumed, the steady-state birth rate may be derived explicitly: it is straightforward to show that if $\beta(b) = \kappa b$, with $\kappa > 0$, then there exists a unique asymptotic steady-state birth rate $\bar{b}$ given by:

$$\bar{b} = \frac{\frac{\theta}{1-\gamma}\frac{1}{\kappa} - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right) d - \rho}{\frac{\gamma\theta}{1-\gamma} - (1-\gamma)\frac{\sigma\lambda}{1-\phi}}, \tag{25}$$

provided that κ is small enough to guarantee $\bar{b} \geq 0$. Consequently, the steady-state population growth rate $\bar{n} = \bar{b} - d$ equals:

$$\bar{n} = \frac{\frac{\theta}{1-\gamma}\left(\frac{1}{\kappa} - d\right) - \rho}{\frac{\gamma\theta}{1-\gamma} - (1-\gamma)\frac{\sigma\lambda}{1-\phi}}. \tag{26}$$

The model at hand is able to deliver the prediction that the steady-state birth rate $\bar{b}$ decreases with ρ, κ, and d, quite in line with intuition.[12] In order to obtain such plausible comparative statics, however, one must impose that the denominator in (25) be positive.

It can be checked that the population growth rate over the long run is positive – and thus the economic growth rate is positive as well – provided

[12]The last relationship is perhaps the least intuitive, but one has to bear in mind that in our model, a greater death rate means not only a shorter lifespan of the children but also of the parents, and that people here have children at a constant rate thoughout their lives. A shorter life leaves them with less time to be allocated to childrearing.

that:

$$\frac{\theta}{1-\gamma}\left(\frac{1}{\kappa}-d\right) > \rho. \tag{27}$$

As far as the auxilliary conditions are concerned, the transversality condition can be now rewritten, inserting (26) into (19), as:

$$\left(\theta+(1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)\left(\frac{1}{\kappa}-d\right) < \rho. \tag{28}$$

Since $\beta''(\bar{b})=0$, the second order condition for optimality simplifies radically in the linear case. It becomes:

$$\theta > 1-\gamma. \tag{29}$$

For the sake of an illustration that for several plausible parameter values, the above described asymptotic steady-state birth rate indeed exists, we proceed by means of a numerical example. The parameter values are going to be fixed at some plausible benchmark levels. We require them to fall into appropriate intervals, but we do not intend to make any direct connections to the empirically observed values whatsoever. Our benchmark set of parameter values is summarized in table 1.

γ	κ	ϕ	σ	λ	ρ	d	θ
0.8	25	0.7	0.5	0.8	0.04	0.02	0.8

Table 1: THE BENCHMARK PARAMETER VALUES. THE LINEAR CASE.

Please note that since the values of η, μ_0, μ_1, and μ_2 do not play any role along the asymptotic BGP, we do not have to fix them.

It is straightforwardly checked that under this parametrization, conditions (27) through (29) are satisfied and the denominator in (25) is positive. Furthermore, the asymptotic steady-state value of the endogenous birth rate is equal to $\bar{b}=0.0336$, which implies a steady population growth rate of $\bar{n}=0.0136$. The economic growth rate approaches $g=0.0182$, with a fraction of time $(1-\beta(\bar{b}))=0.1591$ allocated to productive activity and the remaining fraction $\beta(\bar{b})=0.8409$ to childrearing.

The asymptotic birth rate is unstable in the current case: $\dot{b}_t$ as a function of b_t crosses the zero line at $\bar{b}$ from below. This instability result implies here that there are either (i) no transitional adjustments in b_t apart from those necessary to converge to the asymptotic BGP (so that $u_b/u \to 0$, $u_{bb}/u \to 0$), or (ii) transitional dynamics implying gradual divergence of childrearing costs: $b \to \beta^{-1}(1)$.

3.6 Numerical example with a non-linear childrearing cost function

Let us now posit a different functional form for the increasing childrearing cost function β. We shall assume that

$$\beta(b_t) = \kappa b_t^{\psi}, \qquad \kappa > 0, \quad \psi \neq 1. \tag{30}$$

Using the definition of function β in (30), we note the following. First, the plausibility condition $\bar{b} \leq \beta^{-1}(1)$ becomes $b \leq (1/\kappa)^{1/\psi}$. Second, the sufficient condition for optimality (20) is automatically satisfied if $\psi > 1$; if $\psi < 1$, however, then (20) is satisfied only if

$$\bar{b} > \left(\frac{1}{\kappa + \frac{1-\gamma-\theta}{\theta} \frac{\psi}{\psi-1}} \right)^{\frac{1}{\psi}}. \tag{31}$$

The obvious next step is to fix the parameter values again. Just like in the linear case above, we shall fix them at some plausible benchmark levels without intending to make connections to the empirically observed values. This set of parameter values is summarized in table 2.

γ	κ	ψ	ϕ	σ	λ	ρ	d	θ
0.8	40	1.2	0.7	0.5	0.8	0.08	0.02	0.8

Table 2: The benchmark parameter values. The non-linear case.

Using a numerical routine to find the root of Φ defined in (21) (in this case, it is unique), we arrive at a result that the endogenous birth rate approaches an asymptotic steady-state value of $\bar{b} = 0.0296$, which implies a steady population growth rate of $\bar{n} = 0.0096$. The economic growth rate approaches $g = 0.0128$, with a fraction of time $(1 - \beta(\bar{b})) = 0.4140$ allocated to productive activity and remaining fraction $\beta(\bar{b}) = 0.5860$ to childrearing.

A simple numerical check of the transversality/summability condition (19) and the second order optimality condition (20) confirms that they both hold along the asymptotic BGP.

The dynamics of the birth rate are depicted in figure 1.[13] We see that the asymptotic steady-state birth rate is again unstable. Since in the current example, $1 - \gamma - \theta < 0$, the solution implying $b_t \to \beta^{-1}(1)$ cannot be ruled out.

[13] We have used the parameter choices given in table 2 when producing this figure.

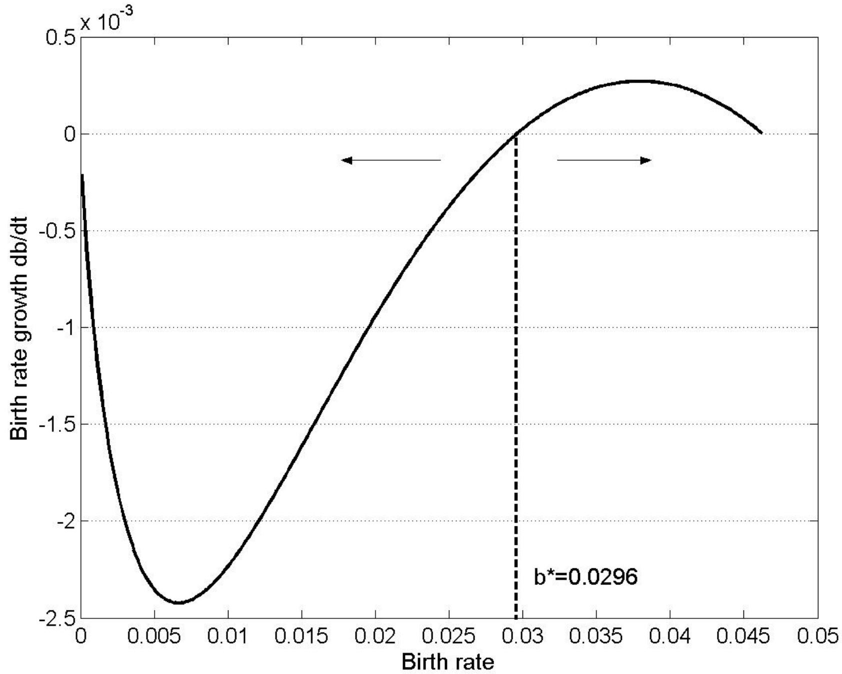

Figure 1: Unstable asymptotic steady-state birth rate.

This numerical example ends the qualitative analysis of our simplified "Barro–Becker–Jones" model. We proceed now to discuss one of the major drawbacks of the methodology we used: it requires the IES in consumption to be greater than one ($\gamma < 1$) which is an empirically doubtful assumption.

4 The IES in consumption

Several growth models with endogenous fertility discussed in literature (notably in Barro and Sala-i-Martin, 1995, Chapter 9; Connolly and Peretto, 2003; Jones, 2003), as long as their long-run dynamics are concerned, rely on the knife-edge assumption that the IES in consumption $1/\gamma$ equals exactly unity. The $\gamma = 1$ assumption is necessary for the results obtained in those works, but it is quite questionable as well. In our setup (as well as in Jones', 2001), this assumption has been substituted with a much less stringent assumption that $\gamma < 1$ – that is, that the IES in consumption exceeds unity. Empirical studies suggest the opposite assumption to be much more plausible, however. We provide a short summary of this literature below. Please note that since the magnitude of the true IES in consumption has been estimated in a wide range of empirical works, we can name just a small percentage of all contributions here.

The discussion began with the work of Hall (1988) who concluded that the IES in consumption in the United States were very small, and possibly zero – that is, that for sure $\gamma > 1$.

Patterson and Pesaran (1992) upgraded Hall's methodology by assuming that the slope coefficient of the MA process, governing per capita consumption, is not known a priori, as Hall presumed. This modification helped them obtain the result of the American IES being around 0,213 and significantly different from zero.

Hall's results have also been criticized on other grounds. It has been argued that the Euler equation he estimated was misspecified, and unsuitable for the country-level aggregate data whatsoever. Beaudry and van Wincoop (1996) used a panel of U.S. states instead (spanning 1953-1991, or in the other estimation round, 1978-1991), and modified the Hall's original Euler equation. They have achieved a large improvement in the estimation precision, and obtained a result of IES being around 0.7-1.1 (depending on the estimation method and the set of instrumental variables): clearly different from zero and not significantly different from one.

Guvenen (2006) added another dimension to this discussion, pointing out that in reality, as opposed to most theoretical approaches, agents are heterogenous. In particular, a large fraction of households does not participate in stock markets at all. Moreover, the IES in consumption varies significantly across individuals, increasing with income. Consumption is much more evenly distributed than wealth. This asymmetry accounts, according to Guvenen, for a serious underestimation of the IES in all previous studies. He concludes, that among non-stockholders, IES is indeed around 0.1 (as e.g. Hall suggested); but among stockholders, it is rather expected to fall into the interval (0.8, 1.2). And it is the stockholders who effectively determine the real interest rate of the economy. This makes the variant IES=1 again possible. However, Guvenen himself states that "a plausible range for this parameter is possibly (0,1)".

Favero (2005) merged the Euler equation with the (linearized) budget constraint of the households, and used the resultant equation to estimate the IES. He obtained IES=0.78, with a standard deviation of 0.11 (i.e. clearly smaller than one).

Harashima (2005) went in a different direction. He overthrew the assumption that in the estimated Euler equation, the real interest rate is taken as given (the "endowment" economy assumption). Instead, he proposed to consider a closed "production" economy, in which the IES is obtained directly from some version of Euler equation. He concluded, that in such case, the IES would be again very low, around 0.09.[14]

The above literature review is by no means complete; it is included here to point out, that although the assumption of $\gamma = 1$ is knife-edge and disputable, the alternative assumption of $\gamma < 1$ is also quite troublesome. Empirical investigations bring somewhat convincing evidence, that $\gamma > 1$.

A proposed solution to this problem would be then to get out of the CRRA framework and to disconnect the curvature of the flow utility function from its sign. The fact is that for CRRA functions, $\gamma > 1$ is associated with high curvature and negative sign, and $\gamma < 1$ – with low curvature and positive sign. However, we insist that this identification should not be taken too far: the research focus varies from study to study. In the empirical works, the main focus is on curvature; in our analysis, necessity for $\gamma < 1$ stems from the sign requirement imposed on the utility function rather than from its curvature.

5 Conclusion

In this paper, we have studied the dynamic behavior of R&D-based semi-endogenous growth models with endogenous fertility. The number of knife-edge conditions required for sustained growth in this model is just one, and a naturally interpretable one: we just assume that people reproduce in proportion to their number, and thus that the population equation is linear. Per capita GDP growth is obtained here, however, thanks to R&D – in particular, thanks to constantly increasing numbers of researchers.

We have argued that in the involved literature, endogenous fertility is usually treated differently than exogenous fertility is. This is probably because the assumption of population size entering the utility functional multiplicatively brings about cumbersome complications which researchers would rather prefer to omit. We have provided a detailed treatment of the case at hand and thus filled a gap in the literature.

We have analyzed a bare-bones "Barro–Becker–Jones" model that features infinite-horizon ("dynastic") optimization, fertility choice and R&D-based semi-endogenous growth. We have shown that it has the capacity to produce an asymptotic steady-state birth rate, and thus an asymptotic BGP. The long-run growth rate of the economy is fully endogenous and the ultimate

[14] Harashima just calibrated his theoretical growth model, and did not use any econometric methods.

factor driving growth is fertility.[15]

The paper ends with a brief review of empirical literature quantifying the IES in consumption that is required to exceed unity in the class of R&D-based growth models with endogenous fertility and CRRA utility. Since the empirical evidence favors the opposite case of IES<1, we arrive at a puzzle that calls for a resolution.

A Mathematical appendix

Proof of Proposition 1. The second derivatives of the Hamiltonian (15), after substituting the appropriate expression for Λ_t, read:

$$\frac{\partial^2 \mathcal{H}}{\partial N_t^2} = \theta(\theta-1)N_t^{\theta-2}ue^{-\rho t}, \tag{32}$$

$$\frac{\partial^2 \mathcal{H}}{\partial N_t \partial b_t} = (\theta-1)N_t^{\theta-1}[u_b - u_y A_t^{\sigma}\beta'(b_t)]e^{-\rho t}, \tag{33}$$

$$\frac{\partial^2 \mathcal{H}}{\partial b_t^2} = N_t^{\theta}[u_{bb} - u_y A_t^{\sigma}\beta''(b_t) + u_{yy}A_t^{2\sigma}(\beta'(b_t))^2]e^{-\rho t}. \tag{34}$$

The Hessian is negative definite if $\frac{\partial^2 \mathcal{H}}{\partial N_t^2} < 0$ (which is the case since $\theta \in (0,1)$ and $u > 0$) and its determinant $\det(D^2\mathcal{H})$ is positive. The latter statement can be rewritten as:

$$\det(D^2\mathcal{H}) = N_t^{2\theta-2}e^{-2\rho t}(\theta-1) \times \tag{35}$$
$$\left(\theta u(u_{bb} - u_y A_t^{\sigma}\beta''(b_t) + u_{yy}A_t^{2\sigma}(\beta'(b_t))^2) - (\theta-1)(u_b - u_y A_t^{\sigma}\beta'(b_t))^2\right) > 0.$$

Thus, the determinant is positive if the expression in big parentheses is negative. Taking the asymptotic BGP approximation: $\lim_{t\to\infty} \frac{u_y A_t^{\sigma}}{u} = \frac{1-\gamma}{(1-\beta(\bar{b}))}$; $\lim_{t\to\infty} \frac{u_{yy}A_t^{2\sigma}}{u} = -\frac{(1-\gamma)\gamma}{(1-\beta(\bar{b}))^2}$ and rearranging transforms this requirement to a condition that

$$\frac{\beta''(\bar{b})(1-\beta(\bar{b}))}{(\beta'(\bar{b}))^2} > \frac{1-\gamma-\theta}{\theta}. \quad \blacksquare \tag{36}$$

[15] Of course, we have completely set aside the issue of policy effectiveness in our model. Without developing these arguments any further, we indicate here that by saying that growth is “fully endogenous,” we mean that there is a possibility for public interventions (and more systematic public policies) to change the long-run growth rate of the economy. By saying that fertility is the “ultimate factor driving growth,” we mean that the only way a policymaker can influence the long-run economic growth rate, is by influencing the long-run fertility rate.

Proof of Proposition 2. Dividing (18) sidewise by $u > 0$ (which makes both sides of the equation asymptotically stationary), equating $\dot{b}_t$ to zero and taking the limits $\lim_{t\to\infty} u_b/u = \lim_{t\to\infty} u_{bb}/u = 0$ yields:

$$\theta - \frac{u_y A_t^\sigma \beta'(\bar{b})}{u}\left(\rho - \left(\theta + \frac{\sigma\lambda}{1-\phi}\right)(\bar{b}-d)\right) + \\ + \frac{u_{yy} A_t^{2\sigma}(1-\beta(\bar{b}))\beta'(\bar{b})}{u}\frac{\sigma\lambda}{1-\phi}(\bar{b}-d) = 0. \tag{37}$$

Taking further limits: $\lim_{t\to\infty} \frac{u_y A_t^\sigma}{u} = \frac{1-\gamma}{(1-\beta(\bar{b}))}$; $\lim_{t\to\infty} \frac{u_{yy} A_t^{2\sigma}}{u} = -\frac{(1-\gamma)\gamma}{(1-\beta(\bar{b}))^2}$, makes us rewrite the steady-state equation (37) as an equation in a single variable $\bar{b}$. It reads:

$$\frac{\theta}{1-\gamma}\left(\frac{1-\beta(\bar{b})}{\beta'(\bar{b})}\right) = \rho - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\bar{b}-d). \quad \blacksquare \tag{38}$$

Proof of Proposition 3. Note that both sides of (21) are continuous for $\bar{b} \in [0, \beta^{-1}(1)]$. Denote:

$$\Phi(\bar{b}) = \frac{\theta}{1-\gamma}\left(\frac{1-\beta(\bar{b})}{\beta'(\bar{b})}\right) - \rho + \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\bar{b}-d). \tag{39}$$

At the steady state, $\Phi(\bar{b}) = 0$. The boundary values of Φ are as follows:

$$\Phi(0) = \frac{\theta}{1-\gamma}\frac{1}{\beta'(0)} - \rho - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)d, \tag{40}$$

$$\Phi(\beta^{-1}(1)) = -\rho + \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\beta^{-1}(1)-d). \tag{41}$$

Thus, existence of at least one root follows from the intermediate value theorem if (i) $\Phi(0) > 0$ and $\Phi(\beta^{-1}(1)) < 0$ or if (ii) $\Phi(0) < 0$ and $\Phi(\beta^{-1}(1)) > 0$. In case (i), these two conditions are equivalent to (22) and (23), respectively. In case (ii), one has to reverse the signs of both inequalities. ■

Proof of Proposition 4. To assure local asymptotic stability in the single-dimensional setup, we have to show that $\dot{b}_t$ as a function of b_t crosses the zero line from above.

The left-hand side of (18), Ψ, divided by $u > 0$, reads:

$$\bar{\Psi}(\bar{b}) = -\frac{\gamma(\beta'(\bar{b}))^2 + (1-\beta(\bar{b}))\beta''(\bar{b})}{(1-\beta(\bar{b}))^2} < 0. \tag{42}$$

Negativity of $\bar{\Psi}$ at the steady state, equivalent to $\frac{(1-\beta(\bar{b}))\beta''(\bar{b})}{(\beta'(\bar{b}))^2} > -\gamma$, follows from the second order condition:

$$\frac{(1-\beta(\bar{b}))\beta''(\bar{b})}{(\beta'(\bar{b}))^2} > \frac{1-\gamma-\theta}{\theta} > -\gamma, \tag{43}$$

because the last inequality is equivalent to the trivial $(1-\theta)(1-\gamma) > 0$.

It follows that $\dot{b}_t$ crosses the zero line from above if the right-hand side of (18), Φ, divided by $u > 0$, crosses the zero line from below, as in (24).

One final remark is due here: if (22) and (23) hold, we know that Φ crosses the zero line from above at least once (it is surely true for the *first* and *last* time it crosses zero), but it doesn't have to cross the zero line from below. If the signs in (22) and (23) are reversed, then Φ crosses the zero line from below at least once so there exists at least one asympotically stable steady-state birth rate.

In the case the steady state is unstable, we obtain the following. Unless the knife-edge value of b_t implying $\dot{b}_t = 0$ is chosen, two possibilities may emerge. First, if $b \to 0$ implying $u_b \to \infty$, then $\Lambda < 0$ and $\dot{\Lambda} < 0$ (note that $\dot{\Lambda}/\Lambda + \dot{N}/N = \frac{\theta u}{u_b - u_y A_t^\sigma \beta'(b_t)} > 0$ as $u_b \to \infty$). In consequence, $\lim_{t\to\infty} \Lambda N < 0$ so the transversality condition is violated. Second, if $b \to \beta^{-1}(1)$ implying $y \to 0$, $u_y A_t^\sigma \to \infty$, then

$$\lim_{b\to\beta^{-1}(1)} \frac{\gamma\beta'(b)}{1-\beta(b)}\dot{b} = \rho - \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\beta^{-1}(1) - d), \tag{44}$$

and thus

$$\frac{\dot{\Lambda}}{\Lambda} + \frac{\dot{N}}{N} \quad = \quad \left(\theta + (1-\gamma)\frac{\sigma\lambda}{1-\phi}\right)(\beta^{-1}(1) - d) - \rho + \frac{\gamma\beta'(b)}{1-\beta(b)} \to 0. \tag{45}$$

Since we also have that $\dot{\Lambda}/\Lambda + \dot{N}/N > 0$, the transversality condition holds. The second order condition can be analyzed as follows: dividing the determinant of $D^2\mathcal{H}$ as defined in (35) by $u_y A_t^\sigma$ and taking the limit $b \to \beta^{-1}(1)$ simplifies the formula inside the parentheses in (35) as $\beta'(\beta^{-1}(1))^2 \left(\frac{1-\theta-\gamma}{1-\gamma}\right)$. This means that the second order condition fails if $1-\theta-\gamma$ is positive.

Otherwise, the solution implying $b \to \beta^{-1}(1)$ cannot be ruled out of the analysis. ■

References

[1] Arrow, K.J. (1962), The Economic Implications of Learning by Doing, *Review of Economic Studies* 29(3), pp. 155-173.

[2] Barro, R.J., G.S. Becker (1989), Fertility Choice in a Model of Economic Growth, *Econometrica* 57(2), pp. 481-501.

[3] Barro, R.J., X. Sala-i-Martin (1995), *Economic Growth* (McGraw-Hill, Inc.)

[4] Beaudry, P., E. van Wincoop (1996), The Intertemporal Elasticity of Substitution: An Exploration Using a U.S Panel of State Data, *Economica* 63, pp. 495-512.

[5] Becker, G.S. (1981), *A Treatise on the Family* (Harvard University Press, Cambridge).

[6] Becker, G.S., C. Mulligan (1997), The Endogenous Determination of Time Preference, *Quarterly Journal of Economics* 112, pp. 729-758.

[7] Connolly, M., P. Peretto (2003), Industry and the Family: Two Engines of Growth, *Journal of Economic Growth* 8(1), pp. 114-148.

[8] Das, M. (2003), Optimal Growth With Decreasing Marginal Impatience, *Journal of Economic Dynamics and Control* 27, pp. 1881-1898.

[9] Favero, C.A. (2005), Consumption, Wealth, the Elasticity of Intertemporal Substitution and Long-Run Stock Market Returns, CEPR Working Paper No. 5110.

[10] Growiec, J. (2006), Fertility Choice and Semi-Endogenous Growth: Where Becker Meets Jones, CORE Discussion Paper 2006/23.

[11] Guvenen, M.F. (2006), Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective, *Journal of Monetary Economics* 52(7), pp. 1451-1472.

[12] Hall, R.E. (1988), Intertemporal Elasticity of Substitution in Consumption, *Journal of Political Economy* 96, pp. 339-357.

[13] Hall, R.E., C. I. Jones (2007), The Value of Life and the Rise in Health Spending, *Quarterly Journal of Economics* 122(1), pp. 39-72.

[14] Harashima, T. (2005), An Estimate of the Elasticity of Intertemporal Substitution in a Production Economy, mimeo, University of Tsukuba.

[15] Jones, C. I. (1995), R&D-Based Models of Economic Growth, *Journal of Political Economy* 103, pp. 759-784.

[16] Jones, C. I. (2001), Was an Industrial Revolution Inevitable? Economic Growth Over the Very Long Run, *Advances in Macroeconomics* Vol. 1, No. 2, Article 1.

[17] Jones, C. I. (2003), Population and Ideas: A Theory of Endogenous Growth, in: P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford, eds., *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps* (Princeton University Press).

[18] Jones, C. I. (2005), Growth and Ideas, in: P. Aghion and S.N. Durlauf, eds., *Handbook of Economic Growth* (North-Holland, Amsterdam).

[19] Malthus, T.R. (1798), *An Essay on the Principle of Population* (printed for J. Johnson, in St. Paul's Church-Yard, London).

[20] Patterson K.D., B. Pesaran (1992), The Intertemporal Elasticity of Substitution in Consumption in the United States and the United Kingdom, *Review of Economics and Statistics* 74(4), pp. 573-584.

[21] Pimentel, D., O. Bailey, P. Kim, E. Mullaney, J. Calabrese, L. Walman, F. Nelson, X. Yao (1999), Will Limits of the Earth's Resources Control Human Numbers?, in: B. Nath, L. Hens and D. Pimentel, eds., *Environment, Development and Sustainability*, pp. 19-39 (Springer Science+Business Media B.V.).

[22] Strulik, H. (2005), The Role of Human Capital and Population Growth in R&D-Based Models of Economic Growth, *Review of International Economics* 13, pp. 129-145.

[23] van de Kaa, D. J. (1997), Options and Sequences: Europe's Demographic Patterns, *Journal of the Australian Population Association* 14 (1), pp. 1-30.