Final exam "Optimization" 17H-19H. 4 december

QEM. Delay : 2H, no documents, no computers, no electronic devices, no cellphone !

Exercise 1

Consider 11 cities A, B, C, D, B', C', D', B'', C'', D'' and E. Some Cities are connected by roads, with the following distances :
$AB = 1, AC = 2, AD = 6, BC = 2, CD = 2, BB' = 5, CC' = 3, DD' = 1, B'C' = 1, C'D' = 1, B'B'' = 5, C'C'' = 5, D'D'' = 2, B''C'' = 1, C''D'' = 2, B''E = 1, D''E = 3, C''E = 2$.
This means, for example, you can go directly from A to B in 1 km. But B and C' (for example) are not directly connected.

For every city M, call $V(M)$ the distance of the shortest path from M to E. Compute, by a dynamic programming argument (backward induction), $V(M)$ for every point M. Find one shortest path from A to E.

(**Remark : Please**, draw a graph to represent the problem, each point of the graph being a city, an edge being a road ; represent the distance between two cities by a number on the corresponding edge, and represent $V(M)$ for every M by a number on (or close to) M. But also explain the computation of $V(M)$).

Exercise 2

Let $0 < \alpha$, $0 < \beta < 1$ and $k > 0$. Consider the following optimization problem

$$V(k) = \sup_{k_0 = k, \forall t \in \mathbb{N}, 0 < k_{t+1} < k_t^\alpha} \sum_{t=0}^{+\infty} \beta^t ln(k_t^\alpha - k_{t+1}).$$

a) Recall quickly why $\sum_{t=0}^{+\infty} t.\beta^t$ converges.
b) For every $k > 0$, let $C(k) = \{(k_t)_{t \in \mathbb{N}} : k_0 = k, \forall t \in \mathbb{N}, 0 < k_{t+1} < k_t^\alpha\}$ the set of constraints. Prove that for every $(k_t)_{t \in \mathbb{N}} \in C(k)$, $\sum_{t=0}^{+\infty} \beta^t ln(k_t^\alpha - k_{t+1})$ is well defined.
c) For every function $f :]0, +\infty[\to \mathbb{R}$, define a new function $T(f)$ from $]0, +\infty[\to \mathbb{R}$ by : $\forall k > 0, T(f)(k) = \sup\{ln(k^\alpha - x) + \beta f(x), 0 < x < k^\alpha\}$. Prove that V satisfies the Bellman equation $T(V) = V$.
d) Recall the assumptions, in the course, which insures that the Bellman equation has a unique solution. Are these assumptions true here ?

Exercise 3

a) Give and prove Banach fixed point theorem.

b) Let (X, d) a complete metric space and Λ a metric space. Consider a mapping $f : X \times \Lambda \to X$ such that :

- For every $x \in X$, the mapping from $\Lambda \to X$, $\lambda \mapsto f(x, \lambda)$ is continuous ;

- $d(f(x, \lambda), f(y, \lambda)) \leq \frac{1}{2} d(x, y)$ for every $(\lambda, x, y) \in \Lambda \times X \times X$.

1 - Prove that for every $\lambda \in \Lambda$, there exists a unique $a_\lambda \in X$ such that $f(a_\lambda, \lambda) = a_\lambda$.

2 - Prove explicitely (for example using a sequence) that the mapping $\lambda \mapsto a_\lambda$ is continuous.

Exercise 4

Using Kuhn and Tucker conditions, solve the following Optimization problem :

$$\max f(x, y, z) = xyz + z$$

under the constraint $x^2 + y^2 + z \leq 6, x \geq 0, y \geq 0, z \geq 0$.

Exercise 5

Solve $\max f(x, y) = xy$ under the constraint $x + 4y = 16$. Represent some level curves of f, the set of constraints, and explain the relationship between the gradient of f and the set of constraints at a solution.