Consider the following optimization problem with a finite time horizon $T>0$:

$$\max_{\{c_t\}_0^T} \sum_{t=0}^{T} \ln c_t \quad \text{s.t.} \quad A_{t+1} = \left(A_t - c_t(1-\tau^c)\right)(1+r)(1-\tau^w),$$

where $r>0$ – interest rate on assets A_t, $\tau^c \in (0,1)$ – tax on consumption, $\tau^w \in (0,1)$ – tax on wealth. Assume $A_0 > 0$ – given.

- Find the optimal time paths $\{c_t\}_{t=0}^T$ and $\{A_t\}_{t=0}^T$.
- Identify the policy function and the value function.

- The problem satisfies the requirements of dynamic programming (e.g., time separability of the objective).
- We may proceed by dynamic programming $\Rightarrow$ backward induction.

- Scheme:

  1 – solve for $T, T-1, T-2, \ldots$
  2 – identify a pattern
  3 – prove that the pattern characterizes the policy and value function
  4 – Solve for $\{c_t\}$, $\{A_t\}$.

- For $t=T$ the problem is straightforward:

$$\max_{c_T} \{\ln c_T\} \quad \text{s.t.} \quad A_{T+1} \geq 0.$$

In optimum, $A_{T+1}=0 \Rightarrow A_T - c_T(1-\tau^c) = 0 \Rightarrow \underbrace{c_T = \frac{A_T}{1-\tau^c}}_{\text{policy fct.}}$.

Hence, $\underbrace{V_T(A_T)}_{\text{value fct}} = \ln \underbrace{c_T(A_T)}_{\text{policy fct}} = \ln A_T - \ln(1-\tau^c)$.

- For $t=T-1$:

Write down the Bellman equation, $A_T=(A_{T-1}-c_{T-1}(1-\tau^c))(1+r)(1-\tau^w)$

$$V_{T-1}(A_{T-1}) = \max_{c_{T-1}} \left\{ \ln c_{T-1} + V_T(A_T) \right\} =$$

$$= \max_{c_{T-1}} \left\{ \ln c_{T-1} + \ln\left(A_{T-1}-c_{T-1}(1-\tau^c)\right) + \ln(1+r)(1-\tau^w) - \ln(1-\tau^c) \right\}.$$

Maximizing the term in brackets we obtain:

$$\frac{1}{c_{T-1}} = \frac{1-\tau^c}{A_{T-1}-c_{T-1}(1-\tau^c)}$$

$$(1-\tau^c)c_{T-1} = A_{T-1}-(1-\tau^c)c_{T-1}$$

$$\underbrace{c_{T-1} = \frac{A_{T-1}}{2(1-\tau^c)}} \rightarrow \text{policy fct.}$$

Hence, $V_{T-1}(A_{T-1}) = \ln\left[\frac{A_{T-1}}{2(1-\tau^c)}\right] + \ln\left[\frac{A_{T-1}}{2}\right] + \ln\left[\frac{(1+r)(1-\tau^w)}{1-\tau^c}\right] =$

$$= 2\ln A_{T-1} \underbrace{- 2\ln 2 - 2\ln(1-\tau^c) + \ln(1+r)(1-\tau^w)}_{\text{call this constant term “}\alpha_{T-1}\text{”.}} . \rightarrow \text{value fct.}$$

- For $t=T-2$, fully analogous calculations yield

$$c_{T-2} = \frac{A_{T-2}}{3(1-\tau^c)}, \quad V_{T-2}(A_{T-2}) = 3\ln A_{T-2} + \underbrace{\alpha_{T-2}}_{\text{can be derived precisely, but is tedious.}} .$$

- These results allow to hypothesize that for all $t=0,\dots,T$, the policy function is

$$c_t(A_t) = \frac{A_t}{(T-t+1)(1-\tau^c)},$$

and the value function is

$$V_t(A_t) = (T-t+1)\ln A_t + \underbrace{\alpha_t}_{\text{independent of any } \{c_\tau\}, \{A_\tau\}}.$$

- This hypothesis can be verified by induction, i.e., by exploiting the recursive character of the Bellman equation.

We obtain:

$$V_t(A_t) = \max_{c_t}\left\{\ln c_t + V_{t+1}(A_{t+1})\right\} \overset{\text{induction assumption}}{=}$$

$$= \max_{c_t}\left\{\ln c_t + (T-t)\ln\left[A_t - c_t(1-\tau^c)\right] + (T-t)\ln(1+r)(1-\tau^w) + \alpha_{t+1}\right\}.$$

Maximizing within brackets we get:

$$\frac{1}{c_t} = (T-t)\frac{1-\tau^c}{A_t - c_t(1-\tau^c)}$$

$$(T-t)(1-\tau^c)c_t = A_t - c_t(1-\tau^c)$$

$$c_t = \frac{A_t}{(T-t+1)(1-\tau^c)}. \qquad \text{(as assumed)}$$

And hence,

$$V_t(A_t) = \ln\left[\frac{A_t}{(T-t+1)(1-\tau^c)}\right] + (T-t)\ln\left(\frac{T-t}{T-t+1}\right)A_t + \{\text{additional terms}\} =$$

$$= (T-t+1)\ln A_t + \alpha_t \qquad \square$$

[The sequence $\{\alpha_t\}$ could be identified precisely, but this requires tedious calculations and is not necessary for finding the optimal solution.]

- Inserting the policy function into the assets' equation of motion, we get

$$A_{t+1} = \left(\frac{T-t}{T-t+1}\right) A_t (1+r)(1-\tau^w).$$

- This implies that

$$A_1 = \frac{T}{T+1} A_0 (1+r)(1-\tau^w)$$ and, by iterating the equation,

$$\ldots, \quad A_t = \left(\frac{T}{T+1}\right)\left(\frac{T-1}{T}\right)\left(\frac{T-2}{T-1}\right) \cdots \left(\frac{T-t+1}{T-t+2}\right) \cdot A_0 (1+r)^t (1-\tau^w)^t,$$

and so $$\boxed{A_t = \left(\frac{T-t+1}{T+1}\right) A_0 (1+r)^t (1-\tau^w)^t.}$$

- By the policy function,

$$\boxed{C_t = \left(\frac{A_0}{T+1}\right) \cdot \frac{(1+r)^t (1-\tau^w)^t}{1-\tau^c}.}$$

- The last two equations provide the explicit model solution.

OBSERVATIONS:

- Consumption grows/declines geometrically $\Leftrightarrow (1+r)(1-\tau^w) \gtrless 1$.
- The formulas of the value and policy functions are <u>not</u> time-invariant (as t enters them directly). The model is <u>not stationary</u>, and the next-period's problem is different from the current period's problem (precisely because it is one period closer to the end of time).
- (!) Wealth tax τ^w affects the <u>growth rate</u> of consumption whereas consumption tax τ^c only affects its <u>level</u>.
- Solving the model with Lagrangeans would lead to exactly the same optimal solution, but would not yield the policy function (which could only be derived posterior to optimization).
- Wealth tax τ^w diminishes the effective return on assets. Consumption tax τ^c doesn't have this effect.