Lectures notes on: Dynamical Optimisation[1]

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Chapter 1

Metric spaces

1.1 Metric spaces

In this chapter, we will extend the result presented in Chapter 2 from Euclidean spaces to metric spaces. A particular case of metric spaces are normed linear spaces. So, all the results apply to normed linear spaces. In the next chapter, we will provide some additional results concerning specifically the normed linear spaces.

Keep in mind that, even if we can extend many definitions and results from Euclidean spaces to normed linear spaces or metric spaces, there are a lot of traps since key topological properties of Euclidean spaces are no more true. So be very careful in applying some well-known results for finite dimensional spaces to infinite dimensional spaces. In the following, we put in red some warnings to help the reader to detect the most common traps.

For many applications, we are considering data which are not structured as a linear spaces on which addition and multiplication by a scalar are defined. For example, we can optimise on a graph, on a discrete set of persons, of words, of integers, on the Grassmannian set of all linear subspaces of a given dimension, of the set of closed sets of a compact sets, on the sphere, ... So, we will define the notion of distance between two elements of a sets and extending the notion of convergence, continuity, completeness presented in Euclidean spaces.

Definition 1 Let X be a set. A distance d on X is a function from $X \times X$ to $\mathbb{R}$ satisfying the following properties:

$$\forall (x, y) \in X \times X,\ d(x, y) \geq 0;$$

$$\forall (x, y) \in X \times X,\ d(x, y) = 0 \text{ if and only if } x = y;$$

$$\forall (x, y, z)) \in X \times X \times X,\ d(x, z) \leq d(x, y) + d(y, z).$$

Remark 1 A normed linear space (E, N) is a metric space with the distance $d(x, y) = N(x - y)$. If X is a set, the function d from $X \times X$ to $\mathbb{R}$ defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$ is a distance on X. So we can define a distance on any set. This particular distance is the atomic distance. If G is a connected graph, we can define a distance on the set of nodes of the graph by considering the shortest path between two nodes and defining the distance as the number of edges crossed by the shortest path. If X is the set of one dimensional linear subspaces (lines) of a norm vector spaces, we can define the distance between to lines L and L' as the minimum of $\|u - u'\|$ and $\|u + u'\|$, where $u \in L$, $\|u\| = 1$ and $u' \in L$, $\|u'\| = 1$.

Remark 2 For a given set X, which is not a singleton, there is an infinite number of distance on X. So, it is important to recall the one that we consider if the context is not obvious.

From the triangular inequality, we deduces the following inequality which is useful in many proofs.

Proposition 1 *Let (X, d) be a metric space. Then, for all $(x, y, z) \in X \times X \times X$,*

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Remark 3 Let (X, d) be a metric space. Then if Y is a subset of X, the function δ, which is the restriction of d to $Y \times Y$ is a distance on Y and (Y, δ) is a metric space.

Exercise 1 Let φ be a concave, continuous, strictly increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying $\varphi(0) = 0$.
1) Show that for all $(t, t') \in \mathbb{R}_+ \times \mathbb{R}_+$, such that $0 < t < t'$,

$$\frac{\varphi(t')}{t'} \geq \frac{\varphi(t') - \varphi(t)}{t' - t} \geq \frac{\varphi(t + t') - \varphi(t)}{t'}$$

and deduce that $\varphi(t + t') \leq \varphi(t) + \varphi(t')$.
2) Let (X, d) be a metric space. Show that $\varphi \circ d$ is a distance on X.
3) Let $\varphi(t) = \frac{t}{1+t}$ defined on $\mathbb{R}_+$. Show that φ satisfies the assumptions of the exercise.
4) Let (X, d) be a metric space. Show that δ from $X \times X$ to $\mathbb{R}_+$ defined by $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a distance on X.

Let $((X^i, d^i)_{i=1}^p$ be p metric spaces. Let X be the product of this spaces $X = \prod_{i=1}^p X^i$. Then one can define a distance d on X as follows: for all $(x = (x^i), y = (y^i)) \in X \times X$,

$$d(x, y) = \sum_{i=1}^{p} d^i(x^i, y^i)$$

We can define many other distances on X as in the following exercise. Later, we will show that the topological properties of X are the same, whatever is the chosen distance according to the procedure below.

Exercise 2 Let $((X^i, d^i)_{i=1}^p$ be p metric spaces. Let N be a norm on $\mathbb{R}^p$ such that for all $(\xi, \zeta) \in \mathbb{R}^p_+ \times \mathbb{R}^p_+$, if $\xi \geq \zeta$, that is $\xi_i \geq \zeta_i$ for all $i = 1, \ldots, p$, then $N(\xi) \geq N(\zeta)$. Show that the function δ_N defined by: for all $(x = (x^i), y = (y^i)) \in X \times X$,

$$\delta(x, y) = N\left((d^i(x^i, y^i))_{i=1}^p\right)$$

is a distance on X.

Metric on a countable product of bounded metric spaces

Let $((X^i, d^i)_{i \in \mathbb{N}}$ be a countable family of metric spaces. Let X be the product of this spaces $X = \prod_{i \in \mathbb{N}} X^i$. We assume that the metric spaces are bounded in the sense that for all i, for all $(x^i, y^i) \in X^i \times X^i$, there exists $M^i \in \mathbb{R}$ such that $d^i(x^i, y^i) \leq M^i$. Then one can define a distance d on X as follows: for all $(x = (x^i)_{i \in \mathbb{N}}, y = (y^i)_{i \in \mathbb{N}}) \in X \times X$,

$$d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^i M^i} d^i(x^i, y^i)$$

The definitions and results presented below are exactly the same as the ones presented for Euclidean spaces where the distance replaces the norm. The proofs are very similar.

Definition 2 Let (X, d) be a metric space. Let r be a non-negative real number and $\bar{x}$ an element of X.

a) the closed ball of center $\bar{x}$ and radius r is the set

$$\bar{B}(\bar{x}, r) = \{x \in X \mid d(x, \bar{x}) \leq r\}$$

b) the open ball of center $\bar{x}$ and radius r is the set

$$B(\bar{x}, r) = \{x \in X \mid d(x, \bar{x}) < r\}$$

Remark 4 Contrary to a normed vector space, the closed ball of center $\bar{x}$ and radius r may not be the closure of the open ball of center $\bar{x}$ and radius r and, conversely the open ball of center $\bar{x}$ and radius r may not be the interior of the closed ball of center $\bar{x}$ and radius r. Indeed, take X with more than one element and d be the atomic distance, which is $d(x, y) = 1$ if $x \neq y$ and 0 otherwise. Then, the closure of the open ball $B(\bar{x}, 1)$ is itself, that is the singleton $\{\bar{x}\}$ and the interior of the closed ball $\bar{B}(\bar{x}, 1)$ is itself, that is the set X, whereas the open ball $B(\bar{x}, 1)$ is the singleton $\{\bar{x}\} \neq X$.

1.2 Sequences

Definition 3 Let (X, d) be a metric space. A sequence is a mapping from $\mathbb{N}$ to X.

A sequence is often denoted (u_ν) where u_ν is the image of $\nu \in \mathbb{N}$.

Definition 4 Let (X, d) be a metric space. A sequence (u_ν) is bounded if there exists $\bar{x} \in X$ and $r > 0$ such that for all $\nu \in \mathbb{N}$, $u_\nu \in \bar{B}(\bar{x}, r)$.

A sequence (u_ν) converges to a limit $\ell \in X$ if for all $r > 0$, there exists an integer $\nu_r \in \mathbb{N}$ such that for all $\nu \geq \nu_r$, $u_\nu \in B(\ell, r)$ or equivalently $d(u_\nu, \ell) < r$.

If a sequence converges to a limit, we say that it is convergent and the limit is denoted by $\lim_{\nu\to\infty} u_\nu$.

Proposition 2 *(i) If a sequence is convergent, it has unique limit.*

(ii) The sequence (u_ν) converges to the limit ℓ if and only if the real sequence $(d(u_\nu, \ell))$ converges to 0.

(iii) If the sequence (u_ν) is convergent, then it is bounded.

Proposition 3 *Let $((X^i, d^i)_{i=1}^p$ be p metric spaces and d the distance defined above on $X = \prod_{i=1}^p X^i$. Let $(u_\nu = (u_\nu^i))$ be a sequence of X. Then the sequences (u_ν) converges for the distance d if and only if the p sequences (u_ν^i) converges for the distance d^i.*

Proposition 4 *Let $((X^i, d^i)_{i\in\mathbb{N}}$ be a countable family of bounded metric spaces. Let X be the product of this spaces and d be the distance defined above on X, that is: for all $(x = (x^i)_{i\in\mathbb{N}}, y = (y^i)_{i\in\mathbb{N}}) \in X \times X$,*

$$d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^i M^i} d^i(x^i, y^i)$$

A sequence $(u_\nu = (u_\nu^i)_{i\in\mathbb{N}}$ is convergent in X for the distance d if and only if for all $i \in \mathbb{N}$, the sequence (u_ν^i) is convergent in X^i for the distance d^i.

Proof. Let $(u_\nu = (u_\nu^i)_{i\in\mathbb{N}})$ be a convergent sequence of X and $\bar{u} = (\bar{u}^i)_{i\in\mathbb{N}}$ its limit. Then for all i, $d^i(u_\nu^i, \bar{u}^i) \leq 2^i M^i d(u_\nu, \bar{u}_\nu)$. So, since the real sequence $(d(u_\nu, \bar{u}))$ converges to 0, the real sequence $(d^i(u_\nu^i, \bar{u}^i))$ converges to 0, that is the sequence (u_ν^i) converges to $\bar{u}^i$ in X^i for the distance d^i.

Conversely, assume that each sequence (u_ν^i) is convergent in X^i for the distance d^i and let us denote $\bar{u}^i$ its limit. let $\bar{u} = (\bar{u}^i)_{i\in\mathbb{N}} \in X$. We show that the sequence (u_ν) converges to $\bar{u}$.

Let $r > 0$. Since the series $(\frac{1}{2^i})$ is absolutely convergent in $\mathbb{R}$, there exists $\bar{k} \in \mathbb{N}$ such that $\sum_{i=\bar{k}}^{\infty} \frac{1}{2^i} < r/2$. For all $i = 0, \ldots, \bar{k} - 1$, the sequence $(d^i(u_\nu^i, \bar{u}^i))$ converges to 0. So the sequence $(\sum_{i=0}^{\bar{k}-1} \frac{1}{2^i M^i} d^i(u_\nu^i, \bar{u}^i))$ converges to 0 in $\mathbb{R}$. So, there exists $\bar{\nu} \in \mathbb{N}$, such that for all $\nu \geq \bar{\nu}$, $\sum_{i=0}^{\bar{k}-1} \frac{1}{2^i M^i} d^i(u_\nu^i, \bar{u}^i) < r/2$. Gathering these inequalities, we get for all $\nu \geq \bar{\nu}$,

$$d(u_\nu, \bar{u}) = \sum_{i=0}^{\infty} \frac{1}{2^i M^i} d^i(u_\nu^i, \bar{u}^i) \leq \sum_{i=0}^{\bar{k}-1} \frac{1}{2^i M^i} d^i(u_\nu^i, \bar{u}^i) + \sum_{i=\bar{k}}^{\infty} \frac{1}{2^i} < r$$

Since the above inequality if true for all $r > 0$ for a suitable choice of $\bar{\nu}$, one concludes that the sequence $(d(u_\nu, \bar{u}))$ converges to 0, which means that the sequence (u_ν) converges to $\bar{u}$ in X. $\Box$

Exercise 3 Let X be a set and d be the distance defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that a sequence is convergent for this distance if and only if it is constant after a given rank, that is, for a sequence (u_ν), there exists $\underline{\nu} \in \mathbb{N}$ such that for all $\nu \geq \underline{\nu}$, $u_\nu = u_{\underline{\nu}}$.

Cauchy Criterion: We can extend the Cauchy Criterion to a metric space:

Definition 5 A sequence (u_ν) of a metric space (X, d) satisfies the Cauchy Criterion if :

$$\forall r > 0, \exists \nu_r \in \mathbb{N}, \forall \nu, \mu \geq \nu_r, d(u_\nu u_\mu) \leq r$$

Proposition 5 *Let $((X^i, d^i)_{i=1}^p$ be p metric spaces and d the distance defined above on $X = \prod_{i=1}^p X^i$. Let $(u_\nu = (u^i_\nu))$ be a sequence of X. Then the sequences (u_ν) satisfies the Cauchy Criterion for the distance d if and only if the p sequences (u^i_ν) satisfy the Cauchy Criterion for the distance d^i.*

Proposition 6 *A sequence (u_ν) of a metric space (X, d) satisfying the Cauchy Criterion is bounded.*

Proposition 7 *If a sequence (u_ν) of a metric space (X, d) is convergent, it satisfies the Cauchy criterion.*

Remark 5 A fundamental difference between a general metric space and an Euclidean space is the fact that a sequence satisfying the Cauchy criterion may not converge.

For example, let us consider the set ℓ_0 of real sequences with a finite number of non zero terms. We define a norm on this linear space by $N^1((u_\nu)) = \sum_{\nu \in \mathbb{N}} \frac{1}{2^\nu} |u_\nu|$ and the associated distance. For all $i \in \mathbb{N}$, let u^i be the real sequence such that the $i+1$ first terms are equal to 1 and the remaining ones are equal to 0. Clearly, for all $i < j$, $N^1(u^j - u^i) = \sum_{\nu=i+1}^j \frac{1}{2^\nu} \leq \frac{1}{2^i}$, so the sequence $(u^i)_{i \in \mathbb{N}}$ satisfies the Cauchy criterion. But, this sequence $(u^i)_{i \in \mathbb{N}}$ of ℓ_0 does not have a limit in ℓ_0. Indeed, if $v \in \ell^0$, then there exists an integer k such that $v_\nu = 0$ for all $\nu \geq k$. So, for all $i \geq k$, $N^1(u^i - v) \geq \sum_{\nu \in k}^j \frac{1}{2^\nu} \geq \frac{1}{2^k} > 0$. So, the real sequence $(N^1(u^i - v))$ does not converges to 0 and consequently the sequence $(u^i)_{i \in \mathbb{N}}$ of ℓ_0 does not converge to v in ℓ_0.

Definition 6 A metric space such that all sequences satisfying the Cauchy criterion is convergent is called a complete metric space.

Proposition 8 *Let (X, d) be a complete metric space and F be a closed subset of X. Then F with the distance d_F, which is the restriction of d to $F \times F$, is a complete metric space.*

Exercise 4 Let X be a set and d be the distance defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that (X, d) is a complete metric space.

We now consider a space X and a complete metric space (Y, d). We denote by $\mathcal{B}(X, Y)$ the set of bounded mappings from X to Y, that is the mapping f from X to Y such that there exists an element $\bar{y} \in Y$ and $r > 0$ such that for all $x \in X$, $f(x) \in \bar{B}(\bar{y}, r)$. Then we define the uniform distance d_∞ on $\mathcal{B}(X, Y)$ as follows:

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$$

We have the important following result, which combined with the previous proposition, is a suitable way to prove the completeness of many metric spaces.

Proposition 9 *The space $(\mathcal{B}(X, Y), d_\infty)$ is complete.*

Proof. Let (f_ν) be a sequence of $\mathcal{B}(X, Y)$ satisfying the Cauchy criterion. Then, one easily shows that for all $x \in X$, the sequence $(f_\nu(x))$ of Y satisfies the Cauchy criterion. As Y is complete, the sequence $(f_\nu(x))$ is convergent in Y and we denote by $\bar{f}(x)$ its limit.

Since (f_ν) satisfies the Cauchy criterion, it is bounded. So, there exists $\bar{g} \in \mathcal{B}(X, Y)$ and $\bar{r} > 0$ such that for all $\nu \in \mathbb{N}$, $f_\nu \in \bar{B}(\bar{g}, \bar{r})$. Then for all $x \in X$ and for all $\nu \in \mathbb{N}$, $f_\nu(x) \in \bar{B}(\bar{g}(x), \bar{r})$ and the limit $\bar{f}(x)$ belongs also to $\bar{B}(\bar{g}(x), \bar{r})$. Since $\bar{g}$ is bounded, there exists $\bar{y}_0 \in Y$ and $r_0 > 0$ such that for all $x \in X$, $\bar{g}(x) \in \bar{B}(\bar{y}_0, r_0)$. Consequently, $\bar{f}(x) \in \bar{B}(\bar{y}_0, r_0 + \bar{r})$, which shows that $\bar{f}$ is bounded.

To complete the proof, we now show that the real sequence $(d_\infty(f_\nu, \bar{f}))$ converges to 0. Let $r > 0$. Since (f_ν) satisfies the Cauchy criterion, there exists $\bar{\nu}$, such that for ν, μ satisfying $\bar{\nu} \leq \nu < \mu$, $d_\infty(f_\nu, f_\mu) \leq r$. This implies that for all $x \in X$, $d(f_\nu(x), f_\mu(x)) \leq r$. Keeping ν constant and taken the limit for μ at ∞, we get that $d(f_\nu(x), \bar{f}(x)) \leq r$. Since, this is true for all $x \in X$, $d_\infty(f_\nu, \bar{f}) \leq r$. We remark that this inequality holds true for all $\nu \geq \bar{\nu}$. Hence, we have proved that $\lim_{\nu \to \infty} d_\infty(f_\nu, \bar{f}) = 0$. □

Exercise 5 Let ℓ^∞ be the set of bounded real sequences. We define the norm on ℓ^∞ as follows: $\|(u_\nu)\|_\infty = \sup\{|u_\nu| \mid \nu \in \mathbb{N}\}$. Show that $(\ell^\infty, \|\cdot\|_\infty)$ is a complete metric space.

Exercise 6 Let us now consider the following norm N on ℓ^∞:

$$N((u_\nu)) = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} |u_\nu|$$

The purpose of the exercise is to show that ℓ^∞ is not complete for the norm N. Let us consider the sequence $(u^i = (u^i_\nu)_{\nu \in \mathbb{N}})_{i \in \mathbb{N}}$ of ℓ^∞ defined by: for all $i \in \mathbb{N}$,

$$u^i_\nu = \nu \text{ if } \nu \leq i, i \text{ otherwise}$$

1) Show that this sequence satisfies the Cauchy criterion for the norm N.
2) Show that for all $v \in \ell^\infty$, the real sequence $N(u^i - v)$ is bounded below by a non negative number for all i large enough and conclude that the sequence (u^i) is not convergent for the norm N.

Subsequences

From a given sequence (u_ν), we can build many others by picking only some terms of it.

Definition 7 Let (u_ν) be a sequence of a metric space (X, d). A subsequence of (u_ν) is a sequence (v_ν) defined by a strictly increasing mapping φ from $\mathbb{N}$ to itself and for all $\nu \in \mathbb{N}$, $v_\nu = u_{\varphi(\nu)}$.

Proposition 10 *If (u_ν) is a converging sequence of a metric space (X, d), then all subsequences of (u_ν) are convergent and they are converging to the same limit.*

Remark 6 A fundamental difference between a general metric space and an Euclidean space is the fact that the bounded sequence may not have a convergent subsequence. So, the closed ball may not be compact.

For example, let us consider the set ℓ_0 of real sequences with a finite number of non zero terms. We define a norm on this linear space by $\|(u_\nu)\|_\infty = \max\{|u_\nu| \mid \nu \in \mathbb{N}\}$ and the associated distance. For all $i \in \mathbb{N}$, let v^i be the real sequence such that all terms are equal to 0 but the ith one which is equal to 1. Clearly $\|(v^i)\|_\infty = 1$, so the sequence $(v^i)_{i\in\mathbb{N}}$ in ℓ_0 is a bounded sequence. But we remark that the distance between two different elements of this sequence $\|(v^i - v^j)\|_\infty$ is also equal to 1. So, no subsequence of $(v^i)_{i\in\mathbb{N}}$ can satisfy the Cauchy criterion. Indeed, if φ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, then for all $(i, j) \in \mathbb{N}\times\mathbb{N}$, $i \neq j$, $\|v^{\varphi(i)} - v^{\varphi(j)}\| = 1$.

Definition 8 Let (u_ν) be a sequence of a metric space (X, d). $x \in X$ is a cluster point of (u_ν) if for all $r > 0$, the set $\{\nu \in \mathbb{N} \mid u_\nu \in B(c, r)\}$ is infinite.

Proposition 11 *Let (u_ν) be a sequence of a metric space (X, d). $x \in X$ is a cluster point of (u_ν) if and only if there exists a convergent subsequence (v_ν) of (u_ν) such that x is the limit of (v_ν).*

1.3 Basic topology of a metric space

Definition 9 a) A subset F of a metric space (X, d) is closed if for all convergent sequences (u_ν) such that $u_\nu \in F$ for all $\nu \in \mathbb{N}$, then the limit of (u_ν) belongs to F.

b) A subset U of a metric space (X, d) is open if for all convergent sequences (u_ν) such that the limit belongs to U, then there exists $\nu_0 \in \mathbb{N}$ such that $u_\nu \in U$ for all $\nu \geq \nu_0$.

Remark 7 A closed ball is closed. An open ball is open. Let $((X^i, d^i)_{i=1}^p$ be p metric spaces and d the distance defined above on $X = \prod_{i=1}^p X^i$. If for all $i = 1, \ldots, p$, F^i is a closed subset of X^i, then $\prod_{i=1}^p F^i$ is a closed subset of X. If for all $i = 1, \ldots, p$, U^i is an open subset of X^i, then $\prod_{i=1}^p U^i$ is an open subset of X.

Proposition 12 *a) A subset F of a metric space (X, d) is closed if and only if F^c, its complement in X, is open.*

b) A subset U of a metric space (X, d) is open if and only if U^c, its complement in X, is closed.

c) A subset U of a metric space (X, d) is open if and only if for all $x \in U$, there exists $r > 0$ such that $B(x, r) \ \subset U$.

Proposition 13 *a) A finite union of closed sets is closed.*

b) An intersection of finitely many or infinitely many closed sets is closed.

c) A finite intersection of open sets is open.

d) A union of finitely many or infinitely many open sets is open.

Definition 10 Let A be a subset of a metric space (X, d).

a) The closure of A is the set of vectors $x \in X$ such that there exists a sequence (u_ν) converging to x and satisfying $u_\nu \in A$ for all $\nu \in \mathbb{N}$. The closure of A is denoted cl$A$ or $\overline{A}$.

b)The interior of A is the set $a \in A$ for which there exists $r > 0$ such that $B(a, r) \subset A$. The interior of A is denoted int$A$ or $\overset{\circ}{A}$.

Proposition 14 *Let A be a subset of a metric space (X, d).*

a) $A \subset \overline{A}$;

b) $\overline{A}$ is a closed subset of X;

c) $\overline{A}$ is the smallest closed subset of X containing A, that is, if F is closed and $A \subset F$, then $\overline{A} \subset F$;

d) $\overline{A}$ is the intersection of all closed subsets of X containing A.

Proposition 15 *Let A be a subset of $\mathbb{R}^n$.*

a) int$A \subset A$*;*

b) int$A$ *is an open subset of X;*

c) int$A$ *is the largest open subset of X included in A, that is, if U is open and $U \subset A$, then $U \subset$* int$A$*;*

d) int$A$ *is the union of all open subsets of* X *included in* A*.*

Definition 11 Let A be a subset of a metric space (X, d). The boundary of A denoted bd$A$ is the set $\overline{A} \cap \overline{A^c}$, that is the intersection of the closure of A with the closure of the complement of A in X.

Remark 8 An element b belongs to the boundary of A if and only if it is a limit of a sequence of elements of A and a limit of a sequence of elements not in A.

Proposition 16 *Let* A *be a subset of a metric space* (X, d)*.*

a) The boundary of A *is a closed set.*

b) A *is closed if and only if the boundary of* A *is included in* A*.*

c) A *is open if and only if the intersection of the boundary of* A *and* A *is empty,* bd$A \cap A = \emptyset$*.*

1.3.1 Compact metric space

Definition 12 A subset K of a metric space (X, d) is compact if all sequences of K have a converging subsequence in K.

Proposition 17 *Let* K *be a subset of a metric space* (X, d)*. The set* K *is compact if one of the following equivalent conditions is satisfied:*

If (u_ν) *is a sequence such that* $u_\nu \in K$ *for all* ν*, then it has a converging subsequence with a limit in* K*.*

If $(U_i)_{i\in I}$ *is a family of open subsets of* X *such that* $K \subset \cup_{i\in I} U_i$*, there exists a finite subset* $J \subset I$ *such that* $K \subset \cup_{i\in J} U_i$*.*

If $(F_i)_{i\in I}$ *is a family of closed subsets of* X *such that* $K \cap (\cap_{i\in I} F_i) = \emptyset$*, there exists a finite subset* $J \subset I$ *such that* $K \cap (\cap_{i\in J} F_i) = \emptyset$*.*

Proposition 18 *Let* K *be a compact subset of* (X, d)*. Then* K *is bounded, closed and complete.*

Remark 9 A fundamental difference between a general metric space and an Euclidean space is the fact that a bounded closed subset of a metric space may not be compact.

For an example, it suffices to take the closed unit ball in ℓ_0 for the norm $\|\cdot\|_\infty$. Indeed, we provide above an example of a sequence in this set having no convergent subsequence: (u^i) is defined by $u^i_\nu = 0$ if $\nu \neq i$ and $u^i_i = 1$. The closed unit ball is clearly closed and bounded.

Proposition 19 *Let* K *be a compact subset of* X *and* F *be a closed subset of* X*. Then* $K \cap F$ *is a compact subset of* X*.*

The characterisation of compact subsets in infinite dimensional spaces is a key issue with a lot of results which are far beyond the scope of this course. Some results are considering the norm compact sets, some others are considering weaker topologies to get more compact sets. We provide only one useful result about the countable product of compact spaces.

Countable product of compact spaces

Let $(A^i, d^i)_{i\in\mathbb{N}}$ be a countable family of compact metric spaces. Since a compact metric space is bounded, for each i there exists $M^i > 0$ such that for all $(a^i, b^i) \in A^i \times A^i$, $d^i(a^i, b^i) \leq M^i$. Let $A = \prod_{i\in\mathbb{N}} A^i$. We define as above a distance d on A as follows: for all $(a, a') \in A \times A$,

$$d(a, a') = \sum_{i=0}^{\infty} \frac{1}{2^i M^i} d^i(a^i, a'^i)$$

Theorem 1 *A is compact for the distance d.*

Proof. Let $u_\nu = (u^i_\nu) \in A$, be a sequence of A, that is, for all $i \in \mathbb{N}$, $u^i_\nu \in A^i$. We prove that this sequence has a converging subsequence.

For $i = 0$, the sequence (u^0_ν) has a converging subsequence $(u^0_{\varphi_0(\nu)})$. We denote its limit by $\bar{u}^0$. For $\nu = 1$, the bounded sequence $(u^1_{\varphi_0(\nu)})$ has a converging subsequence $\left(u^1_{\varphi_0\circ\varphi_1(\nu)}\right)$. We denote its limit by $\bar{u}^1$. Iterating the same process, for i, the bounded sequence $\left(u^i_{\varphi_0\circ\varphi_1\circ\ldots\circ\varphi_{i-1}(\nu)}\right)$ has a converging subsequence $\left(u^i_{\varphi_0\circ\varphi_1\ldots\circ\varphi_{i-1}\circ\varphi_i(\nu)}\right)$. We denote its limit by $\bar{u}^i$.

We now define $\psi(k) = \varphi_0\circ\varphi_1\ldots\circ\varphi_k(k)$. We check that ψ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. Indeed, $\psi(k+1) = \varphi_0\circ\varphi_1\ldots\circ\varphi_k\circ\varphi_{k+1}(k+1)$. Since φ_{k+1} is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, then $\varphi_{k+1}(k+1) \geq k+1$. So, since φ_k is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$, $\varphi_k(\varphi_{k+1}(k+1)) \geq \varphi_k(k+1) > \varphi_k(k)$. Hence

$$\psi(k+1) = \varphi_0\circ\varphi_1\ldots\circ\varphi_k\circ\varphi_{k+1}(k+1) > \varphi_0\circ\varphi_1\ldots\circ\varphi_k(k) = \psi(k)$$

Let us take a given $i \in \mathbb{N}$. We prove that the sequence $\left(u^i_{\psi(\nu)}\right)$ converges to $\bar{u}^i$. Indeed, for all $\nu > i$, then

$$\psi(\nu) = \varphi_0\circ\ldots\circ\varphi_i[\varphi_{i+1}\circ\ldots\circ\varphi_\nu(\nu)]$$

With the same argument as above, the mapping $k \to \varphi_{i+1}\circ\ldots\circ\varphi_k(k)$ is increasing for $k > i$. So, the sequence $(u^i_{\psi(\nu)})$ is, after a finite number of steps, a subsequence of the sequence $(u^i_{\varphi_0\circ\varphi_1\ldots\circ\varphi_{i-1}\circ\varphi_i(\nu)})$, which converges to $\bar{u}^i$. So, $(u^i_{\psi(\nu)})$ is a converging subsequence and its limit is $\bar{u}^i$. From Proposition 4, one concludes that the sequence $(u_{\psi(\nu)})$, which is a subsequence of (u_ν), converges in A. $\square$

In a complete metric space X, we have a simpler criterion to check that a closed subset A is compact.

Proposition 20 *Let (X, d) be a complete metric space and A be a closed subset of X. Then A is compact if and only if for all $r > 0$, there exists a finite family $(x^1, x^1, \ldots, x^k)$ of A such that $A \subset \cup_{j=1}^k B(x^j, r)$.*

Proof. If A is compact, it satisfies the property of finite covering by balls of radius r since $(B(a, r))_{a \in A}$ is an open covering of A from which one can extract a finite covering.

Conversely, we show that if (a_ν) is a sequence of A, it has a convergent subsequence. Since A is closed in X complete, A is complete, and then, it suffices to show that the subsequence satisfies the Cauchy Criterion. From the finite covering property, for all $i \in \mathbb{N}$, there exists k_i elements of A, $(x^{i,1}, \ldots, x^{i,k_i})$ such that $E \subset \cup_{k=1}^{k_i} B(x^{i,k}, 2^{-i})$. So, there exists $k^0 \in \{1, \ldots, k_0\}$ such that an infinite number of terms of the sequence (a_ν) belongs to $B(x^{0,k^0}, 1)$. Hence, there exists φ_0 strictly increasing from $\mathbb{N}$ to itself such that for all ν, $a_{\varphi_0(\nu)} \in B(x^{0,k^0}, 1)$.

We repeat the same argument with the sequence $(a_{\varphi_0(\nu)})$, from which one deduces that there exists $k^1 \in \{1, \ldots, k_1\}$ and φ_1 strictly increasing from $\mathbb{N}$ to itself such that for all ν, $a_{\varphi_0 \circ \varphi_1(\nu)} \in B(x^{1,k^1}, 2^{-1})$. Iterating this process, for all $i \in \mathbb{N}$, there exists $k^i \in \{1, \ldots, k_i\}$ and φ_i strictly increasing from $\mathbb{N}$ to itself such that for all ν, $a_{\varphi_0 \circ \varphi_1 \circ \ldots \varphi_i(\nu)} \in B(x^{i,k^i}, 2^{-i})$.

Let ψ from $\mathbb{N}$ to itself defined by $\psi(i) = \varphi_0 \circ \varphi_1 \circ \ldots \varphi_i(i)$. As in the proof of the previous theorem, we show that ψ is strictly increasing. We conclude the proof by showing that the sequence $(a_{\psi(\nu)})$ satisfies the Cauchy criterion. Let i and j two integers such that $i < j$. Then

$$\psi(j) = \varphi_0 \circ \ldots \circ \varphi_i[\varphi_{i+1} \circ \ldots \circ \varphi_j(j)]$$

so $a_{\psi(j)}$ belongs to $B(x^{i,k^i}, 2^{-i})$ and $a_{\psi(i)}$ also. Hence, $d(a_{\psi(j)}, a_{\psi(i)} \leq 2^{i-1}$. This implies that the sequence $\left(a_{\psi(\nu)}\right)$ satisfies the Cauchy criterion since for all ν, i, j in $\mathbb{N}$, if $\nu \leq i$ and $\nu \leq j$, then $d(a_{\psi(j)}, a_{\psi(i)} \leq 2^{\nu-1}$. □

Equivalences on distances

To end this section, we provide two definitions of equivalence between distances on the same set X. The basic idea is to check if two different distances leads to the same topological structure, convergent sequences, open or closed sets, compact subsets, ...

Definition 13 Let X be a set, d and δ, two distances on X.

1) d and δ are equivalent if there exists two non negative real numbers a and b such that for all $(x, y) \in X \times X$,

$$ad(x, y) \leq \delta(x, y) \leq bd(x, y)$$

2) d and δ are topologically equivalent if all open sets U for the distance d is also open for the distance δ and conversely.

Remark 10 One easily shows that if two distances are equivalent, they are topologically equivalent. The converse is not true. On $\mathbb{N}$, we can define the distance between two integers as $d(p,q) = |p-q|$ and the distance δ by $\delta(p,q) = 0$ if $p = q$ and 1 otherwise. The two distances are not equivalent since the ratio $\frac{d(p,q)}{\delta(p,q)} = d(p,q)$ for $p \neq q$ is not upper bounded on $\mathbb{N} \times \mathbb{N}$. Nevertheless the two distances are topologically equivalent since all subsets of $\mathbb{N}$ are open for both distances.

Exercise 7 On $\mathbb{R}$, show that the distance d defined by $d(x,y) = |x-y|$ and δ defined by $\delta(x,y) = \frac{d(x,y)}{1+d(x,y)}$ are not equivalent but topologically equivalent.

Exercise 8 Let X be a set and d and δ two topologically equivalent distances on X. Show that a sequence (u_ν) of X is convergent for d if and only if it is convergent for δ.

Since two equivalent distances leads to the same topological properties, it is convenient for some problems to choose the most suitable distance among those which are equivalent.

Remark 11 Be careful of the fact that topological concepts are relative to the space we consider. If (X,d) is a metric space and Y a subset of X, then Y may not have the same properties as a subset of X of if we consider the metric space (Y, d_F) where d_F is the restriction of d to $Y \times Y$. Indeed, in $\mathbb{R}$, the segment $[0,1[$ is neither open nor closed. But if we consider the metric space $Y = [0,1[$ with the distance defined by the absolute value restricted to Y, then Y is open and closed!

1.4 Mappings

In this section, we consider two metric spaces (X,d) and (Y,δ) and we extend the definition of continuity and the properties of the continuous mappings to the mapping from U, a subset of X to Y.

Definition 14 Let f be a mapping from $U \subset X$ to Y.

The image of U by f is the set $\{y \in Y \mid \exists x \in U, y = f(x)\}$.

f is bounded if there exists $r > 0$ and $\bar{y} \in Y$ such that the image of f is included in $\bar{B}_Y(\bar{y}, r)$.

Limit of a mapping

Definition 15 Let f be a mapping from X to Y. Let x_0 an element of the closure of U. The function f has a limit y_0 at x_0 if for all sequences (x_ν) satisfying $x_\nu \in U$ for all ν and $\lim_{\nu\to\infty} x_\nu = x_0$, then the sequence $(f(x_\nu))$ is convergent in Y and its limit is y_0.

Proposition 21 *Let f be a mapping from $U \subset X$ to Y. Let x_0 be an element of the closure of U.*

The function f has at most one limit at x_0.

The function f has a limit y_0 at x_0 if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in B_X(x_0, \rho) \cap U$, $f(x) \in B_Y(y_0, r)$.

Cauchy criterion: if the set Y is complete, the function f has a limit at x_0 if and only if for all $r > 0$, there exists $\rho > 0$ such that for all pair (x, x') in $B_X(x_0, \rho) \cap U$, $\delta(f(x), f(x')) < r$.

Limits and closed sets

Proposition 22 *Let f be a mapping from $U \subset X$ to Y. Let $x_0 \in \overline{U}$. We assume that f has a limit y_0 at x_0. If there exists $r > 0$ and F a closed subset of Y such that for all $x \in U \cap B_X(x_0, r)$, $f(x) \in F$, then $y_0 \in F$.*

Limit of the composition of two mappings

We consider a third metric space Z.

Proposition 23 *Let f be a mapping on $U \subset X$ to Y and $x_0 \in \overline{U}$. Let g be a mapping on $V \subset Y$ to Z. We assume that for all $x \in U$, $f(x) \in V$. Let $y_0 = \lim_{x \to x_0} f(x)$. One easily checks that $y_0 \in \overline{V}$. Let $z_0 = \lim_{y \to y_0} g(y)$. Then the limit of $g \circ f$ at x_0 exists and is equal to z_0.*

Continuous mappings

Definition 16 Let f be a mapping on $U \subset X$ to Y. f is continuous at a point $x_0 \in U$, if the limit of f at x_0 exists and is equal to $f(x_0)$. f is continuous on U if f is continuous at every point of U.

Remark 12 For all fixed $\bar{x}$ in X, the function $d(\bar{x}, \cdot)$ from X to $\mathbb{R}$ is continuous. The function d from $X \times X$ to $\mathbb{R}$ is continuous for the distance D on $X \times X$ defined by $D((x, y), (x', y')) = d(x, x') + d(y, y')$. If d is the distance on X defined by $d(x, y) = 0$ if $x = y$ and 1 otherwise, all mappings from X to any metric space is continuous.

A particular class of continuous mapping is the class of **Lipschitzian** mappings, that is the function f from $U \subset X$ to Y such that there exists $k \geq 0$, for all $(x, x') \in U \times U$, $\delta(f(x), f(x')) \leq k d(x, x')$.

Proposition 24 *Let f be a mapping on $U \subset X$ to Y. f is continuous on U if one of the two equivalent following conditions is satisfied:*

For all open set V of Y, the set $f^{-1}(V) = \{x \in U \mid f(x) \in V\} = W \cap U$ where W is an open set of X.

For all closed set F of Y, the set $f^{-1}(F) = \{x \in U \mid f(x) \in F\} = G \cap U$ where G is a closed set of X.

Proposition 25 *Let f and g be two continuous mappings on $U \subset X$ to Y. Then the function $x \to \delta(f(x), g(x))$ is continuous on U.*

Proposition 26 *Let f be a continuous mapping from $U \subset X$ to Y. Let g be a continuous mapping from $V \subset Y$ to Z. We assume that for all $x \in U$, $f(x) \in V$. Then $g \circ f$ is continuous on U.*

With these basic operations, we are able to show almost always that the usual functions are continuous.

We now consider the set $\mathcal{C}_b(X,Y)$ of bounded continuous mappings from X to Y. This set is a subset of the bounded mappings from X to Y. We show that this set is complete for the uniform distance if Y is complete.

Proposition 27 *Let $\mathcal{C}_b(X,Y)$ be the set of bounded continuous mappings from X to Y. Let d_∞ be the distance on $\mathcal{C}_b(X,Y)$ defined by $d_\infty(f,g) = \sup\{\delta(f(x), g(x)) \mid x \in X\}$. Then, if Y is complete, $\mathcal{C}_b(X,Y)$ is a complete metric space for the distance d_∞.*

Proof. From Propositions 8 and 9, it suffices to show that $\mathcal{C}_b(X,Y)$ is a closed subset of $\mathcal{B}(X,Y)$. Let (f_ν) be a sequence of $\mathcal{C}_b(X,Y)$, which converges for the distance d_∞ to $\bar{f} \in \mathcal{B}(X,Y)$. We prove that $\bar{f}$ is continuous.

Let $\bar{x} \in X$ and (x_μ) a sequence of X converging to $\bar{x}$. Let $r > 0$. Since (f_ν) converges to $\bar{f}$, there exists $\bar{\nu} \in \mathbb{N}$ such that $d_\infty(f_{\bar{\nu}}, \bar{f}) < r/3$. Since $f_{\bar{\nu}}$ is continuous at $\bar{x}$, there exists $\mu_0 \in \mathbb{N}$ such that for all $\mu \geq \mu_0$, $\delta(f_{\bar{\nu}}(x_\mu), f_{\bar{\nu}}(\bar{x})) < r/3$. So, for all $\mu \geq \mu_0$,

$$\delta(\bar{f}(x_\mu), \bar{f}(\bar{x})) \leq \delta(\bar{f}(x_\mu), f_{\bar{\nu}}(x_\mu)) + \delta(f_{\bar{\nu}}(x_\mu), f_{\bar{\nu}}(\bar{x})) + \delta(f_{\bar{\nu}}(\bar{x}), \bar{f}(\bar{x}))$$

Since, $d_\infty(f_{\bar{\nu}}, \bar{f}) < r/3$, the first and the third terms of the right side of the inequality is smaller than $r/3$ and the term in the middle is also smaller than $r/3$. So, one concludes that $\delta(\bar{f}(x_\mu), \bar{f}(\bar{x})) < r$ for all $\mu \geq \mu_0$. So the sequence $(\bar{f}(x_\mu))$ converges to $\bar{f}(\bar{x})$ which shows that $\bar{f}$ is continuous. □

Exercise 9 Let $(A^i, d^i)_{i\in\mathbb{N}}$ be a countable family of bounded metric spaces. For each i, let $M^i > 0$ such that for all $(a^i, b^i) \in A^i \times A^i$, $d^i(a^i, b^i) \leq M^i$. Let $A = \prod_{i\in\mathbb{N}} A^i$. We define as above a distance d on A as follows: for all $(a, a') \in A \times A$,

$$d(a, a') = \sum_{i=0}^{\infty} \frac{1}{2^i M^i} d^i(a^i, a'^i)$$

For all i, let g^i be a continuous mapping from A^i to a complete normed linear space (E, N). We assume that there exists a common bound on the mapping g^i, that is a non negative real number $\bar{M}$, so that for all i, for all $a^i \in A^i$,

$g^i(a^i) \in \bar{B}_E(0, \bar{M})$. Let $\beta \in [0, 1[$ be a discount factor. We consider the mapping g from A to E defined by

$$g(a) = \sum_{i=0}^{\infty} \beta^i g^i(a^i)$$

1) Show that the mapping g is well defined.
2) Show that the mapping g is continuous.

Exercise 10 Let (X, d) be a metric space and F be a nonempty subset of X. We define the function distance to F, d_F, by $d_F(x) = \inf\{d(x, y) \mid y \in F\}$. We prove that this function is Lipschitz continuous of rank 1.

Let $(x, y) \in X \times X$. We assume without any loss of generality that $d_F(x) \geq d_F(y)$.
1) Let $r > 0$. Show that there exists $\zeta \in F$ such that $d(y, \zeta) \leq d_F(y) + r$.
2) Show that $d_F(x) - d_F(y) \leq d(x, \zeta) - d(y, \zeta) + r$.
3) Deduce from the previous question that $d_F(x) - d_F(y) \leq d(x, y) + r$.
4) Conclude.

Exercise 11 Let (X, d) be a metric space and F be a nonempty closed subset of X. We define the function distance to F, d_F, by $d_F(x) = \inf\{d(x, y) \mid y \in F\}$.
1) Show that $d_F(x) = 0$ if and only if $x \in F$.

We now assume that the interior of F, $\text{int}F$, is nonempty and $F = \overline{\text{int}F}$. We consider the function δ_F defined by $\delta_F(x) = d_F(x) - d_{F^c}(x)$ where F^c is the complement of F in X.
2) Show that $\delta_F(x) > 0$ if $x \in F^c$, $\delta_F(x) = 0$ if $x \in \text{bd}F$, the boundary of F, and $\delta_F(x) < 0$ if $x \in \text{int}F$.

1.5 Continuous function on a compact set

Theorem 2 *Let $K \subset X$ be a compact subset of X and f be a continuous mapping from K to Y. Then $f(K)$ is a compact subset of Y.*

Corollary 1 *Weierstrass Theorem. Let $K \subset X$ be a compact subset of X and f be a continuous mapping from K to $\mathbb{R}$. Then there exists $\overline{x} \in K$ and $\underline{x} \in K$ such that for all $x \in K$, $f(\underline{x}) \leq f(x) \leq f(\overline{x})$.*

Theorem 3 *Heine's Theorem Let $K \subset X$ be a compact subset of X and f be a continuous mapping from K to Y. Then f is uniformly continuous on K, which means that for all $r > 0$, there exists $\rho > 0$, such that for all $(x, x') \in K \times K$ such that $d(x, x') \leq \rho$, then $\delta(f(x), f(x') \leq r$.*

Remark 13 If (X, d) is a compact metric space, then all continuous mappings are bounded. So if (Y, δ) is a complete metric space, then $\mathcal{C}(X, Y)$, the space of continuous mapping from X to Y is a complete metric space for the uniform distance.

1.6 Banach fixed point theorem

Theorem 4 *Let f be a mapping from $U \subset X$ to U. We assume that U is complete and f is a contraction, that is, there exists $k \in [0,1[$ such that for all $(x,x') \in U \times U$, $d(f(x), f(x')) \leq kd(x,x')$. Then there exists a unique element (fixed point) $\bar{x} \in U$ such that $f(\bar{x}) = \bar{x}$ and for all $x_0 \in U$, the sequence (u_ν) defined by $u_0 = x_0$ and for all $\nu \in \mathbb{N}$, $u_{\nu+1} = f(u_\nu)$ converges to $\bar{x}$.*

Proof. The uniqueness of the fixed point is proved easily by contraposition. Now, let $x_0 \in U$, and the sequence (u_ν) defined by $u_0 = x_0$ and for all $\nu \in \mathbb{N}$, $u_{\nu+1} = f(u_\nu)$. We prove that this sequence satisfies the Cauchy Criterion. Indeed, for all ν, $d(u_{\nu+1}, u_\nu) = d(f(u_\nu), f(u_{\nu-1})) \leq kd(u_\nu, u_{\nu-1})$. So, by a backward induction, one deduces that $d(u_{\nu+1}, u_\nu) \leq k^\nu d(u_1, u_0)$. So, for all integers ν and μ such that $\mu > \nu$, we get:

$$\begin{aligned} d(u_\mu, u_\nu) &\leq d(u_\mu, u_{\mu-1}) + d(u_{\mu_1}, u_{\mu-2}) + \ldots + d(u_{\nu+1}, u_\nu) \\ &\leq (k^{\mu-1} + \ldots + k^\nu) d(u_1, u_0) \leq \frac{k^\nu}{1-k} d(u_1, u_0) \end{aligned}$$

Let $r > 0$ and $\bar{\nu}$ large enough so that $\frac{k^{\bar{\nu}}}{1-k} d(u_1, u_0) < r$. Then for all integers ν and μ such that $\mu > \nu \geq \bar{\nu}$, $d(u_\mu, u_\nu) < r$. Hence, the sequence (u_ν) converges since it satisfies the Cauchy Criterion and U is complete. Let $\bar{u}$ its limit. Then, since f is continuous, $f(\bar{u}) = \lim_{\nu \to \infty} f(u_\nu) = \lim_{\nu \to \infty} u_{\nu+1} = \bar{u}$. □

The following result provides a sufficient condition on a function on $\mathcal{B}(X, \mathbb{R}^n)$, the linear space of bounded functions from X to $\mathbb{R}^n$, to be Lipschitz continuous. We consider the sup-norm on $\mathbb{R}^n$, that is $\|u\|_\infty = \max\{|u_i| \mid i = 1, \ldots, n\}$ and we define the norm on $\mathcal{B}(X, \mathbb{R}^n)$ as $\|f\|_\mathcal{B} = \sup\{\|f(x)\|_\infty \mid x \in X\}$. We also use the following notations: if u and v are two vectors of $\mathbb{R}^n$, then $u \leq v$ means that $u_i \leq v_i$ for all i or equivalently that $v - u \in \mathbb{R}^n_+$. Then if g and h are two mappings of $\mathcal{B}(X, \mathbb{R}^n$, $h \leq g$ means that $h(x) \leq g(x)$ for all $x \in X$.

We denote by $\mathbf{1}_n$ the vector of $\mathbb{R}^n$ whose components are equal to 1 and by $\mathbf{1}_X$ the constant mapping from X to $\mathbb{R}^n$ such that for all $x \in X$, $\mathbf{1}_X(x) = \mathbf{1}_n$.

Theorem 5 *(Blackwell's Theorem) Let F be a mapping from $\mathcal{B}(X, \mathbb{R}^n)$ to itself. We posit the following assumptions:*

(i) for all $(g,h) \in \mathcal{B}(X, \mathbb{R}^n) \times \mathcal{B}(X, \mathbb{R}^n)$, if $h \leq g$ then $F(h) \leq F(g)$;

(ii) there exists $k \geq 0$ such that for all $g \in \mathcal{B}(X, \mathbb{R}^n)$, for all $t \in \mathbb{R}_+$, $F(g + t\mathbf{1}_X) \leq F(g) + kt\mathbf{1}_X$.

Then F is a Lipschitz continuous mapping of rank k.

In the applications, k is smaller than 1, so F is contracting and we can apply the Banach fixed point Theorem to F.

Proof. Let $(g,h) \in \mathcal{B}(X, \mathbb{R}^n) \times \mathcal{B}(X, \mathbb{R}^n)$. For all $x \in X$, for all $i = 1, \ldots, n$, $|g_i(x) - h_i(x)| \leq \|g - h\|_\mathcal{B}$. So, $g(x) \leq h(x) + \|g - h\|_\mathcal{B} \mathbf{1}_n(x)$ and $h(x) \leq g(x) + \|g -$

$h\|_{\mathcal{B}}\mathbf{1}_n(x)$. So, from the first assumption on F, $F(g) \leq F(h + \|g-h\|_{\mathcal{B}}\mathbf{1}_X)$ and $F(h) \leq F(g + \|g-h\|_{\mathcal{B}}\mathbf{1}_X)$. Thus, from the second assumption, $F(g) \leq F(h) + k\|g-h\|_\infty\mathbf{1}_X$ and $F(h) \leq F(g)+k\|g-h\|_\infty\mathbf{1}_X$. Consequently, for all $x \in X$, for all $i = 1,\ldots,n$, $|F(g)_i(x) - F(h)_i(x)| \leq k\|g-h\|_{\mathcal{B}}$, so $\|F(g) - F(h)\|_{\mathcal{B}} \leq k\|g-h\|_{\mathcal{B}}$. □

1.7 Normed linear spaces

We consider in this section a linear space E and a distance derived from a norm N on E. So a normed linear space is a particular case of a metric space. Hence all results above can be applied to normed linear spaces. Nevertheless, we will provide some additional properties and some warnings about the possible traps when we consider infinite dimensional spaces.

First of all, we prove that all norms on a finite dimensional linear space are equivalent. For this, we consider a particular norm on a finite dimensional space E and we prove that all other norms are equivalent to it. Let E be a finite dimensional linear space and $(\epsilon^1,\ldots,\epsilon^n)$ be a basis of E. Then, for all $u \in E$, we let $\|u\| = \max\{|u_i| \mid i = 1,\ldots,n\}$ where $(u_i) \in \mathbb{R}^n$ are the components of u in the given basis.

Theorem 6 *Let E be a finite dimensional linear space and N a norm on E. Then N is equivalent to $\|\cdot\|$.*

Proof. For all $u \in E$, $N(u) = N(\sum_{i=1}^n u_i\epsilon^i) \leq \sum_{i=1}^n |u_i| N(\epsilon^i) \leq \|u\| \left(\sum_{i=1}^n N(\epsilon^i)\right)$. Let $k = \left(\sum_{i=1}^n N(\epsilon^i)\right)$. Then, for all $(u,v) \in E \times E$, from the triangular inequality, we deduces that $|N(u) - N(v)| \leq N(u-v)$. So, $|N(u)-N(v)| \leq k\|u-v\|$, which means that N is a Lipschitz continuous function on E. Consequently, since the sphere $S = \{u \in E \mid \|u\| = 1\}$ is bounded and closed, so compact, there exists $\underline{u} \in S$, such that $N(\underline{u}) \leq N(u)$ for all $u \in S$. Since $\underline{u} \neq 0$, $N(\underline{u}) > 0$. From the positive homogeneity of the norms, one deduces that $\|u\| \leq \frac{1}{N(\underline{u})}N(u)$. Hence the two norms are equivalent. □

Remark 14 Let (E,N) be a finite dimensional normed linear space. Let $(\epsilon^1,\ldots,\epsilon^n)$ be a basis of E. Then one can define a norm on $\mathbb{R}^n$ by $N_n(x) = N(\sum_{i=1}^n x_i\epsilon^i)$. This norm is equivalent to the Euclidean norm associated to the canonical inner product on $\mathbb{R}^n$. So all topological properties of $\mathbb{R}^n$ with the Euclidean norm are transposed to $(\mathbb{R}^n, N_n)$ and then to (E,N) through the one-to-one onto linear mapping $x \to \sum_{i=1}^n x_i\epsilon^i$ which preserves the norm.

We deduce from this result that a finite dimensional subspace of a normed linear space is closed.

Proposition 28 *Let (E,N) be a normed linear space and F be a finite dimensional subspace of E. Then F is closed.*

Proof. Let (u_ν) be a sequence of elements of F converging to $\bar{u} \in E$. Let N_F be the restriction of the norm N to F. One easily check that it is a norm on F and (F, N_F) is a finite dimensional normed linear space, so it is complete. As a converging sequence, (u_ν) satisfies the Cauchy criterion in E, so in F too. Hence (u_ν) is converging to $\bar{v}$ in F. But, this obviously implies that (u_ν) converges to $\bar{v}$ in E. Hence, the unicity of the limit implies that $\bar{u} = \bar{v}$, so F is closed. □

Remark 15 This property is not obvious since many subspaces of an infinite dimensional linear space are not closed, contrary to the finite dimensional case.

Let us consider the space ℓ^∞ of bounded real sequences with the norm $\|u\|_\infty = \sup\{|u_\nu|_\infty \mid \nu \in \mathbb{N}\}$. Let ℓ_0 be the space of real sequences with a finite number of non-zero terms. One easily checks that ℓ_0 is a subspace of ℓ^∞ which is not closed. Indeed, the sequence $(u^i)_{i\in\mathbb{N}}$ defined by $u^i_\nu = \frac{1}{\nu+1}$ if $\nu \leq i$ and 0 otherwise is a sequence of ℓ_0 and its limit which is the sequence $\bar{u}$ defined by $\bar{u}_\nu = \frac{1}{\nu+1}$ for all ν does not belong to ℓ_0.

Exercise 12 Let (E, N) be a normed linear space. We define the norm N^2 on $E \times E$ by $N^2(x, y) = N(x) + N(y)$. We define the norm $\tilde{N}$ on $\mathbb{R} \times E$ by $\tilde{N}(t, x) = |t| + N(x)$.
1) Show that the mapping Σ from $E \times E$ to E defined by $\Sigma(x, y) = x + y$ is continuous for the norms N^2 and N.
2) Show that the mapping Π from $\mathbb{R} \times E$ to E defined by $\Pi(t, x) = tx$ is continuous for the norms $\tilde{N}$ and N.

Exercise 13 **Exercise 14** Let $((X^i, d^i)_{i=1}^p$ be p metric spaces. Let N and N' be two norms on $\mathbb{R}^p$ such that for all $(\xi, \zeta) \in \mathbb{R}^p_+ \times \mathbb{R}^p_+$, if $\xi \geq \zeta$, that is $\xi_i \geq \zeta_i$ for all $i = 1, \ldots, p$, then $N(\xi) \geq N(\zeta)$ (resp. $N'(\xi) \geq N'(\zeta)$). Let δ_N and $\delta_{N'}$ be the two distances on $X = \prod_{i=1}^p X^i$ defined by: for all $(x = (x^i), y = (y^i)) \in X \times X$,

$$\delta(x, y) = N\left((d^i(x^i, y^i))_{i=1}^p\right), \quad \delta'(x, y) = N'\left((d^i(x^i, y^i))_{i=1}^p\right)$$

Show that δ and δ' are equivalent on X.

A fundamental difference between the finite dimensional case and the infinite one is the fact that the closed unit ball of an infinite dimensional normed linear space is never compact. So, it is very important to be particularly cautious when we try to extend some well known results or reasoning from the finite dimensional spaces to the infinite ones. Indeed, many results implicitly used the fact that the closed balls are compact.

Theorem 7 *(Riesz) Let (E, N) be an infinite dimensional normed linear space. Then, $\bar{B}(0, 1)$ is not compact.*

Proof. We prove that the ball $\bar{B}(0, 1)$ cannot be covered by a finite union of open balls of radius 1/2. Indeed, let us assume that $\bar{B}(0, 1) \subset \cup_{j=1}^p B(x_j, 1/2)$.

We will prove that $\bar{B}(0,1)$ is included in the space F spanned by the vectors $(x_j)_{j=1}^p$, which contradicts that E has an infinite dimension.

Let $x \in \bar{B}(0,1)$, there exists j_0 such that $N(x - x_{j_0}) < 1/2$. If $x = x_{j_0}$, we stop since $x \in F$. If not, we consider the element $y_1 = \frac{1}{N(x-x_{j_0})}(x - x_{j_0})$. There exists j_1 such that $N(y_1 - x_{j_1}) < 1/2$. So, we remark that

$$x = x_{j_0} + N(x - x_{j_0})x_{j_1} + N(x - x_{j_0})(y - x_{j_1})$$

and $N(N(x - x_{j_0})(y - x_{j_1})) < 1/4$. This means that there exists a vector in F at a distance less than $1/4$ of x. Repeating the same argument with $x - (x_{j_0} + N(x - x_{j_0})x_{j_1})$ if this vector is not equal to 0, we find a vector of F at a distance less than $1/8$ of x. By induction, we build a sequence of elements of F which converges to x. Since F is closed, as proved above, x belongs to F. $\square$

Exercise 15 Mimicking the proof of the theorem, prove the following result. Let F be a closed subspace of a normed linear space (E, N), with $F \neq E$. Then, for all $k < 1$, there exists $v \in E$ such that $N(v) = 1$, $N(v - u) \geq k$ for all $u \in F$.

Exercise 16 Let E be a linear space and N_1 and N_2 two norms on E. We assume that the associated distances are topologically equivalent. We will show that N_1 and N_2 are equivalent.
1) Show that there exists $r_1 > 0$ and $r_2 > 0$ such that $B_{N_1}(0, r_1) \subset B_{N_2}(0,1)$ and $B_{N_2}(0, r_2) \subset B_{N_1}(0,1)$. Hint: $B_{N_2}(0,1)$ is an open set for the topology derived from N_1.
2) Conclude.

Exercise 17 Let (E, N) be a normed linear space and F a linear subspace of E. Show that F has a nonempty interior if and only if $F = E$.

Exercise 18 Let us consider the Banach space $\mathcal{C}([0,1])$ of the continuous function from $[0,1]$ to $\mathbb{R}$ equipped with the uniform norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$. The Riesz Theorem tells us that the closed unit ball $\bar{B}_\infty(0,1)$ is not compact. Now, we consider the subset $\mathcal{C}_1$ of this ball defined as the set of Lipschitz continuous mapping of rank $k \leq 1$. The aim of this exercise is to prove that C_1 is compact.
1) Show that $\mathcal{C}_1$ is closed and complete.

We now choose $\nu \in \mathbb{N}^*$ and we prove that we can find a finite family H of elements of $\mathcal{C}_1$ such that $\mathcal{C}_1 \subset \cup_{h \in H} B(h, 2\nu^{-1}\}$. For this, we consider the grid $(0, \frac{1}{\nu}, \frac{2}{\nu}, \ldots, \frac{\nu-1}{\nu}, 1)$ of $[0,1]$ and the grid $(-1, -\frac{\nu-1}{\nu}, \ldots, 0, \frac{1}{\nu}, \frac{2}{\nu}, \ldots, \frac{\nu-1}{\nu}, 1)$ on $[-1,1]$.

We consider the finite set H of functions h_j with $j \in \{-\nu, -(\nu-1), \ldots, 0, 1, \ldots, \nu\}$ which are affine on each segment $[\frac{i}{\nu}, \frac{i+1}{\nu}]$, $h_j(0) = \frac{j}{\nu}$ and such that $h_j(\frac{j+1}{\nu})$ is equal either to $h_j(\frac{i}{\nu})$, $h_j(\frac{i}{\nu}) + \frac{1}{\nu}$ or $h_j(\frac{i}{\nu}) + \frac{1}{\nu}$.
2) Show that the functions in H are Lipschitz continuous of rank $k \leq 1$.

Let g be a function of $\bar{B}_\infty(0,1)$ Lispschitz continuous of rank $k \leq 1$. We define the function $h \in H$ as follows: for all $i = 0, \ldots, \nu$, let $j_\nu \in \{-\nu, \ldots, 0, \ldots, \nu\}$ such that $g(\frac{i}{\nu}) \in [\frac{j_\nu}{\nu}, \frac{j_\nu+1}{\nu}[$, then $h(\frac{1}{\nu}) = \frac{j}{\nu}$.

3) Check that h belongs to H, that is, for all i, $h(\frac{j+1}{\nu})$ is equal either to $h(\frac{i}{\nu})$, $h(\frac{i}{\nu}) + \frac{1}{\nu}$ or $h(\frac{i}{\nu}) + \frac{1}{\nu}$.
4) Show that $\|g - h\|_\infty \leq 2/\nu$.
5) Conclude using Proposition 20.

Exercise 19 Let $\mathcal{C}_L([0,1])$ be the space of Lipschitz continuous functions on $[0,1]$ and $\mathcal{C}_k([0,1])$ be the set of Lipschitz continuous functions on $[0,1]$ of rank less or equal to k.
1) Show that $\mathcal{C}_L([0,1])$ is not closed in $\mathcal{C}([0,1])$, the space of the continuous function from $[0,1]$ to $\mathbb{R}$ equipped with the uniform norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$. Hint: Consider the sequence (f_ν) defined by

$$f_\nu(t) = \begin{cases} t\sqrt{\nu+1} \text{ if } t \in [0, \frac{1}{\nu+1}] \\ \sqrt{t} \text{ if } t \in [\frac{1}{\nu+1}, 1] \end{cases}.$$

2) Show that the space $\mathcal{C}_k([0,1])$ is closed in $\mathcal{C}([0,1])$.
3) Using the result of the previous exercise which says that $C_1 \cap \bar{B}_\infty(0,1)$ is compact for the norm $\|\cdot\|_\infty$, show that for all $k \geq 0$ and for all $R > 0$, the set $A_{kR} = \mathcal{C}_k([0,1]) \cap \bar{B}_\infty(0,R)$ is compact.

Hint: show that the image of A_{kR} by the mapping $f \to \frac{1}{kR} f$ is a subset of $C_1 \cap \bar{B}_\infty(0,1)$. Show that this image is compact and prove that A_{kR} is the image of this compact set by a continuous function.

Exercise 20 Let $\mathcal{C}_L([0,1])$ be the space of Lipschitz continuous functions on $[0,1]$ and $\mathcal{C}_k([0,1])$ be the set of Lipschitz continuous functions on $[0,1]$ of rank less or equal to k.

For each $f \in \mathcal{C}_L([0,1])$, we let

$$K(f) = \inf\{k \geq 0 \mid \forall (t,t') \in [0,1] \times [0,1], |f(t) - f(t')| \leq k|t - t'|\}$$

1) Show that $K(f)$ is well defined meaning that it is a non negative real number. Show that $K(f) = 0$ if and only if f is constant over $[0,1]$. Show that for all $(t,t') \in [0,1] \times [0,1]$, $|f(t) - f(t')| \leq K(f)|t - t'|$ and $|f(t)| \leq |f(0)| + K(f)t$.

We now define the function N from $\mathcal{C}_L([0,1])$ to $\mathbb{R}_+$ by $N(f) = |f(0)| + K(f)$.
2) Show that N is a norm on $\mathcal{C}_L([0,1])$.
3) Show that $\mathcal{C}_k([0,1])$ is a closed subset of $\mathcal{C}_L([0,1])$ for the norm N.
4) Show that for all $f \in \mathcal{C}_k([0,1])$, $\|f\|_\infty \leq N(f)$. Deduce that if (f_ν) is a converging sequence of $\mathcal{C}_L([0,1])$ for the norm N, then it is a converging sequence for the norm $\|\cdot\|_\infty$. Show that the function $f \to \|f\|_\infty$ is a Lipschitz continuous function on $\mathcal{C}_L([0,1])$ for the norm N.
5) Let $\bar{B}_N(0,1)$ be the closed unit ball of $\mathcal{C}_L([0,1])$ for the norm N. Show that $\bar{B}_N(0,1)$ is a closed subset of $\bar{B}_\infty(0,1)$, the closed unit ball of $\mathcal{C}([0,1])$ for the norm $\|\cdot\|_\infty$. Using Exercise 18, show that $\bar{B}_N(0,1)$ is compact for the norm $\|\cdot\|_\infty$. Is $\bar{B}_N(0,1)$ compact for the norm N?

Exercise 21 The purpose of this exercise is to show that $\mathcal{C}_L([0,1])$ with the norm N is a complete space.

Let (f_ν) be a sequence of $\mathcal{C}_L([0,1])$ satisfying the Cauchy criterion for the norm N.

1) Show that (f_ν) satisfies the Cauchy criterion for the norm $\|\cdot\|_\infty$ and deduce that the sequence (f_ν) converges to a limit $\bar{f}$ in $\mathcal{C}([0,1])$ for the norm $\|\cdot\|_\infty$.
2) Show that the sequence $(f_\nu(0))$ converges to $\bar{f}(0)$.
3) Show that $(N(f^\nu))$ is a Cauchy sequence and converges to a limit denoted $\bar{k}$.
4) Using the fact that for all $(t,t') \in [0,1]\times[0,1]$, $|\bar{f}(t)-\bar{f}(t')| = \lim_{\nu\to\infty} |f_\nu(t) - f_\nu(t')|$, show that f is $\bar{k}$ Lipschitz continuous.
5) Using the fact that (f_ν) satisfies the Cauchy Criterion, show that for all $r > 0$, there exists $\underline{\nu} \in \mathbb{N}$, such that for all $\nu \geq \underline{\nu}$, for all $(t,t') \in [0,1]\times[0,1]$, $|f_\nu(t) - \bar{f}(t) - (f_\nu(t') - \bar{f}(t'))| \leq r|t-t'|$ and deduces that $\lim_{\nu\to\infty} N(f_\nu - \bar{f}) = 0$.
6) Conclude that the sequence (f_ν) converges to a limit $\bar{f}$ in $\mathcal{C}_L([0,1])$ for the norm N and that $\mathcal{C}_L([0,1])$ with the norm N is a complete space.

We now study the continuity of the linear mappings. Indeed, all linear mappings between finite dimensional spaces are continuous but a linear mapping from E to F may not be continuous when E is infinite dimensional. Even more, there always exists a non-continuous linear mapping from E to $\mathbb{R}$ when E is infinite dimensional. Let us consider the following example. Let $\mathcal{C}([0,1],\mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. Let N_1 be the norm defined by $N_1(f) = \int_0^1 |f(t)|dt$. Then the linear mapping $f \to f(0)$ is not continuous. Indeed, let us consider the sequence of functions (f_ν) defined by $f_\nu(t) = 1-(\nu+1)t$ for $t \in [0, \frac{1}{\nu+1}]$ and $f_\nu(t) = 0$ otherwise. Then, the real sequence $\left(N_1(f_\nu) = \frac{1}{2(\nu+1)}\right)$ converges to 0, which means that the sequence (f_ν) converges to the null function. But the sequence $(f_\nu(0) = 1)$ does not converge to 0.

A key property of linear mappings is the fact that they are continuous if and only if they are Lipschitz continuous.

Theorem 8 *Let (E, N_E) and (F, N_F) two normed linear spaces. Let f be a linear mapping from E to F. Then f is continuous if and only if it is Lispchitz continuous.*

Proof. We only prove that f is Lipschitz continuous if f is continuous. Since f is continuous, $f^{-1}(B_F(0_F,1))$ is an open subset of E containing 0_E. So, there exists $r>0$ such that $B_E(0_E,r) \subset f^{-1}(B_F(0_F,1))$. Hence, for all $u \in E$, $u \neq 0_E$, $\frac{r}{2N_E(u)}u \in B_E(0_E,r)$, so $f\left(\frac{r}{2N_E(u)}u\right) \in B_F(0_F,1)$, that is $N_F\left(f(\frac{r}{2N_E(u)}u\right) < 1$, which implies $N_F(f(u)) \leq \frac{2}{r}N_E(u)$, so f is Lipschitz continuous with a rank $\frac{2}{r}$. □

1.7.1 Norm on the space of continuous linear mappings

As we did in finite dimensional spaces, we can define a norm on the set of continuous linear mapping from (E, N_E) and (F, N_F) denoted $\mathcal{L}(E,F)$ as follows. Let

f be a continuous linear mapping from E to F:

$$N_{\mathcal{L}}(f) = \sup\{N_F(f(x)) \mid x \in \bar{B}_E(0,1)\}$$

$N_{\mathcal{L}}$ is well defined since f is Lipschitz continuous.

We leave the reader checks that $N_{\mathcal{L}}$ is a norm. We just provides two useful properties of this norm, which are the same as the ones proved for finite dimensional spaces.

Proposition 29 *Let f be a continuous linear mapping from E to F. For all $u \in E$, $N_F(f(u)) \leq N_{\mathcal{L}}(f)N_E(x)$.*

Proposition 30 *Let f be a continuous linear mapping from E to F and g be a continuous linear mapping from F to a normed linear space G. Then, $N_{\mathcal{L}}(g \circ f) \leq N_{\mathcal{L}(E,F)}(f)N_{\mathcal{L}(F,G)}(g)$.*

Exercise 22 Let E be a linear space and (F, N_F) and (G, N_G) be two linear subspaces of E with a norm. We assume that $E = F \oplus G$, that is, for all $x \in E$, there exists a unique element $(y, z) \in F \times G$, such that $x = y + z$. Then, we define the mapping N from E to $\mathbb{R}_+$ by $N(x) = N_F(y) + N_G(z)$.
1) Show that N is a norm on E.
2) Show that F and G are closed in E for the norm N.
3) Let (Γ, N_Γ) be a normed linear space. Show that a linear mapping f from E to Γ is continuous if and only if the restrictions of f to F and G are continuous.

Exercise 23 Let E and F be two normed linear spaces and $f \in \mathcal{L}(E, F)$. We assume that f is regular and $f^{-1} \in \mathcal{L}(F, E)$. Show that $\|f^{-1}\|_{\mathcal{L}(F,E)} \geq \frac{1}{\|f\|_{\mathcal{L}(E,F)}}$.

Exercise 24 Let (E, N_E), (F, N_F) and (G, N_G) be 3 normed linear spaces and φ from $E \times F$ to G be a bilinear mapping, that is, a mapping satisfying: for all $((x, y), (x', y'), t) \in (E \times F) \times (E \times F) \times \mathbb{R}$,

1) $\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$;

2) $\varphi(tx, y) = t\varphi(x, y)$;

3) $\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$;

4) $\varphi(x, ty) = t\varphi(x, y)$.

1) Show that $\varphi(0_E, y) = \varphi(x, 0_F) = 0_G$ for all $(x, y) \in E \times F$.
2) Show that φ is continuous for the norm $N_{E\times F} = \max\{N_E(x), N_F(y)\}$ if and only if it exists a constant $k \geq 0$ such that for all $(x, y) \in E \times F$, $N_G(\varphi(x, y)) \leq kN_E(x)N_F(y)$. Hint: adapt the proof showing that a continuous linear mapping is Lipschitz continuous.

Exercise 25 Let f be a continuous linear mapping from (E, N_E) to (F, N_F). Show that the kernel of f is a closed linear subspace of E.

Let $E = \ell^1$ and $F = \ell^\infty$ with their respective norms. Note that $E \subset F$. Let f be the linear mapping from E to F defined by $f(u) = u$.
1) Show that f is continuous.
1) What is the kernel of f?
2) Show that the range of f is not closed.

Exercise 26 We consider the space $\mathcal{C}^1([0,1])$ of $\mathcal{C}^1$ functions on $[0,1]$ with the uniform norm $\|\cdot\|_\infty$. Let Φ be the derivation operator from $\mathcal{C}^1([0,1])$ to $\mathcal{C}([0,1])$ defined by $\Phi(f) = f'$.
1) Show that Φ is a linear mapping.
2) Show that Φ is not continuous if $\mathcal{C}([0,1])$ is also equipped with the uniform norm $\|\cdot\|_\infty$. Hint: consider the sequence $f_\nu(t) = \frac{1}{\nu+1}\sin(2\pi\nu t)$.

Exercise 27 We consider the space $\mathcal{C}([0,1])$ of continuous functions on $[0,1]$ with the uniform norm $\|\cdot\|_\infty$ and the norm $\|f\|_1 = \int_0^1 |f(t)|dt$.
1) Show that the function φ defined by $f \to f(0)$ is a linear function from $\mathcal{C}([0,1])$ to $\mathbb{R}$.
2) Show that φ is continuous for the norm $\|\cdot\|_\infty$.
3) Show that φ is not continuous for the norm $\|\cdot\|_1$. Hint: consider the sequence (f_ν) defined by $f_\nu(t) = \begin{cases} 1 - (\nu+1)t \text{ if } t \in [0, \frac{1}{\nu+1}] \\ 0 \text{ otherwise.} \end{cases}$

1.7.2 On the continuity of convex functions

We just provide two complementary results concerning the continuity of the convex functions and the separation Theorem. The first result has a similar proof than the one in finite dimension except that we do not have to prove that the function is locally upper bounded since this is an assumption. Be careful of the fact that a convex function may be non continuous on the interior of its domain. This is also true for a linear function defined everywhere.

Proposition 31 *Let (E, N) be a normed linear space and f be a convex function from a convex subset U of E to $\mathbb{R}$. Then, f is continuous and even locally Lipschitz on the interior of U if it exists $\bar{x} \in \text{int}U$, $r > 0$ and $m \in \mathbb{R}$ such that for all $x \in B(\bar{x}, r)$, $f(x) \leq m$.*

1.7.3 Separation theorems in normed linear spaces

The second result extend the separation Theorem. It is a consequence of the Hahn-Banach Theorem, the proof of which requires sophisticated argument. Note that the interiority condition is necessary and be careful not to apply this theorem without checking this condition.

Theorem 9 *Hahn-Banach Let (E, N) be a normed linear space and C and D two disjoints convex subsets of E. If the interior of C is nonempty, then it exists a continuous non-zero linear form φ on E such that*

$$\sup\{\varphi(c) \mid c \in C\} \leq \inf\{\varphi(d) \mid d \in D\}$$

The geometric interpretation of this result is the fact that an hyperplan separates weakly C and D. We can extend this theorem with a strict separation when C is compact and D closed as follows.

Corollary 2 *Hahn-Banach Let (E, N) be a normed linear space and C and D two disjoints convex subsets of E. If the C is compact and D is closed, then it exists a non zero continuous linear form φ on E such that*

$$\sup\{\varphi(c) \mid c \in C\} < \inf\{\varphi(d) \mid d \in D\}$$

Proof. We consider the following minimisation problem the distance $\inf\{d(c, D) \mid c \in C\}$. Since this function is continuous (See Exercice 10) and C is compact, this problem has a solution and so, there exists $\underline{c} \in C$ such that $d(\underline{c}, D) \leq d(c, D)$ for all $c \in C$. Since $C \cap D = \emptyset$, $\underline{c} \notin D$ and $d(\underline{c}, D) > 0$ since D is closed (See Exercice 10). Let $r \in]0, d(\underline{c}, D)[$. Let $\tilde{C} = C + B_E(0, r)$. One checks that $\tilde{C}$ is open and $\tilde{C} \cap D = \emptyset$. One applies the previous separation theorem to $\tilde{C}$ and D and we conclude by showing that $\sup\{\varphi(c) \mid c \in C\} < \sup\{\varphi(c) \mid c \in \tilde{C}\}$. Indeed, since φ is not equal to 0, there exists $u \in B_E(0, r)$ such that $\varphi(u) > 0$. From the definition of the supremum, there exists $c \in C$ such that $\varphi(c) > \sup\{\varphi(c) \mid c \in C\} - (\varphi(u)/2)$. Then $c + u \in \tilde{C}$ and $\sup\{\varphi(c) \mid c \in \tilde{C}\} > \varphi(c + u) > \sup\{\varphi(c) \mid c \in C\} + (\varphi(u)/2) > \sup\{\varphi(c) \mid c \in C\}$. □

1.7.4 Examples of Banach spaces

We have already provided an example of a non-complete normed linear space in Exercise 5. The completeness is key property to get the existence of solution for example using the Banach fixed point theorem. A complete norm linear space is usually called a Banach space. So, we now provide the most usual Banach spaces encountered in the applications.

We first provide a general way to build a Banach space. Let (E, N_E) be a normed linear space and (F, N_F) be a Banach space. Then the space of continuous linear mapping from E to F, $\mathcal{L}(E, F)$ is a Banach space with the norm $N_{\mathcal{L}}(E, F)$ defined above. In particular, the set $\mathcal{L}(E, \mathbb{R})$ of continuous linear forms on E is a Banach space.

We now present the norm of the sequence spaces. For $p \in [1, \infty[$, let ℓ^p be the space of real sequences (u_ν) such that the series $(|u_\nu|^p)$ is converging. Then, ℓ^p with the norm $\|(u_\nu)\|_p = (\sum_{\nu=0}^{\infty} |u_\nu|^p)^{1/p}$ is a Banach space.

Let ℓ^∞ be the space of real bounded sequences (u_ν). Then, ℓ^∞ with the norm $\|(u_\nu)\|_\infty = \sup\{|u_\nu| \mid \nu \in \mathbb{N}\}$ is a Banach space.

We can generalise the above examples to sequences in $\mathbb{R}^n$ or to sequences in a Banach space.

Note that the set $\ell_0 = \{(u_\nu) \mid \text{card}\{\nu \mid u_\nu \neq 0\} < \infty\}$ which is a subset of ℓ^p for all $p \in [1, \infty]$ is not a Banach space for the associated norm.

Exercise 28 Prove that ℓ_0 is dense in the ℓ^p spaces for $p \in [1, \infty[$, that is, $\overline{\ell_0} = \ell^p$ when the closure is taken for the associated ℓ^p norms.

Prove that ℓ_0 is not dense in the ℓ^∞

We now present some functional spaces. Let X be a set. Then the space of bounded function from X to $\mathbb{R}$, $\mathcal{B}(X, \mathbb{R})$, is a Banach space for the norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}$. Note that we can generalise this example for bounded mappings to a Banach space.

If (X, d) is a metric space, the space of continuous bounded function from X to $\mathbb{R}$, $\mathcal{C}_b(X, \mathbb{R})$ is a Banach space for the same norm. If X is furthermore compact, then this space is actually the space $\mathcal{C}(X, \mathbb{R})$ of continuous functions from X to $\mathbb{R}$.

Exercise 29 The aim of this exercise is to prove the space $\mathcal{C}([0, 1], \mathbb{R})$ with the norm $\|f\|_1 = \int_0^1 |f(t)|dt$ is not a Banach space.

Let us consider the sequence (f_ν) defined by $f_\nu(t) = 0$ for $t \in [0, \frac{1}{2} - \frac{1}{3(\nu+1)}]$, $f_\nu(t) = \frac{3(\nu+1)}{2}t + \frac{1}{2} - \frac{3(\nu+1)}{4}$ for $t \in [\frac{1}{2} - \frac{1}{3(\nu+1)}, \frac{1}{2} + \frac{1}{3(\nu+1)}]$ and $f_\nu(t) = 1$ for $t \in [\frac{1}{2} + \frac{1}{3(\nu+1)}, 1]$.

1) Show that this sequence satisfies the Cauchy Criterion for the norm $\|\cdot\|_1$. For $\nu < \mu$, note that

$$\|f_\nu - f_\mu\|_1 = \int_{\frac{1}{2} - \frac{1}{3(\nu+1)}}^{\frac{1}{2} + \frac{1}{3(\nu+1)}} |f_\nu(t) - f_\mu(t)|dt$$

Assume that this sequence has a limit $\bar{f}$ in $\mathcal{C}([0, 1])$.

2) Show that for all ν,

$$\|\bar{f} - f_\nu\|_1 = \int_0^{\frac{1}{2} - \frac{1}{3(\nu+1)}} |\bar{f}(t)|dt + \int_{\frac{1}{2} - \frac{1}{3(\nu+1)}}^{\frac{1}{2} + \frac{1}{3(\nu+1)}} |\bar{f}(t) - f_\nu(t)|dt + \int_{\frac{1}{2} + \frac{1}{3(\nu+1)}}^{1} |\bar{f}(t) - 1|dt$$

3) Deduce from the previous question that for all $r \in]0, 1/2[$, $\int_0^{\frac{1}{2} - r} |\bar{f}(t)|dt$ and $\int_{\frac{1}{2} + r}^{1} |\bar{f}(t) - 1|dt$ are equal to 0.

4) Deduce from the previous question that $\bar{f}(t) = 0$ on $[0, 1/2[$ and $\bar{f}(t) = 1$ on $]1/2, 1]$.

4) Show that we get a contradiction.

Now, for the applications in probability and statistics, we consider a measure space (X, Ω, μ) with a sigma-algebra Ω and a positive measure μ. Among the set of measurable functions from X to $\mathbb{R}$, we consider for $p \in [1, \infty[$, the space

$L^p(X, \Omega, \mu)$ of measurable functions f such that $|f|^p$ is integrable and we define the norm $\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p}$. Then, we need to assimilate functions which are equal almost everywhere since they are not distinguishable by the integration process. So, up to this operation, the $L^p(X, \Omega, \mu)$ spaces are Banach spaces for the norm $\| \cdot \|_p$.

The space $L^\infty(X, \Omega, \mu)$ is the space of essentially bounded measurable functions from X to $\mathbb{R}$, that is a function f such that for some $m \in \mathbb{R}_+$, the set $\{x \in X \mid |f(x)| > m\}$ is of measure 0. We then define the essential norm as follows:

$$\|f\|_\infty = \inf\{m \geq 0 \mid \mu(\{x \in X \mid |f(x)| > m\}) = 0\}$$

Then $L^\infty(X, \Omega, \mu)$ is also a Banach space up to the assimilation of functions equal almost everywhere.

Exercise 30 We consider the two Banach spaces ℓ^1 and ℓ^∞ with their respective norms. For every $v \in \ell^\infty$, we define the mapping φ_v from ℓ^1 to $\mathbb{R}$ by $\varphi_v(u) = \sum_{\nu=0}^{\infty} v_\nu u_\nu$.
1) Show that the function φ_v is well defined that is the series $(v_\nu u_\nu)$ is convergent for all $u \in \ell^1$.
2) Show that the function φ_v is linear.
3) Show that the function φ_v is Lipschitz continuous of rank $\|v\|_\infty$.
4) For all $r > 0$ show that there exists $u \in \ell^1$ such that $\|u\|_1 = 1$ and $\varphi_v(u) \geq \|v\|_\infty - r$. Hint: consider the elements ϵ^i in ℓ^1 defined by $\epsilon^i_\nu = 0$ if $\nu \neq i$ and $\epsilon^i_i = 1$.

Let ψ be a continuous linear mapping from ℓ^1 to $\mathbb{R}$. We consider the sequence w defined by $w_\nu = \psi(\epsilon^\nu)$.
5) Show that $w \in \ell^\infty$.
6) Show that $\psi = \varphi_w$, that is, for all $u \in \ell^1$, $\psi(u) = \sum_{\nu=0}^{\infty} w_\nu u_\nu$. Hint: consider first the sequences with a finite numbers of non zero terms and then use a limit argument.

Exercise 31 We consider the two Banach spaces ℓ^1 and ℓ^∞ with their respective norms. For every $v \in \ell^1$, we define the mapping ϕ_v from ℓ^∞ to $\mathbb{R}$ by $\phi_v(u) = \sum_{\nu=0}^{\infty} v_\nu u_\nu$.
1) Show that the function ϕ_v is well defined that is the series $(v_\nu u_\nu)$ is convergent for all $u \in \ell^\infty$.
2) Show that the function ϕ_v is linear.
3) Show that the function ϕ_v is Lipschitz continuous of rank $\|v\|_1$.
4) Show that there exists $u \in \ell^\infty$ such that $\|u\|_\infty = 1$ and $\phi_v(u) = \|v\|_1$. Hint: all components of u are equal to 1 or -1 depending on the sign of v_ν.

Exercise 32 We consider the two Banach spaces ℓ^1 and ℓ^∞ with their respective norms. We define the positive cone of this spaces as the set of sequences with non-negative terms. We denote these cones ℓ^1_+ and ℓ^∞_+.
1) Show that these cones are closed.

2) Show that the interior of ℓ^1_+ is empty.
3) Show that the interior of ℓ^∞_+ is nonempty and show that ℓ^∞_+ is the closure of its interior.

Let us now consider the norm $N(u) = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu}|u_\nu|$ on ℓ^∞.
4) Show that the interior for N of ℓ^∞_+ is empty.

Exercise 33 We consider the Banach space ℓ^∞.
1) Show that the closed unit ball $\bar{B}_\infty(0,1) = \prod_{\nu=0}^{\infty}[-1,1]$.

We define the distance d on $\bar{B}_\infty(0,1)$ as follows: $d(u,u') = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu}|u_\nu - u'_\nu|$.
2) Using a result of the course, show that the metric space $(\bar{B}_\infty(0,1), d)$ is compact.
3) Let (v^i) be the sequence of $\bar{B}_\infty(0,1)$ defined by $v^i = \epsilon^i$ where $\epsilon^i_\nu = 0$ if $\nu \neq i$ and $\epsilon^i_i = 1$. Show that this sequence is converging for d and give the limit.
4) Let (v^i) be the sequence of $\bar{B}_\infty(0,1)$ defined by $v^i = \sum_{k=0}^{i} \epsilon^k$. Show that this sequence is converging for d and give the limit.

Let $v \in \ell^1$ and ϕ_v from $\bar{B}_\infty(0,1)$ to $\mathbb{R}$ defined by $\phi_v(u) = \sum_{\nu=0}^{\infty} v_\nu u_\nu$.
5) Show that this function is continuous for the distance d.
6) Solve the maximisation problem: $\max\{\phi_v(u) \mid u \in \bar{B}_\infty(0,1)\}$.

1.8 Series in Banach spaces

Thanks to the Cauchy criterion, we can extend the notion of series in Banach spaces.

Let (u_ν) be a sequence in a Banach space $(E, \|\cdot\|)$. The series associated to (u_ν) is the sequence (σ_ν) defined by $\sigma_\nu = \sum_{k=0}^{\nu} u_k$.

Definition 17 The series associated to (u_ν) (or, in short, the series (u_ν)) is convergent if the sequence (σ_ν) defined by $\sigma_\nu = \sum_{k=0}^{\nu} u_k$ is convergent.

The series associated to (u_ν) is absolutely convergent if the real sequence $(\sum_{k=0}^{\nu} \|u_k\|)$ is convergent.

Remark 16 One easily shows (exercise) that if the series (u_ν) is convergent, then the sequence (u_ν) converges to 0. The converse is not true.

Using the Cauchy criterion of convergence, one has the fundamental following result.

Proposition 32 *If the series associated to (u_ν) is absolutely convergent, then the series associated to (u_ν) is convergent.*

Since the series associated to a non-negative sequence is increasing, we get the simple convergence criteria.

Proposition 33 *The series associated to the sequence (u_ν) is absolutely convergent if and only if the sequence $(\sum_{k=0}^{\nu} \|u_k\|)$ is bounded above.*

Exercise 34 Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E,E)$ be the Banach space of the continuous linear mapping from E to E with the norm $\|\cdot\|_{\mathcal{L}}$.

Let $f \in \mathcal{L}(E,E)$ such that $\|f\|_{\mathcal{L}} < 1$. We let $f^0 = \mathrm{Id}_E$ and, for all interger $\nu \geq 1$, f^ν is equal to $f \circ f^{\nu-1}$.

1) Show that the series (f^ν) is absolutely convergent in $\mathcal{L}(E,E)$. Hint: show first that $\|f^\nu\|_{\mathcal{L}} \leq (\|f\|_{\mathcal{L}})^\nu$.

Let $\varphi_\nu = \sum_{k=0}^{\nu} f^k$ and $\varphi = \sum_{k=0}^{\infty} f^k$

2) Compute $(\mathrm{Id} - f) \circ \varphi_\nu$ and deduce its limit when ν tends to ∞.

3) Show that $\mathrm{Id} - f$ is regular and that $(\mathrm{Id} - f)^{-1}$ belongs to $\mathcal{L}(E,E)$.

4) The purpose of this question is to show that if $f \in \mathcal{L}(E,E)$ is regular and $f^{-1} \in \mathcal{L}(E,E)$, then for all $g \in B_{\mathcal{L}}(f, \frac{1}{\|f^{-1}\|_{\mathcal{L}}})$, $f + g$ is regular and its inverse belongs to $\mathcal{L}(E,E)$.

a) Let $g \in B_{\mathcal{L}}(f, \frac{1}{\|f^{-1}\|_{\mathcal{L}}})$. Using the inequalities satisfied by $\|\cdot\|_{\mathcal{L}}$, show that $\|f^{-1} \circ g\|_{\mathcal{L}} < 1$.

b) Using the previous question, show that $\mathrm{Id}_E + f^{-1} \circ g$ is regular and its inverse belongs to $\mathcal{L}(E,E)$.

c) Conclude.

Chapter 2

Dynamical programming

2.1 Introduction

In this chapter, we study optimisation over time, which plays a key role in many economic and financial models. The explicit mention of time in the model provides additional structure but, when we deal with infinite horizon models, the solutions belong to infinite dimensional spaces, which requires more sophisticated mathematical tools.

In this first presentation, we will mainly consider the discrete time dynamical optimisation with finite horizon and then with an infinite horizon. The main results deal with the existence of solutions, necessary and sufficient conditions to characterise the solutions and the study of the value function. We use a lot of material already presented in the previous chapters.

For the continuous time optimisation, we will just state the problem and the optimality conditions without any proofs with some examples. Indeed, a complete treatment requires a course on differential equations, integration and differential calculus in infinite dimensional spaces which are beyond the scope of this course.

Let us now start with a very basic example of the intertemporal allocation of wealth. Let us assume that a consumer has an endowments $w_0 > 0$ of a commodity and she wants to share her consumption over today and tomorrow. Her instantaneous utility is represented by a concave utility function u from $\mathbb{R}_+$ to $\mathbb{R}$. A parameter β, called a discount factor, measures her preference for the present. She has also the opportunity to borrow or lend at an interest rate r between today and tomorrow. So, the optimal allocation is a solution of the following optimisation problem:

$$\begin{cases} \text{Maximise } u(c_1) + \beta u(c_2) \\ c_2 = (1+r)(w_0 - c_1) \\ c_1 \geq 0,\, c_2 \geq 0 \end{cases}$$

If we assume that we have an interior solution ($c_1^* > 0$, $c_2^* > 0$), the first order necessary condition are:

$$\begin{cases} u'(c_1^*) = \lambda(1+r) \\ \beta u'(c_2^*) = \lambda \end{cases}$$

If we assume that the function c is strictly increasing with a positive derivative, we can eliminate the multiplier λ and we get

$$u'(c_1^*) = \beta u'(c_2^*)(1+r)$$

Using the Lagrangian of this problem, we can check that this first order condition is sufficient. We can interpret this condition economically by saying that the intertemporal marginal rate of substitution $\frac{u'(c_1^*)}{\beta u'(c_2^*)}$ is equal to $1+r$, which is the return of 1 unit saved in the first period.

Since u is concave, u' is decreasing, so we can check that $c_1^* > c_2^*$ if and only if $(1+r)\beta < 1$. So, the consumption decreases over the two periods if the discount factor representing the time preferences or the impatience is small enough with respect to the interest rate.

Exercise 35 Compute the optimal allocation in the above problem when $u(c) = \sqrt{c}$ and $u(c) = \ln(c)$.

We can easily extend this problem to T periods. The initial endowments w_0 is given. We get the following problem of maximising the intertemporal welfare:

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t u(c_t) \\ w_t = (1+r)(w_{t-1} - c_{t-1}), \text{ for } t = 1, \dots, T \\ w_T \geq 0 \\ c_t \geq 0, \text{ for } t = 1, \dots, T-1 \end{cases}$$

Due to the homogeneous formulation of the constraints, we need to add a terminal condition $w_T \geq 0$. Otherwise, without this constraint, the agent can borrow as many quantities of the consumption good as she wants and never reimburse her debt. So, the problem has no solution and is not economically relevant.

Assuming that the positivity constraints on the consumptions are not binding, interior solution, the first order necessary conditions are as follows:

$$\begin{cases} u'(c_0^*) - \lambda_1(1+r) = 0 \\ \beta^t u'(c_t^*) - \lambda_{t+1}(1+r) = 0 \text{ for } t = 1, \dots, T-1 \\ (1+r)\lambda_{t+1} - \lambda_t = 0 \text{ for } t = 1, \dots, T-1 \\ \lambda_T \geq 0. \ \lambda_T w_T = 0 \end{cases}$$

Note that if the terminal constraint is not binding, that is $w_T > 0$, then all multipliers are equal to 0 and c_t^* is constant over time given by $u'(c_t^*) = 0$. So this is possible only if u has a global maximum on $\mathbb{R}_+$. In this case, the initial endowments is sufficiently large so that the economic agent can consume her optimal consumption at each period. Then the intertemporal problem is reduced to T independent one period identical optimisation problems.

This first order conditions are sufficient since we have a concave objective functions with linear constraints.

We remark that the multipliers are given by a backward equation, which is a general characteristic.

Remark 17 If λ_T is known, then we compute easily the other multipliers using the backward equation $\lambda_t = (1+r)\lambda_{t+1}$. Then, we compute the optimal consumption c_t^* using the inverse of the derivative of u.

We can eliminate the multipliers and we find the following equation for all $t = 0, \ldots, T-2$:

$$u'(c_t^*) = \beta(1+r)u'(c_{t+1}^*)$$

We remark that this is the same as in the two period model. So, for example, c_t^* is decreasing over time if $(1+r)\beta < 1$.

We can interpret the multiplier λ_t, which is equal to $\beta^{t+1}u'(c_{t+1}^*)$, as the shadow price of the wealth w_{t+1} at the period t. Indeed, at the period t, the consumer could increase her consumption c_t^* of one unit, which provides an instantaneous gain of $\beta^t u'(c_t^*)$ but decreases the wealth w_{t+1} of an amount equal to $(1+r)$ and so decreases the future consumptions and the future welfare. So, at an optimal solution c_t^*, these two effects must be equal, otherwise, the consumer could increase her intertemporal welfare. This corresponds to the equality derived from the first order condition:

$$\beta^t u'(c_t^*) = \lambda_{t+1}(1+r) = \lambda_t$$

So, let us analyse the effect of a change of the wealth w_{t+1} on the future optimal welfare. The remaining problem starting at period $t+1$ is the following:

$$\begin{cases} \text{Maximise } \sum_{\tau=t+1}^{T-1} \beta^\tau u(c_\tau) \\ w_\tau = (1+r)(w_{\tau-1} - c_{\tau-1}), \text{ for } \tau = t+2, \ldots, T \\ w_T \geq 0 \\ c_\tau \geq 0, \text{ for } \tau = t+1, \ldots, T-1 \end{cases}$$

where w_{t+1} is given. Using the particular form of the equality constraints, we can eliminate the wealth variables and write this problem with a unique intertemporal budget constraint:

$$\begin{cases} \text{Maximise } \sum_{\tau=t+1}^{T-1} \beta^\tau u(c_\tau) \\ (1+r)^{T-t-1}c_{t+1} + (1+r)^{T-t-2}c_{t+2} + \ldots + (1+r)c_{T-1} \leq (1+r)^{T-t-1}w_{t+1} \\ c_\tau \geq 0, \text{ for } t = t+1, \ldots, T-1 \end{cases}$$

The inequality constraint is equivalent to the following one, where w_{t+1} appears only on the right hand side of the inequality

$$c_{t+1} + (1+r)^{-1}c_{t+2} + \ldots + (1+r)^{-(T-t-2)}c_{T-1} \leq w_{t+1}$$

In the sensitivity analysis, we have shown that the multiplier is equal to the derivative of the value function, noticing that the inequality constraint is binding with a non zero multiplier. Note also that $(c_{t+1}^*, \ldots, c_{T-1}^*)$ is a solution of this truncated problem.The multiplier μ of this problem satisfies for all $\tau = t+1, \ldots, T-1$,

$$\beta^\tau u'(c_\tau^*) = \mu(1+r)^{-(\tau-t-1)}$$

So, one get $\mu = \beta^{t+1} u'(c^*_{t+1}) = \lambda_t$. This is the effect on the global welfare of the increase of one unit of the wealth at the period $t+1$. So, we have formally justified the fact that λ_t is the shadow price of the wealth at period t. At each period, the agent chooses an optimal consumption by balancing the two opposite effects of an immediate gain by increasing the today consumption and a future loss by decreasing the remaining wealth for the future periods.

Remark 18 Note that some authors multiply the constraints by β^t and they get a different formulation of the optimality condition. Actually, this is fully equivalent, since it is just a positive rescaling of the multipliers by a power of β.

Exercise 36 Compute the optimal allocation in the T period problem when $u(c) = \sqrt{c}$ and $u(c) = \ln(c)$. Compute the derivative of the value function with respect to w_0 and check that it is equal to the multiplier λ_0.

2.2 The General model with a finite horizon

We now present the general model of a dynamical optimisation problem. We are are considering a discrete time model where the periods are denoted $t = 0, 1, \ldots, t, \ldots, T$, where T is the finite horizon. At each period, we have a state s_t and an action a_t, also called control. The initial state s_0 is given. The new state at the period $t+1$ is determined by a transition equation $s_{t+1} = g_t(a_t, s_t)$ depending on the current state and the action. At each period, the economic agent receives a payoff $f_t(a_t, s_t) \in \mathbb{R}$. It is computing her intertemporal payoff as the discount sum of the payoffs with a discounting factor β. Furthermore, the action - state pair is constrained to stay in a given set A_t.

So, the optimisation problem is to choose the actions $(a_0, \ldots, a_{T-1})$ in order to maximise the intertemporal payoff taken s_0 as given:

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T f_T(s_T) \\ s_{t+1} = g_t(a_t, s_t),\ t = 0, \ldots, T-1, \\ (a_t, s_t) \in A_t \ t = 0, \ldots, T-1 \\ s_T \in A_T \end{cases}$$

A_T represents a constraint on the final state, which can be for example a non-negativity constraint. In the general case, s_t and a_t belongs to finite dimensional vector spaces. For this first course, we assume that they belongs to $\mathbb{R}$. So A_t is a subset of $\mathbb{R}^2$ and A_T is a subset of $\mathbb{R}$. f_t and g_t are defined on subsets of $\mathbb{R}^2$ and f_T, the final payoff, is defined on $\mathbb{R}$.

Definition 18 For a given s_0, $U(s_0)$ is the subset of $\mathbb{R}^T \times \mathbb{R}^{T+1}$ of feasible finite sequences $\left((a_t)_{t=0}^{T-1}, (s_t)_{t=0}^{T}\right)$ satisfying the initial condition at s_0, the transition equations $s_{t+1} = g_t(a_t, s_t)$ and the constraints $(a_t, s_t) \in A_t$ for $t = 0, \ldots, T-1$ and $s_T \in A_T$.

We implicitly assume that $U(s_0)$ is nonempty, otherwise, the problem have no interest.

2.2.1 First order optimality conditions

Let us consider an optimal solutions (a_t^*) of the above problem. Let us denote by s_t^* the associated sequence of states given by the transition equations $s_{t+1}^* = g_t(a_t^*, s_t^*)$ for $t = 0, \ldots, T-1$ Let us assume that $(a_t^*, s_t^*) \in \text{int}A_t$ for all t and $s_T^* \in \text{int}A_T$, that is (a_t^*) is an *interior solution.* Then, applying the result stated in the previous chapter, we get the first order necessary conditions under the following assumptions:

Assumptions

1) for all $t = 0, \ldots, T-1$, f_t and g_t are $\mathcal{C}^1$ functions defined on open subsets of $\mathbb{R}^2$, f_T is a $\mathcal{C}^1$ function defined on an open interval of $\mathbb{R}$ such that for all $((a_t), (s_t)) \in U(s_0)$, (a_t, s_t) belongs to the domain of definition of f_t and g_t and s_T belongs to the domain of definition of f_T.

2) for all $t = 0, \ldots, T-1$,the partial derivative of g_t with respect to a is not equal to 0 on its domain of definition.

Proposition 34 *Under the above assumptions, at an interior solution, we get the following first order necessary conditions: there exists a vector of multipliers* $\lambda \in \mathbb{R}^T$ *such that*

$$\left\{ \begin{array}{l} \beta^t \frac{\partial f_t}{\partial a}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial a}(a_t^*, s_t^*) = 0, \quad t = 0, \ldots, T-1 \\ \beta^t \frac{\partial f_t}{\partial s}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial s}(a_t^*, s_t^*) = \lambda_t, \quad t = 1, \ldots, T-1 \\ f_T'(s_T^*) = \lambda_T \\ s_{t+1}^* = g_t(a_t^*, s_t^*). \ t = 0, \ldots, T-1 \end{array} \right.$$

Proof. We first check that the gradients of the equality constraints are linearly independent. We rewrite the equality constraints $s_{t+1} - g_t(a_t, s_t) = 0$. Then these gradients are given in the following matrix:

$$\begin{pmatrix} -\frac{\partial g_0}{\partial a_0} & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ -\frac{\partial g_0}{\partial s_0} & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\frac{\partial g_1}{\partial a_1} & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 1 & -\frac{\partial g_1}{\partial s_1} & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -\frac{\partial g_t}{\partial a_t} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & -\frac{\partial g_t}{\partial s_t} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & -\frac{\partial g_{T-1}}{\partial a_t} & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & -\frac{\partial g_{T-1}}{\partial s_t} & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Since $\frac{\partial g_t}{\partial a_t}(a_t^*, s_t^*)$ is non vanishing, the gradients are independent thanks to the particular temporal structure of the constraints.

Now, considering the fact that we have a maximisation problem, the first order are given by the following equalities:

$$\begin{cases} \frac{\partial f_0}{\partial a_0} = & -\lambda_1 \frac{\partial g_0}{\partial a_0} \\ \beta \frac{\partial f_1}{\partial a_1} = & -\lambda_2 \frac{\partial g_1}{\partial a_1} \\ \beta \frac{\partial f_1}{\partial s_1} = & -\lambda_2 \frac{\partial g_1}{\partial s_1} + \lambda_1 \\ \vdots & \vdots \\ \beta^t \frac{\partial f_t}{\partial a_t} = & -\lambda_{t+1} \frac{\partial g_t}{\partial a_t} \\ \beta^t \frac{\partial f_t}{\partial s_t} = & -\lambda_{t+1} \frac{\partial g_t}{\partial s_t} + \lambda_t \\ \vdots & \vdots \\ \beta^T \frac{\partial f_T}{\partial s_T} = & \lambda_T \end{cases}$$

from which one easily derives the given conditions. □.

Positivity constraints on the variables

In many problems, the state and the action variables are supposed to be non negative. These constraints are incorporated in the condition $(a_t, s_t) \in A_t$ as well as possible other constraints. So, we need to carefully check what happens if some of these constraints are binding at the optimal solution, because this requires additional multipliers and, so, more complex first ordre necessary conditions. We now provide the first order necessary conditions for the case where we have non negative constraints $a_t \geq 0$, $s_t \geq 0$ for all t without assuming that these constraints are not binding at the optimal solution.

Proposition 35 *Under the above assumptions, at a solution where the positivity constraints on the states and the actions may be binding, we get the following first order necessary conditions: there exists a vector of multipliers $\lambda \in \mathbb{R}^T$ such that*

$$\begin{cases} \beta^t \frac{\partial f_t}{\partial a}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial a}(a_t^*, s_t^*) \leq 0, \quad t = 0, \ldots, T-1 \\ \left(\beta^t \frac{\partial f_t}{\partial a}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial a}(a_t^*, s_t^*)\right) a_t^* = 0, \quad t = 0, \ldots, T-1 \\ \beta^t \frac{\partial f_t}{\partial s}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial s}(a_t^*, s_t^*) \leq \lambda_t, \quad t = 1, \ldots, T-1 \\ \left(\beta^t \frac{\partial f_t}{\partial s}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial s}(a_t^*, s_t^*) - \lambda_t\right) s_t^* = 0, \quad t = 1, \ldots, T-1 \\ f_T'(s_T^*) \leq \lambda_T \\ (f_T'(s_T^*) - \lambda_T)\, s_T^* = 0 \\ s_{t+1}^* = g_t(a_t^*, s_t^*). \ t = 0, \ldots, T-1 \end{cases}$$

Exercise 37 Write the complete first order necessary conditions when the sets A_t are defined as follows:

$$A_t = \{(a, s) \in \mathbb{R}^2 \mid s \geq 0, a \in [\underline{\alpha}_t(s), \overline{\alpha}_t(s)]\}$$

where $\underline{\alpha}_t$ and $\overline{\alpha}_t$ are continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$ satisfying $\underline{\alpha}_t(s) \leq \overline{\alpha}_t(s)$ for all $s \in \mathbb{R}_+$.

We now provide some additional assumptions on the functions f_t and g_t so that the first order necessary conditions are sufficient. Basically, they imply that the

Lagrangian of the problem is concave with respect to $((a_t), (s_t))$. The Lagrangian is given by the following formula:

$$\mathcal{L}((a_t), (s_t), (\lambda_t)) = \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T f_T(s_T) - \sum_{t=0}^{T-1} \lambda_{t+1}(s_{t+1} - g_t(a_t, s_t))$$

So, it is concave if all functions f_t, g_t are concave and the multipliers are non negative. According to the above first order condition with or without positivity constraints, this happens if the functions f_t, g_t are increasing with respect to the state variables. So, we get the following proposition.

Proposition 36 *We consider the previous dynamical optimisation problem. We maintain the same assumptions and we also assume that:*

1) For all $t = 0, \dots, T-1$, f_t and g_t are concave and f_T is concave;

2) For all $t = 0, \dots, T-1$, f_t and g_t are increasing with respect to the state variable s_t and f_T is increasing with respect to the state variable s_T.

Then, if $((a_t^*), (s_t^*))$ satisfies the first order condition given in the two previous propositions, it is a solution of the dynamical optimisation problem.

Remark 19 Note that the transition equations may be equivalently written $s_{t+1} = g_t(a_t, s_t)$ or $\beta^{t+1}(s_{t+1} - g_t(a_t, s_t)) = 0$. So, the first order necessary conditions have not exactly the same form but they are equivalent. Actually, this leads just to a renormalisation of the multipliers.

2.2.2 Examples

Optimal extraction rate We consider a mine with a stock Q_0 of ore. The mine will be closed after three years of activities. The price of the ore is normalised equal to 1. Q_t is the stock at the beginning of the period t, q_t is the quantity extracted at period t, the cost of extraction is given by q_t^2/Q_t.
1) Show that the problem to be solved is the following:

$$\left\{ \begin{array}{l} \text{Maximise } \sum_{t=0}^{2} q_t - \frac{q_t^2}{Q_t} \\ Q_{t+1} = Q_t - q_t,\ t = 0, 1, 2 \\ Q_3 \geq 0 \\ q_t \geq 0,\ t = 0, 1, 2 \end{array} \right.$$

2) Write the first order necessary condition. Are they sufficient?
3) Compute the optimal solution with $Q_0 = 128$.

Ramsay Optimal growth model
With an initial capital stock $k_0 > 0$, at each period, the agent shares the quantity of capital produced $F(k_t)$, where F is the production function, between

a part c_t devoted to the consumption and a part k_{t+1} devoted to the production. Then, the agent maximises an intertemporal welfare which is represented as a discounted sum of instantaneous utility level $u(c_t)$. We put the following assumptions:

a) $\beta \in]0,1[$;

b) u from $\mathbb{R}_+$ to $\mathbb{R}_+$ is differentiable on $\mathbb{R}_+^*$, strictly concave and strictly increasing with a positive derivative, $\lim_{t\to 0^+} u'(t) = +\infty$ and $u(0) = 0$;

c) f from $\mathbb{R}_+$ to $\mathbb{R}_+$ is differentiable on $\mathbb{R}_+^*$, strictly concave and strictly increasing with a positive derivative, $\lim_{t\to +\infty} f'(t) < 1$, $\lim_{t\to 0^+} f'(t)$ is finite or $+\infty$ and $f(0) = 0$;

The maximisation problem is the following:

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t u(c_t) \\ k_{t+1} = F(k_t) - c_t,\ t = 0, 1, \dots, T-1 \\ k_T \geq 0 \\ c_t \geq 0,\ k_t \geq 0,\ t = 0, 1, \dots, T-1 \end{cases}$$

1) Write the first order necessary condition.
2) Show that the constraint $k_T \geq 0$ is binding.
3) Show that the optimal solution satisfies the Euler's equation

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = F'(k_{t+1}^*)$$

We assume that $\lim_{t\to 0^+} f'(t) > 1$.
4) Show that there exists $\bar{k} > 0$ such that $f(\bar{k}) = \bar{k}$, $f(k) > k$ if $k < \bar{k}$ and $f(k) < k$ if $k > \bar{k}$.
5) Show that if $k_0 < \bar{k}$, then the optimal solution (k_t^*) is decreasing.

Problem of allocation in hotels

This example shows that the index t may represent another variable than time. We consider a travel agency which must distribute travelers in 4 hotels trying to minimise the cost. The cost in each hotel depends on the number of rooms booked in this hotel according to the following table:

Hotels	$\to$	1	2	3	4
Travelers	8	100	80	80	80
⋮	7	80	71	80	70
	6	60	64	72	60
	5	53	55	60	50
	4	48	39	44	40
	3	40	32	11	30
	2	20	20	10	20
	1	8	13	9	10
	0	0	0	0	0

To find the best allocation of 6 travelers, we consider the number of hotels used to accommodate travelers as time. We apply the recursive principle saying that if n travelers are optimally accommodated in the fourth hotel, then the $6-n$ other travelers are accommodated in the three first hotels optimally. So we first compute the optimal cost of allocating 0 to 6 travelers in the two first hotels. Then we compute the optimal cost of allocating 0 to 6 travelers in the three first hotels. Finally, we compute the optimal cost of allocating 6 travelers in the four hotels. Give the optimal cost and the optimal allocation. Show that we can change the order of the hotels without changing the optimal result.

Exhaustible resources

We consider a monopolist extracting an ore at a constant marginal cost c. It faces an inverse demand function $p_t(q_t)$. Q_0 is the initial stock, q_t is the quantity extracted at period t and Q_t is the stock at the beginning of period t. The objective of the monopolist is to maximise the intertemporal profit. The optimisation problem is the following:

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t (p_t(q_t) - c) q_t \\ Q_{t+1} = Q_t - q_t,\ t = 0, 1, \ldots, T-1 \\ Q_T \geq 0 \\ q_t \geq 0,\ Q_t \geq 0,\ t = 0, 1, \ldots, T-1 \end{cases}$$

We assume that p_t is strictly decreasing and differentiable, the instantaneous profit $(p_t(q_t)-c)q_t$ is a concave function of q_t and $p_t(0) > c$ for all $t = 0, \ldots, T-1$.
1) Write the first order necessary condition.
2) Show that if the terminal constraint is not binding, then all multipliers are equal to 0 and the problem is a collection of T independent one dimensional problems. Show that the solution is characterised by $p_t(q_t^*) + p_t'(q_t^*)q_t^* - c = 0$ for all t.

We now consider the case where the terminal constraint is binding: $Q_T^* = 0$.
3) Show that Q_t^* is decreasing and there exists τ such that $Q_0^* = Q_0 > 0$, ..., $Q_\tau^* > 0$, $Q_{\tau+1}^* = 0$, ..., $Q_T^* = 0$.
4) Let us assume that $q_t^* > 0$ for $t = 0, \ldots \tau$. Show that for all $t = 0, \ldots \tau - 1$, $p_t(q_t^*) + p_t'(q_t^*)q_t^* - c = \beta(p_{t+1}(q_{t+1}^*) + p_{t+1}'(q_{t+1}^*)q_{t+1}^* - c)$.
5) Let us assume that the inverse demand function p_t is constant over time equal to p. Show that q_t^* is decreasing , so $q_t^* > 0$ for all $t \leq \tau$.

2.2.3 The maximum principle

Let us consider the same general problem as above. Then we define the Hamiltonian at the period t, H_t, as follows:

$$H_t(a, s, \lambda) = \beta^t f_t(a, s) + \lambda g_t(a, s)$$

Then for an interior solution, the first order condition can be rewritten as follows:

$$\begin{cases} \frac{\partial H_t}{\partial a}(a_t^*, s_t^*, \lambda_{t+1}) = 0 \quad t = 0, \dots, T-1 \\ \frac{\partial H_t}{\partial s}(a_t^*, s_t^*, \lambda_{t+1}) = \lambda_t, \quad t = 1, \dots, T-1 \\ f_T'(s_T^*) = \lambda_T \\ s_{t+1}^* = g_t(a_t^*, s_t^*). \ t = 0, \dots, T-1 \end{cases}$$

The *Maximum principle* states that, at the optimal solution, the action a_t^* maximises the global gain at this period, which is the instantaneous gain through the payoff function f_t and the future gain through the change of the state s_{t+1} via the transition equation g_t. This global gain is approximated by the Hamiltonian in the sense that they have the same derivative with respect to the action a_t as we will see more precisely below. So a_t^* maximises the Hamiltonian for the suitable multiplier λ_{t+1}.

Proposition 37 (Maximum principle) *We consider the previous dynamical optimisation problem. We maintain the same assumptions and we also assume that:*

1) For all $t = 0, \dots, T-1$, f_t and g_t are concave and f_T is concave;

2) For all $t = 0, \dots, T-1$, f_t and g_t are increasing with respect to the state variable s_t and f_T is increasing with respect to the state variable s_T.

Then, $((a_t^*), (s_t^*))$ is an optimal solution if and only if for all $t = 0, \dots, T-1$, a_t^* is a solution of

$$\max\{H(a_t, s_t^*, \lambda_{t+1}) \mid (a_t, s_t^*) \in A_t\}$$

and

$$\begin{cases} \frac{\partial H_t}{\partial s}(a_t^*, s_t^*, \lambda_{t+1}) = \lambda_t, \quad t = 1, \dots, T-1 \\ f_T'(s_T^*) = \lambda_T \\ s_{t+1}^* = g_t(a_t^*, s_t^*). \ t = 0, \dots, T-1 \end{cases}$$

Under our assumptions, the Hamiltonian is concave, so the maximisation for an interior solution is equivalent to the fact that the derivative vanishes. If some constraints are binding, then the result remains true but the proof is more demanding.

We now come back to the evaluation of a change of a_t around an optimal interior solution a_t^*. Actually, at an interior solution, we are in the condition of the sensitivity analysis we did in Section 4.5. Under our assumptions, the gradient vectors of the equality constraints coming from the transition equations are linearly independent. So, we know that the solution and the multipliers are differentiable functions of the parameter. Furthermore, we know that the multiplier associated to a right hand side perturbation is equal to the derivative of the value function with respect to this perturbation.

A change in a_t will have an influence on the future gain $\sum_{\tau=t+1}^{T-1} \beta^t f_(a_t, s_t) + \beta^T f_T(s_T)$ since the initial state s_{t+1} will change according to the equality constraint $s_{t+1} = g_t(a_t, s_t)$. To analyse this question, we reformulate the problem in order to have just a right hand side perturbation. As it is the same kind of problem for the truncated optimisation starting at $t+1$ as for the initial problem starting at $t=0$, we work with the initial problem. So, let us consider the following modified problem where s_0 is now a variable satisfying a linear equality constraint $s_0 = \sigma_0$.

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T f_T(s_T) \\ s_{t+1} = g_t(a_t, s_t),\ t = 0, \dots, T-1, \\ (a_t, s_t) \in A_t\ t = 0, \dots, T-1 \\ s_T \in A_T \\ s_0 = \sigma_0 \end{cases}$$

We can check that the gradient vectors of the equality constraints are still independent since we add a column with just a 1 for the derivative with respect to s_0. Furthermore with this new constraint, the first order necessary condition are the same but there is an additional one concerning the new variable s_0 with a new multiplier λ_0 corresponding to the new constraint, which is:

$$\frac{\partial f_0}{\partial s_0}(a_0^*, s_0^*) = -\lambda_1 \frac{\partial g_0}{\partial s_0}(a_0^*, s_0^*) + \lambda_0$$

So

$$\lambda_0 = \frac{\partial f_0}{\partial s_0}(a_0^*, s_0^*) + \lambda_1 \frac{\partial g_0}{\partial s_0}(a_0^*, s_0^*) = V'(s_0)$$

Hence, the effect of a change of the initial state s_0 on the intertemporal payoff has an immediate part $\frac{\partial f_0}{\partial s_0}(a_0^*, s_0^*)$ on the payoff at date 0 and a future part $\lambda_1 \frac{\partial g_0}{\partial s_0}(a_0^*, s_0^*)$ which is the effect on the next state s_1 given by $\frac{\partial g_0}{\partial s_0}(a_0^*, s_0^*)$ times λ_1. Actually λ_1 is the derivative of V_1, the value of the truncated problem from period 1 to T at s_1^*. Hence λ_1 measures the future consequences on the payoff of a marginal change of s_1. Similarly, λ_t measures the future consequences on the payoff of a marginal change of s_t.

So, the Hamiltonian computed with the multipliers associated to the optimal solution takes into account the global effect of the choice of the action a_t. Indeed, $\lambda_{t+1} g_t(a_t, s_t)$ is a first order approximation of the optimal value $V_{t+1}(g_t(a_t, s_t))$ of the truncated problem:

$$\begin{cases} \text{Maximise } \sum_{\tau=t+1}^{T-1} \beta^\tau f_\tau(a_\tau, s_\tau) + \beta^T f_T(s_T) \\ s_{\tau+1} = g_\tau(a_\tau, s_\tau),\ \tau = t, \dots, T-1, \\ (a_t, s_t) \in A_t\ t = \tau+1, \dots, T-1 \\ s_T \in A_T \\ s_t \text{ given} \end{cases}$$

and the decision at date t is to choose the best action a_t to maximise $\beta^t f_t(a_t, s_t) + V_{t+1}(g_t(a_t, s_t))$.

2.2.4 Bellman principle

We have implicitly used the Bellman principle in the previous parts but we now explicitly state it and we derive the dynamical programming algorithm.

Basically, the Bellman principle tells us that the optimal actions after a date t depends on the state at date t but does not depend on the previous actions before the date t, in other words, does not depend on the previous trajectory.

Let us consider our general problem

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T f_T(s_T) \\ s_{t+1} = g_t(a_t, s_t),\ t = 0, \dots, T-1, \\ (a_t, s_t) \in A_t\ t = 0, \dots, T-1 \\ s_T \in A_T \end{cases}$$

and let us assume that (a_t^*) is an optimal solution, the states being then determined by the transition equations.

Let us now consider the truncated problem $(\mathcal{P}_t)$ at date t

$$(\mathcal{P}_t) \begin{cases} \text{Maximise } \sum_{\tau=t}^{T-1} \beta^\tau f_\tau(a_\tau, s_\tau) + \beta^T f_T(s_T) \\ s_{\tau+1} = g_\tau(a_\tau, s_\tau),\ \tau = t, \dots, T-1, \\ (a_\tau, s_\tau) \in A_\tau\ \tau = t+1, \dots, T-1 \\ s_T \in A_T \\ s_t \text{ given} \end{cases}$$

It $s_t = s_t^*$, then $(a_\tau^*)_{\tau=t}^{T-1}$ is an optimal solution of the truncated problem. So the optimal trajectory starting at the date t only depends on the state s_t^* and not on the previous decisions a_τ^* for $\tau < t$.

Let us denote $V_t(s_t)$ the optimal value of the problème $(\mathcal{P}_t)$ for a given initial state s_t. Then, we get the following **Bellman equation**:

$$V_t(s_t) = \max\{\beta^t f_t(a_t, s_t) + V_{t+1}(g_t(a_t, s_t)) \mid (a_t, s_t) \in A_t\}$$

and a_t^* is a solution of this maximisation problem.

From the Bellman equation, we deduce the **dynamical programming algorithm**, which is working backward starting from the final period and goes step by step to the initial period. At each step, we compute the value function $V_t(s_t)$ for all attainable states using the Bellman equation and the solution of the maximisation problem is the optimal action at date t provided that the current state is s_t.

To initialise the process, we know that the final payoff is given by $\beta^T f_T(s_T)$. Then we solve the following maximisation problem:

$$\begin{cases} \text{Maximise } \beta^{T-1} f_{T-1}(a_{T-1}, s_{T-1}) + \beta^T f_T(g_{T-1}(a_{T-1}, s_{T-1})) \\ (a_{T-1}, s_{T-1}) \in A_{T-1} \end{cases}$$

This is a one dimension problem with the variable a_{T-1} taken the state s_{T-1} as given. We get the value $V_{T-1}(s_{T-1})$ and the optimal solution(s) $\alpha_{T-1}^*(s_{T-1})$.

Then we solve the following maximisation problem:

$$\begin{cases} \text{Maximise } \beta^{T-2} f_{T-2}(a_{T-2}, s_{T-2}) + \beta^T V_{T-1}(g_{T-2}(a_{T-2}, s_{T-2})) \\ (a_{T-2}, s_{T-2}) \in A_{T-2} \end{cases}$$

to get $V_{T-2}(s_{T-2})$ and the optimal solution(s) $\alpha^*_{T-2}(s_{T-2})$ and we repeat the process until the final step to compute $V(s_0) = V_0(s_0)$ and the optimal action $\alpha^*(s_0)$. The algorithm provides the optimal solution by the recursive formula starting from $(a^*_0 = \alpha^*_0(s_0), s_0)$ as follows:

$$(a^*_{t+1}, s^*_{t+1}) = (\alpha^*_{t+1}(g_t(a^*_t, s^*_t)), g_t(a^*_t, s^*_t))$$

Example: intertemporal allocation of wealth

We are applying the dynamical programming algorithm to the intertemporal allocation of wealth. We consider a simple case where $\beta = 1$, the interest rate r equals 0 and the utility function is $\sqrt{c}$. Then we solve this problem:

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \sqrt{c_t} \\ w_t = w_{t-1} - c_{t-1}, \text{ for } t = 1, \ldots, T \\ w_T \geq 0 \\ c_t \geq 0, \text{ for } t = 1, \ldots, T-1 \end{cases}$$

We know that at the optimal solution, the final wealth is equal to 0. So, we know that $V_{T-1}(w_{T-1}) = \beta^{T-1}\sqrt{w_{T-1}}$ and the optimal action is $\alpha^*_{T_1}(w_{T-1}) = w_{T-1}$. Now for the period $T-2$, we solve the following problem:

$$\begin{cases} \text{Maximise } \sqrt{c_{T-2}} + \sqrt{w_{T-2} - c_{T-2}} \\ 0 \leq c_{T-2} \leq w_{T-2} \end{cases}$$

The optimal solution is $\alpha^*_{T_2}(w_{T-2}) = \frac{w_{T-2}}{2}$ and $V_{T_2}(w_{T-2}) = \sqrt{2}\sqrt{w_{T-2}}$. The next step is to solve:

$$\begin{cases} \text{Maximise } \sqrt{c_{T-3}} + \sqrt{2}\sqrt{w_{T-3} - c_{T-3}} \\ 0 \leq c_{T-3} \leq w_{T-3} \end{cases}$$

The optimal solution is $\alpha^*_{T_3}(w_{T-3}) = \frac{w_{T-3}}{3}$ and $V_{T_3}(w_{T-3}) = \sqrt{3}\sqrt{w_{T-3}}$. Repeating the process, we prove by induction that $\alpha^*_{T_\tau}(w_{T-\tau}) = \frac{w_{T-\tau}}{\tau}$ and $V_{T_\tau}(w_{T-\tau}) = \sqrt{\tau}\sqrt{w_{T-\tau}}$.

So, if we start from an initial wealth w_0, the optimal actions and the successive wealths are $(\frac{w_0}{T}, \frac{w_0(T-t)}{T})_{t=0}^{T-1}$, which means that the consumption is the same at each period and the wealth decreases linearly.

Exercise 38 Apply the dynamical programming algorithm to the intertemporal allocation of wealth with $\beta \in]0,1[$, the interest rate r equals to 0 and the utility function is $\ln c$.

2.3 Infinite horizon

We are now considering an infinite horizon dynamical problem. This leads to new questions since now the actions are infinite so the existence of a solution is not guaranteed and even the definition of the objective function is no more granted. Furthermore, since we have no terminal period, we cannot do a backward induction starting from the last period. Nevertheless, under some suitable stationarity conditions, we will show that we are still able to compute the value of the problem as a fixed point of a contracting mapping and to derive the optimal actions by a recursive process.

We maintain the notation of the previous section and we consider the following maximisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } \sum_{t=0}^{\infty} \beta^t f_t(a_t, s_t) \\ s_{t+1} = g_t(a_t, s_t), \quad t \geq 0 \\ (a_t, s_t) \in A_t,\ t \geq 0 \end{array} \right.$$

where s_0 is the initial given state and $\beta \in]0,1[$ is the actualisation factor. The set of feasible action-state pairs is given by:

$$U(s_0) = \{(a_t, s_t)_t \mid \forall t \in \mathbb{N}, s_{t+1} = g_t(a_t, s_t), (a_t, s_t) \in A_t\}$$

The problem is actually to maximise the discount sum of the payoffs $\sum_{t=0}^{\infty} \beta^t f_t(a_t, s_t)$ under the constraint $(a_t, s_t)_{t=0}^{\infty} \in U(s_0)$.

We assume that for all $(a_t, s_t)_t \in U(s_0)$, for all $\tau \in \mathbb{N}$, the function f_τ and g_τ are defined at all attainable pair (a_τ, s_τ).

2.3.1 Existence of an optimal strategy

We first provide some conditions under which the objective function is well defined, continuous and the set $U(s_0)$ is closed in a compact set for the product topology on the sequences of $\mathbb{R}^2$. So, we get the existence of a solution.

Assumption A

a) There exists a positive real sequence (ρ_t) such that $U(s_0) \subset \prod_{t \in \mathbb{N}} \bar{B}(0, \rho_t)$;

b) there exist $k_0, k_1 \geq 0$ such that for all $t \in \mathbb{N}$, for all (a, s) in the domain of definition of f_t, $|f_t(a, s)| \leq k_0 + k_1 \|(a, s)\|$;

c) $\lim_{t \to \infty} \sqrt[t]{k_1 \rho_t + k_0} \beta < 1$;

d) $\forall t$, f_t and g_t are continuous.

e) $\forall t$, the set A_t is closed in $\mathbb{R}^2$.

Remark 20 If f_t is upper bounded by a constant k_0, that is $k_1 = 0$, then Assumption A(c) is satisfied whatever is $\beta \in]0,1[$.

If the sequence (ρ_t) is upper bounded, then Assumption A(c) is satisfied whatever is $\beta \in]0,1[$.

If f_t is concave on $\mathbb{R}^2_+$ and differentiable on $\mathbb{R}^2_{++}$,

$$f_t(a,s) \leq \nabla f_t(1,1) \cdot (a-1, s-1) + f_t(1,1)$$

and

$$\begin{aligned} |\nabla f_t(1,1) \cdot (a-1,s-1)| &\leq \|\nabla f_t(1,1)\| \|(a-1,s-1)\| \\ &\leq \|\nabla f_t(1,1)\| \|(a,s)\| + \|\nabla f_t(1,1)\| \|(1,1)\| \end{aligned}$$

So Assumption A(b) is satisfied if either f_t is constant and does not depend on t or if $(\|\nabla f_t(1,1))$ and $(|f_t(1,1)|)$ are bounded above.

Proposition 38 *Under Assumption A,*

1) *The set $U(s_0)$ is closed in the compact set $\prod_{t\in\mathbb{N}} \bar{B}(0,\rho_t)$ for the product topology on the space of sequences of $\mathbb{R}^2$.*

2) *The function $\Phi((a_t,s_t)) = \sum_{t=0}^{\infty} \beta^t f_t(a_t,s_t)$ is well defined and continuous for the product topology on $U(s_0)$.*

3) *The problem $(\mathcal{P})$ has a solution.*

Proof. 1) Let $((a_t^\nu, s_t^\nu))$ be a sequence of $U(s_0)$ converging to $(\bar{a}_t, \bar{s}_t)$ for the product topology. From the property of the product topology, for all fixed t, the sequence in $\mathbb{R}^2$ $(a_t^\nu, s_t^\nu)_\nu$ converges to $(\bar{a}_t, \bar{s}_t)$. So, since A_t is closed, $(\bar{a}_t, \bar{s}_t)$ belongs to A_t. Furthermore, since g_t is continuous, the limit of $(s_{t+1}^\nu = g_t(a_t^\nu, s_t^\nu))_\nu$ is equal to $g_t(\bar{a}_t, \bar{s}_t)$ and the limit of $(s_{t+1}^\nu)_\nu$ is equal of $\bar{s}_{t+1}$ So, $\bar{s}_{t+1} = g_t(\bar{a}_t, \bar{s}_t)$, which proves that $(\bar{a}_t, \bar{s}_t)$ belongs to $U(s_0)$.

2) From Assumption A(c), there exists $\underline{t} \in \mathbb{N}$ and $\tilde{\beta} \in]0,1[$, such that for all $t \geq \underline{t}$, $\beta^t(k_1\rho_t + k_0) \leq \tilde{\beta}^t$. From Assumption A(b), for all $(a_t,s_t)_{t\in\mathbb{N}} \in U(s_0)$, $|f_t(a_t,s_t)| \leq k_0 + k_1\rho_t$. Thus, for all $\varepsilon > 0$, there exists $t_\varepsilon \in \mathbb{N}$, such that $|\sum_{t=t_\varepsilon}^{\infty} \beta^t f_t(a_t,s_t)| \leq \varepsilon$ for all $(a_t,s_t)_{t\in\mathbb{N}} \in U(s_0)$. So the objective function is well defined on $U(s_0)$.

Let $((a_t^\nu, s_t^\nu))$ be sequence of $U(s_0)$ converging to $(\bar{a}_t, \bar{s}_t) \in U(s_0)$. From the property of the product topology, for all fixed t, the sequence in $\mathbb{R}^2$ $(a_t^\nu, s_t^\nu)_\nu$ converges to $(\bar{a}_t, \bar{s}_t)$. So, $(\sum_{t=0}^{t_\varepsilon} \beta^t f_t(a_t^\nu, s_t^\nu))$ converges to $\sum_{t=0}^{t_\varepsilon} \beta^t f_t(\bar{a}_t, \bar{s}_t)$. Hence there exists $\underline{\nu} \in \mathbb{N}$ such that for all $\nu \geq \underline{\nu}$,

$$\left| \sum_{t=0}^{t_\varepsilon} \beta^t f_t(a_t^\nu, s_t^\nu)) - \sum_{t=0}^{t_\varepsilon} \beta^t f_t(\bar{a}_t, \bar{s}_t) \right| \leq \varepsilon$$

Hence, one deduces that for all $\nu \geq \underline{\nu}$,

$$\begin{aligned} |\Phi((a_t^\nu, s_t^\nu)_{t\in\mathbb{N}}) - \Phi((\bar{a}_t, s_t)_{t\in\mathbb{N}})| &\leq |\textstyle\sum_{t=0}^{t_\varepsilon} \beta^t f_t(a_t^\nu, s_t^\nu)) - \sum_{t=0}^{t_\varepsilon} \beta^t f_t(\bar{a}_t, \bar{s}_t)| \\ &\quad + |\textstyle\sum_{t=t_\varepsilon+1}^{\infty} \beta^t f_t(a_t^\nu, s_t^\nu)| + |\sum_{t=t_\varepsilon+1}^{\infty} \beta^t f_t(\bar{a}_t, \bar{s}_t)| \\ &\leq 3\varepsilon \end{aligned}$$

which shows that Φ is continuous.

3) This is a direct consequence of the two previous parts and the fact that $\prod_{t\in\mathbb{N}} \bar{B}(0,\rho_t)$ is compact for the product topology. □

Examples

Optimal consumption with a given initial wealth:

We are considering the same problem as in the finite horizon case except that we extend the horizon to $+\infty$. For a given $w_0 > 0$, we have the following optimisation problem:

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{t=0}^{\infty} \beta^t u(c_t) \\ w_{t+1} = (1+r)(w_t - c_t), \quad t \geq 0 \\ c_t \in [0, w_t],\ t \geq 0 \end{cases}$$

We assume that u is a differentiable concave strictly increasing function on $\mathbb{R}_+$ satisfying $u(0) = 0$. We remark that $U(w_0) \subset \prod_{t\in\mathbb{N}} \bar{B}(0,(1+r)^t w_0)$. For $\tilde{c} > 0$, $u(c) \leq u'(\tilde{c})c + k_0$. Note that $\lim_{t\to\infty} \sqrt[t]{u'(\tilde{c})(1+r)^t w_0 + k_0}\beta = (1+r)\beta$.

So Assumption A is satisfied if the interest rate r is small enough with respect to the actualisation factor. Then there exists a solution. This condition means that the consumer has a strong enough preference for the present, meaning a small enough actualisation rate β, with respect to the interest rate.

If the interest rate is above the inverse of the discount factor, then maybe no solution exists: for example, if $u(c) = \gamma c$ with $\gamma > 0$. Then if $(1+r) > \frac{1}{\beta}$, $\beta^t \gamma (1+r)^t w_0$ tends to $+\infty$ so the consumer can wait to get a utility level as high as he wants.

Ramsay growth model:

We consider the same framework presented in the previous section but we extend the horizon to $+\infty$. We maintain the same assumptions. So, the problem is now for a given initial stock of capital k_0:

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{t=0}^{\infty} \beta^t u(c_t) \\ k_{t+1} = F(k_t) - c_t\ t \geq 0 \\ c_t \geq 0,\ k_t \geq 0,\ t \geq 0 \end{cases}$$

Show that a feasible sequence (k_t) of stock of capital is bounded above by $\max\{\bar{k}, k_0\}$ where $\bar{k} > 0$ satisfies $F(\bar{k}) = \bar{k}$. Show that the problem has a solution for all $k_0 > 0$.

2.3.2 First order necessary conditions

We show that the first order necessary conditions are the same as the one presented in the previous section with a finite horizon. Let (a_t^*, s_t^*) be an optimal solution of the general problem $(\mathcal{P})$. For a given period $T > 0$, we consider the truncated problem:

$$(\mathcal{P}^T)\left\{\begin{array}{l}\text{Maximise } \sum_{t=0}^{T}\beta^t f_t(a_t,s_t)+\sum_{t=T+1}^{\infty}\beta^t f_t(a_t^*,s_t^*)\\ s_{t+1}=g_t(a_t,s_t), \quad t=0,\dots,T-1\\ s_{T+1}^*=g_T(a_T,s_T)\\ (a_t,s_t)\in A_t,\ t=0,\dots,T\end{array}\right.$$

Then, one easily check that the truncated sequence $(a_t^*,s_t^*)_{t=0}^T$ satisfies the constraints and is a solution of the problem $(\mathcal{P}^T)$. So, one can apply the result of the previous section to this finite horizon problem.

If $(a_t^*,s_t^*)\in \text{int}A_t$ for all $t=0,\dots,T$, then, for all $t=0,\dots,T$.

$$\left\{\begin{array}{l}\beta^t\frac{\partial f_t}{\partial a_t}(a_t^*,s_t^*)=-\lambda_{t+1}\frac{\partial g_t}{\partial a_t}(a_t^*,s_t^*)\\ \beta^t\frac{\partial f_t}{\partial s_t}(a_t^*,s_t^*)=-\lambda_{t+1}\frac{\partial g_t}{\partial s_t}(a_t^*,s_t^*)+\lambda_t\end{array}\right.$$

If some positivity constraints are binding or if we have a functional representation of the set A_t, then we also get the same first order necessary conditions taken into account the other binding constraints.

Exercise 39 *(Ramsay growth model)* 1) Write the first order necessary conditions for the Ramsay growth model at an interior solution (c_t^*,k_t^*), that is $c_t^*\in]0,k_t^*[$ for all t.
2) Derive from these conditions the Euler equation:

$$\beta u'(c_{t+1}^*)F'(k_{t+1}^*)=u'(c_t^*)$$

3) Show that an optimal solution is always an interior solution as a consequence of the Inada condition $u'(0)=+\infty$.

2.3.3 Value function

In this part, we study properties of the value function V of the maximisation of the intertemporal payoffs with an infinite horizon. So we assume that V is defined on an open interval of $\mathbb{R}$. The previous existence result provides sufficient condition under which this holds true.

We first give sufficient conditions under which V is concave with respect to the initial state s_0.

Proposition 39 *Let us assume that for all $t\in\mathbb{N}$,*

a) f_t and g_t are concave functions and increasing with respect to s;

b) A_t is convex and if $(a_t,s_t)\in A_t$ and $s'_t\geq s_t$, then $(a_t,s'_t)\in A_t$.

Then V is concave on its interval of definition and so continuous on the interior of this interval.

Proof. Let s_0 and s'_0 two initial states in the domain of V. Let $(\bar{a}_t, \bar{s}_t)$ be a feasible action - state sequence for s_0 and $(\tilde{a}_t, \tilde{s}_t)$ feasible for s'_0. Let $\tau \in]0,1[$. Then let $a^\tau_t = \tau \bar{a}_t + (1-\tau)\tilde{a}_t$ and $s^\tau_t = \tau \bar{s}_t + (1-\tau)\tilde{s}_t$. (σ^τ_t) is a state sequence defined by $\sigma^\tau_{t+1} = g_t(a^\tau_t, \sigma^\tau_t)$, $\sigma^\tau_0 = s^\tau_0$.

Then for all t, from the concavity of g_t and the fact that it is increasing with respect to s, $\sigma^\tau_t \geq s^\tau_t$. Indeed, $s^\tau_1 = \tau \bar{s}_1 + (1-\tau)\tilde{s}_1 = \tau g_0(\bar{a}_0, s_0) + (1-\tau) g_0(\tilde{a}_0, s'_0) \leq g_0(\tau \bar{a}_0 + (1-\tau)\tilde{a}_0, \tau s_0 + (1-\tau)s'_0) = \sigma^\tau_0$ since g concave.

$s^\tau_2 = \tau \bar{s}_2 + (1-\tau)\tilde{s}_2 = \tau g_1(\bar{a}_1, \bar{s}_1) + (1-\tau) g_1(\tilde{a}_1, \tilde{s}_1) \leq g_1(\tau \bar{a}_1 + (1-\tau)\tilde{a}_1, \tau \bar{s}_1 + (1-\tau)\tilde{s}_1) = g_1(a^\tau_1, s^\tau_1) \leq g_1(a^\tau_1, \sigma^\tau_1) = \sigma^\tau_2$ since g is concave and increasing with respect to s. So by induction, we prove the result for all $t \in \mathbb{N}$.

$(a^\tau_t, \sigma^\tau_t)$ is feasible for $\tau s_0 + (1-\tau)s'_0$ from our assumption on the convexity of A_t and the possibility to increase the state. From the definition of the value function:

$$V(\tau s_0 + (1-\tau)s'_0) \geq \sum_{t=0}^{\infty} f_t(a^\tau_t, \sigma^\tau_t) \geq \sum_{t=0}^{\infty} f_t(a^\tau_t, s^\tau_t)$$

since f_t increasing with respect to s. So,

$$\sum_{t=0}^{\infty} f_t(a^\tau_t, s^\tau_t) \geq \tau \sum_{t=0}^{\infty} f_t(\bar{a}_t, \bar{s}_t) + (1-\tau) \sum_{t=0}^{\infty} f_t(\tilde{a}_t, \tilde{s}_t)$$

since f concave.

One concludes that $V(\tau s_0 + (1-\tau)s'_0) \geq \tau V(s_0) + (1-\tau)V(s'_0)$. □

We now study the differentiability of V on its interval of definition. For this, we use a property of the concave functions which is left as an exercise.

Exercise 40 Let φ be a concave function on an open interval I. Let $\bar{x} \in I$. We assume that there exists a function ψ defined on an open interval J containing $\bar{s}$ such that ψ is differentiable at $\bar{x}$, $\psi(\bar{x}) = \varphi(\bar{x})$ and $\varphi(x) \geq \psi(x)$ for all $x \in J$. Then φ is differentiable at $\bar{x}$ and $\varphi'(\bar{x}) = \psi'(\bar{x})$.

Proposition 40 *We maintain the assumption of the previous proposition. Let* (a^*_0, s^*_0) *be a solution of the problem* $(\mathcal{P})$ *with the initial state* s^*_0. *We assume that the functions* f_0 *and* g_0 *are differentiable on a neighbourhood of* (a^*_0, s^*_0) *and* $\frac{\partial g_0}{\partial a}(a^*_0, s^*_0) \neq 0$.

Then V *is differentiable at* s^*_0 *and*

$$V'(s^*_0) = -\frac{\partial f_0}{\partial a}(a^*_0, \bar{s}^*_0) \frac{\frac{\partial g_0}{\partial s}(a^*_0, s^*_0)}{\frac{\partial g_0}{\partial a}(a^*_0, s^*_0)} + \frac{\partial f_0}{\partial s}(a^*_0, s^*_0)$$

Proof. From the Implicit Function Theorem, there exists a differentiable function α defined on a neighbourhood of s^*_0 such that $g_0(\alpha(s_0), s_0) = s^*_1$ for all s_0, $\alpha(s^*_0) = a^*_0$ and

$$\alpha'(s^*_0) = -\frac{\frac{\partial g_0}{\partial s}(a^*_0, s^*_0)}{\frac{\partial g_0}{\partial a}(a^*_0, s^*_0)}$$

So, for all s_0 in a neighbourhood of s_0^*, $V(s_0) \geq f_0(a(s_0), s_0) + \sum_{t=1}^{\infty} f_t(a_t^*, s_t^*)$ and $V(s_0^*) = f_0(a(s_0^*), s_0^*) + \sum_{t=1}^{\infty} f_t(a_t^*, s_t^*)$. Since V is concave and the function $f_0(a(s_0), s_0) + \sum_{t=1}^{\infty} f_t(a_t^*, s_t^*)$ is differentiable at s_0^*, one conclude that V is differentiable at s_0^* and

$$\begin{aligned} V'(s_0^*) &= \tfrac{\partial f_0}{\partial a}(a_0^*, \bar{s}_0^*)\alpha'(s_0^*) + \tfrac{\partial f_0}{\partial s}(a_0^*, s_0^*) \\ &= -\tfrac{\partial f_0}{\partial a}(a_0^*, \bar{s}_0^*)\frac{\frac{\partial g_0}{\partial s}(a_0^*, s_0^*)}{\frac{\partial g_0}{\partial a}(a_0^*, s_0^*)} + \tfrac{\partial f_0}{\partial s}(a_0^*, s_0^*) \end{aligned}$$

□

Remark 21 *Multipliers and derivatives of the value functions* Let $V^1(s_1)$ be the value function of the truncated problem starting at period 1 with the initial state s_1:

$$(\mathcal{P}^1) \begin{cases} \text{Maximise } \sum_{t=1}^{\infty} \beta^t f_t(a_t, s_t) \\ s_{t+1} = g_t(a_t, s_t), \quad t \geq 1 \\ (a_t, s_t) \in A_t,\ t \geq 1 \end{cases}$$

Under the same assumptions as above, V^1 is a concave differentiable function and

$$V^{1\prime}(s_1^*) = \beta \left(-\frac{\partial f_1}{\partial a}(a_1^*, s_1^*) \frac{\frac{\partial g_1}{\partial s}(a_1^*, s_1^*)}{\frac{\partial g_1}{\partial a}(a_1^*, s_1^*)} + \frac{\partial f_1}{\partial s}(a_1^*, s_1^*) \right)$$

From the first order necessay condition:

$\beta \frac{\partial f_1}{\partial a}(a_1^*, s_1^*) = -\lambda_2 \frac{\partial g_1}{\partial a}(a_1^*, s_1^*)$

$\beta \frac{\partial f_1}{\partial s}(a_1^*, s_1^*) = -\lambda_2 \frac{\partial g_1}{\partial s}(a_1^*, s_1^*) + \lambda_1$

one deduces that

$$\lambda_1 = V^{1\prime}(s_1^*)$$

Using the same reasoning, one proves that $\lambda_t = V^{t\prime}(s_t^*)$ at each period.

So, as in the finite horizon case, the Hamiltonian $H_t(a, s, \lambda) = f_t(a, s) + \lambda\ g_t(a, s)$ is maximised at the optimal action a_t^* for the given state s_t^* and for the suitable multiplier λ_{t+1}. Indeed, the action a_t is chosen in order to maximise $f_t(a_t, s_t^*) + V^{t+1}(g_t(a_t, s_t^*))$ that is the current payoff plus the optimal future payoff which depends on the action at date t through the transition equation g_t which determines the initial state of the problem $(\mathcal{P}^{t+1})$. So, the Hamiltonian $H_t(a_t, s_t^*, \lambda_{t+1})$ is a first order approximation the objective function at date t since it has the same derivative than the objective function at the optimal solution a_t^*.

Exercise 41 *Ramsay growth model:* Check that the above assumptions are satisfied. Show that at an interior solution:

$$V'(k_0^*) = -u'(c_0^*)F'(k_0^*)$$

2.4 Stationary optimisation problem

Contrary to the finite horizon model, we cannot use the dynamical programming algorithm since we have no final period. In this section, we show that we can compute the value function for stationary optimisation problems as the fixed point of a contracting operator.

From now on, we assume that f_t, g_t and A_t are independent of t and they are denoted f, g and A.

So, our dynamical optimisation problem is now

$$(\mathcal{P}) \begin{cases} \text{Maximise } \sum_{t=1}^{\infty} \beta^t f(a_t, s_t) \\ s_{t+1} = g(a_t, s_t), \quad t \geq 0 \\ (a_t, s_t) \in A, \, t \geq 0 \end{cases}$$

and the feasible trajectories are given by:

$$U(s_0) = \{(a_t, s_t) \mid \forall t \in \mathbb{N}, s_{t+1} = g(a_t, s_t), (a_t, s_t) \in A\}$$

$V(s_0)$ denotes the value of the above problem with the initial condition s_0.

We now consider stronger assumptions to get the recursive equation satisfied by the value function.

Assumption B There exists an interval I of $\mathbb{R}$, there exists $\rho > 0$, such that for all $s_0 \in I$,

a) for all a such that $(a, s_0) \in A$, then $g(a, s_0) \in I$;

b) the set $\{a \mid (a, s_0) \in A\}$ is compact;

c) $U(s_0) \subset \prod_{t \in \mathbb{N}} \bar{B}(0, \rho)$;

d) f and g are continuous.

One deduces from this assumption that $\exists k_0 \geq 0$, $|f(a, s)| \leq k_0$ on $\bar{B}(0, \rho)$, so the objective function is well defined for all $s_0 \in I$.

2.4.1 Bellman Equation

Proposition 41 *V satisfies the Bellman equation: for all s_0,*

$$V(s_0) = \max\{f(a_0, s_0) + \beta V(s_1) \mid (a_0, s_0) \in A, s_1 = g(a_0, s_0)\}$$

Proof. We start by showing that $V(s_0) \geq \max\{f(a_0, s_0) + \beta V(s_1) \mid (a_0, s_0) \in A, s_1 = g(a_0, s_0)\}$.

Let $(a_0, s_0) \in A$ and $s_1 = g(a_0, s_0)$. Let $(\alpha_t^*, \sigma_t^*)_{t \geq 1}$ be a solution for the problem $(\mathcal{P}^1)$ with the initial state s_1. Then, $(a_0, s_0, (\alpha_t^*, \sigma_t^*)_{t \geq 1}) \in U(s_0)$, so

$$V(s_0) \geq f(a_0, s_0) + \sum_{t=1}^{\infty} \beta^t f(\alpha_t^*, \sigma_t^*) = f(a_0, s_0) + \beta V_1(s_1)$$

Taken the maximum over a_0, we get the result.

For the converse inequality; Let $(a_t^*, s_t^*)_{t\geq 0}$ be a solution for the initial state s_0. Then $(a_t^*, a_t^*)_{t\geq 1}$ is solution for for the problem $(\mathcal{P}^1)$ and the initial state s_1^*. If not, there exists $(\alpha_t^*, \sigma_t^*)_{t\geq 1}$, $\sum_{t=1}^{\infty} \beta^t f(\alpha_t^*, \sigma_t^*) > \sum_{t=1}^{\infty} \beta^t f(a_t^*, s_t^*)$, then $f(a_0^*, s_0) + \sum_{t=1}^{\infty} \beta^t f(\alpha_t^*, \sigma_t^*) > f(a_0^*, s_0) + \sum_{t=1}^{\infty} \beta^t f(a_t^*, s_t^*)$, which is in contradiction with $(a_t^*, s_t^*)_{t\geq 0}$ solution for s_0.

So,

$$\begin{aligned} V(s_0) &= f(a_0^*, s_0) + \textstyle\sum_{t=1}^{\infty} \beta^t f(a_t^*, s_t^*) \\ &= f(a_0^*, s_0) + \beta V(s_1^*) \\ &\leq \max\{f(a_0, s_0) + \beta V(s_1) \mid (a_0, s_0) \in A, s_1 = g(a_0, s_0)\} \end{aligned}$$

□

Proposition 42 *V is the unique continuous solution to the Bellman equation, satisfying the following transversality condition for all $s_0 \in I$:*

$$\lim_{T\to\infty} \beta^T V(s_0) = 0$$

Proof. Let us first show that V satisfies the transversality condition. From our assumption, $|V(s_0)| \leq \sum_{t=0}^{\infty} k_0 \beta^t \leq \frac{k_0}{1-\beta}$. So $\lim_{T\to\infty} \beta^T V(s_0) = 0$

Let W be another solution of the Bellman equation satisfying the transversality condition. From our assumptions, there exists $\tilde{a}_0$ such that $W(s_0) = f(\tilde{a}_0, s_0) + \beta W(s_1)$. By induction, there exists a sequence $(\tilde{a}_t, \tilde{s}_t)$ such that

$$W(s_0) = \sum_{t=0}^{T-1} \beta^t f(\tilde{a}_t, \tilde{s}_t) + \beta^T W(\tilde{s}_T)$$

From the transversality condition, $W(s_0) = \sum_{t=0}^{\infty} \beta^t f(\tilde{a}_t, \tilde{s}_t) \leq V(s_0)$.

Conversely, for any feasible sequence in $U(s_0)$,

$$W(s_0) \geq f(a_0, s_0) + \beta W(s_1)$$
$$W(s_1) \geq f(a_1, s_1) + \beta W(s_2)$$

So, for all T,

$$W(s_0) \geq \sum_{t=0}^{T-1} f(a_t, s_t) + \beta^T W(s_T)$$

Using the transversality condition, $W(s_0) \geq \sum_{t=0}^{\infty} f(a_t, s_t)$. Since this inequality holds for all feasible sequence, $W(s_0) \geq V(s_0)$. □

Remark 22 From the Bellman equation, one checks that the optimal action at date t can be computed as a solution of the following problem:

$$\max\{f(a_t, s_t) + \beta V(g(a_t, s_t)) \mid (a_t, s_t) \in A\}$$

Exercise 42 *Steady state* We consider the Bellman equation and we denote by $\alpha(s_0)$ the optimal solution given s_0. A fixed point s^* of $g(\alpha(\cdot), \cdot)$ is called a Steady state. Show that if $s_0 = s^*$, then the optimal solution of the problem is the constant sequence$(\alpha(s^*), s^*)_{t\in\mathbb{N}}$.

2.4.2 V is a fixed point of a contracting mapping

We consider the set $\mathcal{B}(I)$ of the bounded functions from I to $\mathbb{R}$ with the uniform norm. We recall that this is a complete metric space. We define an operator T from $\mathcal{B}(I)$ to itself as follows:

$$Th(s) = \sup\{f(a,s) + \beta h(g(a,s)) \mid (a,s) \in A\}$$

Remark 23 Th is well defined since h bounded, $g(a,s) \in I$ by assumption, the set of a such that $(a,s) \in A$ is compact and f is continuous and upper-bounded.

Exercise 43 Show that if $I = \mathbb{R}_+$ and A is defined by

$$A = \{(a,s) \in \mathbb{R}^2 \mid s \geq 0, a \in [\underline{\alpha}(s), \overline{\alpha}(s)]\}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ satisfying $\underline{\alpha}(s) \leq \overline{\alpha}(s)$ for all $s \in \mathbb{R}_+$, then Th is continuous if h is a continuous function of $\mathcal{B}(I)$.

Proposition 43 *From the Blackwell criterion, T is β-contracting operator.*

Proof. Let h and $\tilde{h}$ two functions in $\mathcal{B}(I)$. If $h \leq \tilde{h}$, that is $h(s) \leq h(s')$ for all $s \in I$, then, for all $s \in I$, for all a such that $(a,s) \in A$, $f(a,s) + \beta h(g(a,s)) \leq f(a,s) + \beta \tilde{h}(g(a,s))$, so $\sup\{f(a,s) + \beta h(g(a,s)) \mid (a,s) \in A\} \leq \sup\{f(a,s) + \beta \tilde{h}(g(a,s)) \mid (a,s) \in A\}$, that is $Th \leq T\tilde{h}$.

Let $\mathbf{1}_I$ be the constant function on I equals to 1 for all s. Then

$$\begin{aligned} T(h + \beta \mathbf{1}_I)(s) &= \sup\{f(a,s) + \beta(h(g(a,s)) + 1) \mid (a,s) \in A\} \\ &= \sup\{f(a,s) + \beta h(g(a,s)) \mid (a,s) \in A\} + \beta \end{aligned}$$

hence, $T(h + \beta \mathbf{1}_I) = Th + \beta \mathbf{1}_I$.

So the operator T satisfies the Blackwell criterion from which one concludes that it is β-contracting. □

Proposition 44 *Under Assumption B, from the Banach fixed point theorem, the value function V is the unique fixed point of the operator T in $\mathcal{B}(I)$. Since T send a continuous mapping to a continuous one, one deduces that V is continuous on I.*

Exercise 44 We consider the Ramsay growth model. We consider the Bellman equation and for all $k \geq 0$, we denote by $\alpha(k)$ the optimal solution. Let $\varphi(k) = k - \alpha(k)$.
1) Show that α and φ are continuous.
2) Show that if $k > 0$, then $\varphi(k) > 0$ and $\alpha(k) > 0$.
3) Show that φ is increasing.
4) Show that $f - \varphi$ is increasing.

Exercise 45 We consider the Ramsay growth model.
1) Show that if we choose an interval $I = [0, \hat{k}]$ with $\hat{k} \geq \bar{k}$, where $\bar{k}$ is the fixed point of F, then Assumption B is satisfied.
2) Show that the optimal capital stock (k_t^*) is monotonic;
3) Show that if $F'(0) \leq \frac{1}{\beta}$, then the optimal capital stock (k_t^*) converges to 0;
3) Show that if $F'(0) > \frac{1}{\beta}$, then the optimal capital stock (k_t^*) converges to a steady state K which is strictly positive and satisfies $F'(K) = \frac{1}{\beta}$.

2.5 Continuous Time

We are now considering the continuous time case, which is a good approximation when we consider smaller and smaller time periods. Furthermore, the powerful tools of mathematical analysis and dynamical systems allow to derive interesting properties of the optimal trajectories with valuable economic interpretations.

Exercise 46 *Actualisation factor in continuous time* Let us consider a discrete actualisation factor $\beta \in]0, 1[$, which means that 1 euro today is equivalent to $\frac{1}{\beta}$ euro at the next period. This denotes the preference for the present or the impatience of the economic agent. Conversely, if r is the interest rate, then 1 euro invested today provides a return of $1 + r$ euro at the next period. So the economic agent with the actualisation factor β is indifferent between consuming 1 euro now or investing it for one period and consuming $1 + r$ euro tomorrow if $\beta = \frac{1}{1+r}$.

If the period is divided in n sub-periods with an interest rate $\frac{r}{n}$ on each sub-period, then the placement of 1 euro provides a return of $(1+\frac{r}{n})^n$ and the economic agent is indifferent if $\beta = \frac{1}{(1+\frac{r}{n})^n}$.

Show that at the limit when the duration of the sub-periods tends to 0, then the instantaneous interest rate equivalent to the discount factor β is equal to $\bar{r} = -\ln \beta$.

Show that the actualisation factor between to dates t and $t' > t$ is equal to $e^{-\bar{r}(t'-t)}$.

2.5.1 Finite horizon continuous time dynamical problem

The formulation of a dynamical problem in continuous time considers an initial given state s_0 and a period $[0, T]$. Then the transition equation is replaced by a differential equation $\dot{s} = g(a(t), s(t), t)$ where g is supposed to be sufficiently regular to insure that we have a unique solution on the interval $[0, T]$ for regular enough action function $a(\cdot)$. For this, we use the Cauchy-Lipschitz Theorem on the existence of solutions for a differential equation. For example, it is assumed that g is locally Lipschitz continuous.

Then we have a function f representing the instantaneous payoff and a terminal payoff given by f_T. So, the problem is the following

$$(\mathcal{P}) \begin{cases} \text{Maximise } \int_0^T e^{-rt} f(a(t), s(t), t) dt + e^{-rT} f_T(s(T)) \\ \dot{s} = g(a(t), s(t), t) \ t \geq 0 \\ (a_t, s_t) \in A, \ t \geq 0 \end{cases}$$

Then the Lagrange multipliers are now a function $\lambda(\cdot)$ defined on $[0, T]$. We can define the Lagrangian of the problem:

$$\begin{aligned} \mathcal{L}(a, s, t, \lambda) = & \int_0^T e^{-rt} f(a(t), s(t), t) dt + e^{-rT} f_T(s(T)) \\ & - \int_0^T e^{-rt} \lambda(t)(\dot{s} - g(a(t), s(t), t)) dt \end{aligned}$$

Defining the Hamiltonian as

$$H(a(t), s(t), t, \lambda(t)) = f(a(t), s(t), t) + \lambda(t)(\dot{s} = g(a(t), s(t), t))$$

we get

$$\mathcal{L}(a, s, t, \lambda) = \int_0^T e^{-rt} H(a(t), s(t), t, \lambda(t)) dt + e^{-rT} f_T(s(T)) - \int_0^T e^{-rt} \lambda(t) \dot{s} dt$$

Assuming that λ is differentiable and Integrating by part the last term, we get

$$\begin{aligned} \int_0^T e^{-rt} \lambda(t) \dot{s} dt = & \ e^{-rT} \lambda(T) s(T) - \lambda(0) s(0) - \int_0^T e^{-rt} s(t) \dot{\lambda}(t) dt \\ & + r \int_0^T e^{-rt} s(t) \lambda(t) dt \end{aligned}$$

So the Lagrangian can now be written as:

$$\begin{aligned} \mathcal{L}(a, s, t, \lambda) = & \int_0^T e^{-rt} \Big(H(a(t), s(t), t, \lambda(t)) + s(t) \dot{\lambda}(t) - r s(t) \lambda(t) \Big) dt \\ & + e^{-rT} f_T(s(T)) - e^{-rT} \lambda(T) s(T) + \lambda(0) s(0) \end{aligned}$$

The first order conditions provided by the partial derivatives of the Lagrangian are:

$$e^{-rt} D_a H(a(t), s(t), t, \lambda(t)) = 0$$

$$e^{-rt} \Big(D_s H(a(t), s(t), t, \lambda(t)) + \dot{\lambda}(t) - r \lambda(t) \Big) = 0$$

$$e^{-rt} \left(f_T'(s(T)) - \lambda(T) \right) = 0$$

which implies

$$D_a H(a(t), s(t), t, \lambda(t)) = 0$$

$$\dot{\lambda}(t) = r \lambda(t) - D_s H(a(t), s(t), t, \lambda(t)) = 0$$

$$\lambda(T) = f_T'(s(T))$$

We also have the transition equation on the state

$$\dot{s} = g(a(t), s(t), t)$$

More generally, we can prove that the maximum principle holds true, that is, at almost all $t \in [0, T]$, $a(t)$ maximises the Hamiltonian $H(a(t), s(t), t, \lambda(t))$.

So, we can state the first order necessary conditions for this continuous time finite horizon dynamical problem.

Proposition 45 *Under suitable regularity conditions on the the functions f and g, if a^* is an optimal solution of the problem $(\mathcal{P})$, then it exists an almost everywhere differentiable function λ such that*

$a^*(t)$ maximises the Hamiltonian $H(a, s(t), t, \lambda(t))$ almost everywhere;

$$\dot{\lambda}(t) = r\lambda(t) - D_s H(a^*(t), s^*(t), t, \lambda(t)) = 0$$
$$\lambda(T) = f_T'(s^*(T))$$
$$\dot{s} = g(a^*(t), s^*(t), t)$$

2.5.2 Calculus of variations

The classical calculus of variations deals with the problems of the following forms:

$$(\mathcal{Q}) \left\{ \text{ Maximise } \int_0^T e^{-rt} f(\dot{s}(t), s(t), t)dt + e^{-rT} f_T(s(T)) \right.$$

for a given s_0.

By considering the additional differential equation $\dot{s} = a(t)$, we get an equivalent standard dynamical optimisation problem in continuous time:

$$(\tilde{\mathcal{Q}}) \left\{ \begin{array}{l} \text{Maximise } \int_0^T e^{-rt} f(a(t), s(t), t)dt + e^{-rT} f_T(s(T)) \\ \dot{s} = a(t) \end{array} \right.$$

The Hamiltonian is now simply

$$H(a(t), s(t), t, \lambda(t)) = f(a(t), s(t), t) + \lambda(t)a(t)$$

The necessary optimality conditions are

$$D_a f(a(t), s(t), t) + \lambda(t) = 0$$

$$\dot{s} = a(t)$$
$$\dot{\lambda} = -D_s f(a(t), s(t), t)$$

If we consider the derivative of the first equation with respect to t, we get:

$$\dot{\lambda} = -D_t D_a f(a(t), s(t), t)$$

From which, one derives the Euler Equation:

$$D_s f(\dot{s}(t), s(t), t) = D_t D_a f(\dot{s}(t), s(t), t)$$

2.5.3 A remark for the infinite horizon

If we are considering an infinite horizon problem like

$$(\mathcal{P}) \begin{cases} \text{Maximise } \int_0^\infty e^{-rt} f(a(t), s(t), t) dt \\ \dot{s} = g(a(t), s(t), t) \ t \geq 0 \\ (a_t, s_t) \in A, \ t \geq 0 \end{cases}$$

then, as in the discrete case and using the Bellman principle, the necessary conditions for the finite horizon problem are still valid. So, under suitable regularity conditions on the the functions f and g, if a^* is an optimal solution of the problem $(\mathcal{P})$, then it exists an almost everywhere differentiable function λ such that

$a^*(t)$ maximises the Hamiltonian $H(a, s(t), t, \lambda(t))$ almost everywhere;

$$\dot{\lambda}(t) = r\lambda(t) - D_s H(a^*(t), s^*(t), t, \lambda(t)) = 0$$

$$\dot{s} = g(a^*(t), s^*(t), t)$$

2.5.4 Examples

Optimal growth model In continuous time with an infinite horizon, the Ramsey growth model becomes for a given stock $k(0)$ of capital

$$(\mathcal{P}) \begin{cases} \text{Maximise } \int_0^\infty e^{-rt} u(c(t)) dt \\ \dot{k} = F(k(t)) - c(t) \\ c(t) \geq 0, \ k(t) \geq 0 \text{ for all } t \geq 0 \end{cases}$$

The Hamiltonian is

$$H(a(t), s(t), t, \lambda(t)) = u(c(t)) + \lambda(t)(F(k(t)) - c(t))$$

The necessary conditions are for an interior solution:

$$u'(c(t)) - \lambda(t) = 0$$

$$\dot{\lambda} = (r - F'(k(t)))\lambda(t)$$

$$\dot{k} : F(k(t)) - c(t)$$

One deduces from $u'(c(t)) = \lambda(t)$ that $\dot{\lambda} = u''(c(t))\dot{c}$. Using this equation, we derive the Euler equation

$$u''(c(t))\dot{c} = (r - F'(k(t)))u'(c(t))$$

Under the usual assumption that u is concave and increasing as well as F, we derive the fact that the sign of c is the same as the sign of $r - F'(k(t))$. So the consumption is increasing if the marginal productivity is higher than the instantaneous interest rate, which is similar to what we get in a discrete time model.

Investment in a competitive firm We consider a firm producing an output from an input "capital" according to a production function $F(k)$. The "capital" good is long lived with a depreciation. The initial stock of capital is $k(0)$. The constant depreciation rate is denoted δ. The firm can invest by purchasing the capital good on the market at the constant price q. The output price is $p(t)$ at time t. The firm maximises its intertemporal profit, that is, chooses an optimal investment policy $I(t)$ in order to solve the following maximisation problem:

$$(\mathcal{P}) \begin{cases} \text{Maximise } \int_0^\infty e^{-rt} \left(p(t)F(k(t)) - qI(t)\right) dt \\ \dot{k} = I(t) - \delta k(t) \\ k(t) \geq 0,\ I(t) \geq 0 \text{ for all } t \geq 0 \end{cases}$$

Show that the necessary optimality condition for an optimal investment is:

$$p(t)F'(k(t)) = (r + \delta)q$$