Lectures notes on:
Optimisation in Euclidean Spaces[1]

Jean-Marc Bonnisseau[2]

September 8, 2020

[1]EMJMD QEM, First semester, first year, 2020-2021

[2]Paris School of Economics, Université Paris 1 Panthéon Sorbonne, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France, Jean-marc.Bonnisseau@univ-paris1.fr

Contents

Optimisation

General Presentation: This course introduces to Optimization in finite dimensional spaces (first part), and infinite dimensional spaces (second part). This is motivated by models in Economics, Finance, Macroeconomics, Statistics, etc., where these tools are very important to study existence of Optimization problems, uniqueness, properties of the value, method to compute the solution, etc...

Part I: Optimization in Euclidean spaces.

1. Presentation of Optimization, examples (micro, macro, statistics), vocabulary.

2. Optimisation in the one-dimensional case. Basic differential calculus in $\mathbb{R}$. First-order necessary condition, second-order sufficient condition, convex or concave functions and application to optimisation.

3. Topology in Euclidean spaces: norm, distance, continuity, closed subset, compact sets, open subset, boundary, and interior points. Notion of level curves.

4. The Euclidean case: existence results (if compactness: Weierstrass; otherwise, coercivity). Applications.

5. Differentiable unconstrained optimization problem in Euclidean space: reminders about differentiability (differential, first-order development, $\mathcal{C}^k$ functions, Hessian matrix, semi-definite matrices, second-order development, etc...). First-order necessary condition, second-order necessary condition. Optimisation with linear equality constraint.

6. Constrained optimization with equality constraints. Reminder on the implicit function theorem. First-order necessary condition: Existence of Lagrange multipliers. Lagrangian function. Interpretation of the multipliers as the derivatives of the value function. The enveloppe theorem.

7. Convex (concave) functions. Basic properties, continuity of convex function, properties of the gradient and the hessian matrix of a convex function. Necessary and sufficient optimality conditions for convex (concave) objectives functions.

8. Convex sets, projection and separation theorems. Basic properties of convex sets, convex cone, Carathéodory's Theorems, projection on a convex set, properties, characterisation. Separation theorems. Polar cone, bipolar Theorem, Farkas Lemma.

9. Constrained optimization with equality and inequality constraints. Constraint qualification (rank condition, affine case, Slater's condition, etc),

Karush-Kuhn-Tucker necessary conditions, second order necessary condition. Sufficient optimality condition for convex (concave) functions. Interpretation of the multipliers as the derivative of the value function.

Part II: Dynamical optimization.

1. Reminders about infinite dimensional normed spaces. Examples of norms in sequence spaces, functions spaces (l_p-norm, l_∞-norm). Equivalence of norms in finite dimensional spaces. Non-compactness of a ball in infinite dimensional spaces (Riesz). Continuity of linear function in normed space and link with Lipschitz function.

2. Metric spaces. Convergences, continuity, compactness, completeness. Examples of sequence spaces. Basic example of a sequence space which is not compact and another one which is compact. Product metric on a countable product of metric spaces, link with the product topology. Compactness of a countable product of compact.

3. Complete spaces (Cauchy sequences, link with convergence, examples for sequence spaces, function spaces,...). Banach fixed-point theorem. Blackwell fixed-point theorem.

4. Dynamic programming with a finite horizon. Framework. Decision variables. Time consistency. Backward induction. Examples.

5. Dynamic programming with an infinite horizon. Bellman equation, necessary optimality condition. Interpretation of the multipliers with the value function. General discounted optimization problem under some feasibility conditions. Existence of a solution, existence of the value function. Properties of the value function. Some applications in growth models.

Chapter 1

Presentation of Optimization

1.1 Mathematical presentation

Let us consider a function f from a set A to $\mathbb{R}$. A can be a subset of $\mathbb{R}$ or of $\mathbb{R}^n$, or $\mathbb{N}$, or a set of persons, of dates, of locations ... Let C be a subset of A. The problem consists in finding the maximum (respectively the minimum) of f, called the objective function on C. The set C is the set of feasible points (admissible points), it is often described by a finite list of constraints.

We will note

$$(\mathcal{P}) \max_{x \in C} f(x) \qquad \textit{resp.} \ (\mathcal{Q}) \min_{x \in C} f(x)$$

Definition 1 The point $\overline{x}$ is solution of $(\mathcal{P})$ (respectively of $(\mathcal{Q})$) if $\overline{x} \in C$ and if for all x in C, $f(x) \leq f(\overline{x})$ (respectively $f(x) \geq f(\overline{x})$).

Definition 2 If we can define a distance on A, $d : A \times A \to \mathbb{R}_+$, the point $\overline{x}$ is a local solution of $(\mathcal{P})$ (respectively of $(\mathcal{Q})$) if $\overline{x} \in C$ and if there exists $r > 0$ such that for all x in C such that $d(x, x') < r$, $f(x) \leq f(\overline{x})$ (respectively $f(x) \geq f(\overline{x})$).

We will denote $\mathcal{S}ol(\mathcal{P})$ for the set of solutions of $\mathcal{P}$.

Definition 3 We define the value of Problem $(\mathcal{P})$ (respectively of $(\mathcal{Q})$) the supremum (respectively the infimum) of the set $\{f(x) \mid x \in C\}$. This value is either finite or infinite.

If the domain C is empty, we will let by convention $\mathrm{val}(\mathcal{P}) = -\infty$ and $\mathrm{val}(\mathcal{Q}) = +\infty$. One should distinguish between c a solution of $\mathcal{P}$ which is a vector of A and $v = f(c)$ the corresponding value which is an element of $[-\infty, +\infty]$. There might exist several solutions c while the value v is unique.

Example 1

$$(\mathcal{P}) \max_{x \in \mathbb{R}} \sin x$$

$\mathrm{val}\,\mathcal{P} = 1$ while $\mathcal{S}ol\mathcal{P} = \{\pi/2 + 2k\pi \mid k \in \mathbb{Z}\}$

Example 2

$$(\mathcal{P}) \max_{x \in \mathbb{R}} \frac{x^2}{1+x^2}$$

$\operatorname{val} \mathcal{P} = 1$ while $\mathcal{S}ol\mathcal{P} = \emptyset$

Proposition 1 *Let $f : A \to \mathbb{R}$. Let us consider the optimization problem $\mathcal{P}$, $\max_{x \in C} f(x)$ whose value is α, we have $C \cap \{x \in A \mid f(x) > \alpha\} = \emptyset$ and*

$$\mathcal{S}ol\mathcal{P} = f^{-1}(\alpha) \cap C \text{ and } = \{x \in A \mid f(x) \geq \alpha\} \cap C.$$

In particular, if $\operatorname{val} \mathcal{P} \notin \mathbb{R}$, then $\mathcal{S}ol\mathcal{P} = \emptyset$.

Definition 4 If $\mathcal{P}$ and $(\mathcal{Q})$ are two optimization problems, we say that they are equivalent if their sets of solution are equal (but in general, their values are not equal).

Exercise 1 Let $X \subset Y$ be two subsets of A and f be a function from A to $\mathbb{R}$. Let us consider two optimization problems:

$$(\mathcal{P}_X) \left\{ \begin{array}{l} \max f(x) \\ x \in X \end{array} \right. \qquad (\mathcal{P}_Y) \left\{ \begin{array}{l} \max f(x) \\ x \in Y \end{array} \right.$$

1) Show that $\operatorname{val}(\mathcal{P}_X) \leq \operatorname{val}(\mathcal{P}_Y)$.
2) If $\overline{y} \in \mathcal{S}ol(\mathcal{P}_Y)$, and if $\overline{y} \in X$, show that $\overline{y} \in \mathcal{S}ol(\mathcal{P}_X)$. This means :

$$\mathcal{S}ol(\mathcal{P}_Y) \cap X \subset \mathcal{S}ol(\mathcal{P}_X)$$

3) If $\overline{x} \in \mathcal{S}ol(\mathcal{P}_X)$, and if $\operatorname{val}(\mathcal{P}_X) = \operatorname{val}(\mathcal{P}_Y)$, then prove that $\overline{x} \in \mathcal{S}ol(\mathcal{P}_Y)$. This means :

$$\operatorname{val}(\mathcal{P}_X) = \operatorname{val}(\mathcal{P}_Y) \Rightarrow \mathcal{S}ol(\mathcal{P}_X) \subset \mathcal{S}ol(\mathcal{P}_Y)$$

4) We assume that for all y in the set Y, there exists $x \in X$ such that $f(x) \geq f(y)$, show that $\operatorname{val}(\mathcal{P}_X) = \operatorname{val}(\mathcal{P}_Y)$.
5) In the following example, show that $\mathcal{S}ol(\mathcal{P}_Y) \not\subset \mathcal{S}ol(\mathcal{P}_X)$.

$$(\mathcal{P}_X) \left\{ \begin{array}{l} \max 1 - x^2 \\ x \in \,]2, 3[\end{array} \right. \qquad (\mathcal{P}_Y) \left\{ \begin{array}{l} \max 1 - x^2 \\ x \in \mathbb{R} \end{array} \right.$$

6) Same question for $\mathcal{S}ol(\mathcal{P}_X) \not\subset \mathcal{S}ol(\mathcal{P}_Y)$.

$$(\mathcal{P}_X) \left\{ \begin{array}{l} \max e^x(\sin x + 2) \\ x \in [0, \pi] \end{array} \right. \qquad (\mathcal{P}_Y) \left\{ \begin{array}{l} \max e^x(\sin x + 2) \\ x \in \mathbb{R} \end{array} \right.$$

Exercise 2 Let $\mathcal{P}$ be the optimization problem $\max_{x \in C} f(x)$. Let us define the set

$$D = \{(x, y) \in C \times \mathbb{R} \mid y \leq f(x)\}$$

Let us define g on $C \times \mathbb{R}$ par $g(x, y) = y$ and the optimization problem $(\mathcal{Q})$, $\max_{(x,y) \in D} g(x, y)$.
1) Show that $(\mathcal{P})$ and $(\mathcal{Q})$ have the same value.
2) Show that if $\overline{x} \in \mathcal{S}ol\mathcal{P}$, then $(\overline{x}, f(\overline{x})) \in \mathcal{S}ol(\mathcal{Q})$.
3) Prove that if $(\overline{x}, \overline{y}) \in \mathcal{S}ol(\mathcal{Q})$, then $\overline{x} \in \mathcal{S}ol\mathcal{P}$, and $\overline{y} = f(\overline{x})$.

Exercise 3 Let f be a function defined on C. Let us suppose that $\varphi : X \subset \mathbb{R} \to \mathbb{R}$ is an increasing function and $f(c) \in X$ for all $c \in C$.

$$(\mathcal{P}_1) \left\{ \begin{array}{l} \max f(x) \\ x \in C \end{array} \right. \quad (\mathcal{P}_2) \left\{ \begin{array}{l} \max \varphi(f(x)) \\ x \in C \end{array} \right. \quad (\mathcal{P}_3) \left\{ \begin{array}{l} \min -\varphi(f(x)) \\ x \in C \end{array} \right.$$

1) Prove that the three following problems are equivalents, that is that their sets of solutions are the same.
2) Prove that if φ is continuous and $\mathrm{val}(\mathcal{P}_1) \in X$, then $\mathrm{val}(\mathcal{P}_2) = \varphi(\mathrm{val}(\mathcal{P}_1))$.
3) Show that if there exists a solution, then $\mathrm{val}(\mathcal{P}_2) = \varphi(\mathrm{val}(\mathcal{P}_1))$.
4) Let us consider $f(x) = x$, $C =]0, 1[$ and φ equal to the ceiling function, that is$\varphi(x)$ is the smallest element of $\mathbb{Z}$ greater or eqal to x, or $\varphi(x) = \min\{z \in \mathbb{Z} \mid x \leq z\}$. Compute $\mathrm{val}(\mathcal{P}_1)$, $\mathrm{val}(\mathcal{P}_2)$ and $\varphi(\mathrm{val}(\mathcal{P}_1))$.

Definition 5 Let us consider the optimization problem $\mathcal{P}$, $\max_{x \in C} f(x)$. The sequence $(x_k)_k$ is said to be a maximizing sequence for $\mathcal{P}$ if for all k, $x_k \in C$ and if the limit of $f(x_k)$ exists (either finite or infinite) and is equal to the value of the problem $\mathcal{P}$.

Exercise 4 Let X be a nonempty set, and f be a function from X to $\mathbb{R}$. Let us consider the following optimization problem:

$$(\mathcal{P}_X) \left\{ \begin{array}{l} \max f(x) \\ x \in X \end{array} \right.$$

1) Prove that there exists a maximizing sequence.
2) Prove that there exists a non-decreasing maximizing sequence.
3) Let us suppose morever that $\mathcal{S}ol\mathcal{P} = \emptyset$, prove that there exists an increasing maximizing sequence.

1.2 Examples of economic optimisation problems

1.2.1 Consumer theory

In microeconomics, we suppose that u is a utility function from $\mathbb{R}^\ell_+$ to $\mathbb{R}$. let us give a price vector $p = (p_1, \ldots, p_\ell)$ and a wealth $w \geq 0$. The consumer's demand is the set of solutions of

$$(\mathcal{P}_X) \left\{ \begin{array}{l} \max u(x) \\ p_1 x_1 + \ldots + p_\ell x_\ell \leq w \\ x_1 \geq 0, \ldots, x_\ell \geq 0 \end{array} \right.$$

1.2.2 Producer Theory

In microeconomics, we consider a firm which produces the good ℓ, using goods $(1, \ldots, \ell - 1)$ as inputs. We describe the production set with f, production function from $\mathbb{R}_+^{\ell-1}$ to $\mathbb{R}$. Let us give a price vector $p = (p_1, \ldots, p_{\ell-1})$ of the inputs and a level of production $y_\ell \geq 0$, The cost function $c((p_1, \ldots, p_{\ell-1}), y_\ell)$ of the firm is the value fo the problem

$$(\mathcal{P}) \begin{cases} \min p_1 y_1 + \ldots + p_{\ell-1} y_{\ell-1} \\ y_\ell = f(y_1, \ldots, y_{\ell-1}) \\ y_1 \geq 0, \ldots, y_{\ell-1} \geq 0 \end{cases}$$

The firm's demand of inputs corresponds to the set of solutions of $\mathbb{P}$. In addition, if we consider the price p_ℓ of the unique output, the total offer (with usual signs' convention) of the firm is the set of solutions of

$$(\mathcal{Q}) \begin{cases} \max p_\ell y_\ell - c((p_1, \ldots, p_{\ell-1}), y_\ell) \\ y_\ell \geq 0 \end{cases}$$

1.2.3 Finance theory

In finance, there are S possible states of the world tomorrow with the corresponding probabilities $\pi_1, \ldots \pi_S$. Today, we can buy or sell J assets with price $q_1, \ldots, q_J$. If we own one unit of asset j, we will receive if state s occurs, the amount (possibly negative) a_s^j. The investor will try to maximize the expected value of his stochastic income with respect to his initial capital w. He will buy a portfolio $(z_1, \ldots, z_J)$, solution of

$$\begin{cases} \max \sum_{s=1}^S \pi_s \sum_{j=1}^J a_s^j z_j \\ \sum_{j=1}^J q_j z_j \leq w \end{cases}$$

1.2.4 Game theory

In game theory, the best response of a player is the solution of the maximisation of the payoff function with respect to the strategy of this agent, taken the strategies of the other agents as given.

1.2.5 Statistics

In statistics, we determine an estimator using the maximum of the likehood, and we determine the regression's lines by minimizing the sum of the squares of the "distance to the line" among all possible lines. Note that the mean of n real numbers $(x_1, x_2, \ldots, x_n)$ is the solution of the following minimisation problem on $\mathbb{R}$: minimise $\sum_{i=1}^m (m - x_i)^2$.

1.2.6 Transportation problems

Let us consider a firm with m units of production $P_1, \ldots, P_m$, which produce quantities $q_1, \ldots, q_m$ of a certain good. There are n markets $M_1, \ldots, M_n$ to provide whose respective demands are $\delta_1, \ldots, \delta_n$. In order to transport one unit of good from the the unity i to the market j, there is a cost γ_{ij}. We try to provide all the markets at the lowest transportation cost. We have to determine all the flows x_{ij} (quantity moved from P_i to M_j) solution of

$$\begin{cases} \min \sum_{i=1}^m \sum_{j=1}^n \gamma_{ij} x_{ij} \\ \sum_{i=1}^m x_{ij} \geq \delta_j \text{ for all } j, \\ \sum_{j=1}^n x_{ij} \leq q_i \text{ for all } i, \\ x_{ij} \geq 0, \text{ for all } i \text{ and all } j \end{cases}$$

1.2.7 Constant returns to scale

Let us consider a firm using m processes $P_1, \ldots, P_m$ of production. The process P_j is characterized by a vector $\alpha^j \in \mathbb{R}^\ell$. For a single level of activity, there are α_h^j units of good h produced by the firm if $\alpha_h^j \geq 0$ and α_h^j units of good h used by the firm in the process if $\alpha_h^j \leq 0$. The total amount of activity of Process P_j will be denoted by $x_j \geq 0$.

A first class of problem consists in furnishing the demand at minimal cost. There are ℓ markets (one for each good) with respective demands $\delta_1, \ldots, \delta_n$ and the marginal cost of Process P_j is γ_j. The problem consists in determining all activity levels x_j solutions of

$$\begin{cases} \max \sum_{j=1}^m \gamma_j x_j \\ \sum_{i=1}^m \alpha_h^j x_j \geq \delta_i \text{ for all } i, \\ x_j \geq 0, \text{ for all } j \end{cases}$$

A second class of problem consists in maximizing the income. We assume that the planer owns an initial stock $\sigma_1, \ldots, \sigma_\ell$ of inputs and that the marginal income of process j is r_j. The problem consists in determining all activity levels x_j solutions of

$$\begin{cases} \max \sum_{j=1}^m r_j x_j \\ \sum_{i=1}^m \alpha_h^j x_j \leq \sigma_h \text{ for all } h = 1, \ldots, \ell, \\ x_j \geq 0, \text{ for all } j \end{cases}$$

To conclude this introduction, we briefly present the main topics we will address in this course and the mathematical methods to answer to the different issues.

First we will provide sufficient conditions for the existence of a solution, which mainly rest upon topological properties of the objective function and the set of feasible points defined by the constraints. We will later address the question of the uniqueness of the solution thanks to the strict convexity or concavity of the objective function.

Then, the main part of the course will be devoted to the study of necessary optimality conditions. These conditions are almost always derived from the following remark. If $\bar{x}$ is a solution or a local solution of a maximisation problem and $u(t)$ is a differentiable mapping satisfying $u(0) = 0$ and $\dot{u}(t) = \bar{u}$, then by analysing the limit at 0^+ of $(1/t)(f(\bar{x}+u(t))-f(\bar{x}))$, we conclude that this limit is non positive if $\bar{x}+u(t)$ satisfies the constraint of the problem for t small enough. In the simplest case with linear constraint, it is enough to take $u(t)=t\bar{u}$. Then, using tools of differentiable calculus, we can conclude that the limit is equal to the inner product of the gradient of f at $\bar{x}$. Analysing the constraints around $\bar{x}$, we can characterise the cone of vectors $\bar{u}$ such that there exists a derivable mapping $u(t)$ satisfying $u(0)=0$, $\dot{u}(t)=\bar{u}$ and $\bar{x}+u(t)$ satisfies the constraint of the problem for t small enough. Using a duality theorem coming from convex analysis, which is simply a property of orthogonal linear spaces when we have only equality constraints, we deduce the gradient of f at $\bar{x}$ is a linear combination of the gradients of the constraints, the coefficient being called Lagrange or Karush-Kuhn-Tucker multipliers. The existence of theses multipliers is the necessary optimality condition.

The convexity of the objective function and of the set of feasible points allows us to show that this necessary condition is then sufficient. We derive from this property a sufficient optimality condition, which involves the second order derivative of the objective function.

Finally, we will also state some sensitivity analysis, which means that we study the behaviour of the solution or of the value when the problem depends on a parameter. The key tool for this purpose is the implicit function theorem.

The optimisation problems tackle in this course are first in Euclidean spaces with a finite number of variables. The last part of the course is devoted to dynamical optimisation, which eventually deals with an infinite number of variables when the horizon is infinite or we are working in continuous time. We will then briefly present some topological properties of infinite dimension spaces.

1.3 One variable optimization

1.3.1 Existence of a solution

Definition 6 Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. f is coercive if $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$.

We consider the following minimisation problem:

$$(\mathcal{P})\left\{\begin{array}{l}\text{Minimise } f(x)\\ x\in C\end{array}\right.$$

Proposition 2 *The problem $(\mathcal{P})$ has a solution if f is continuous, coercive on $\mathbb{R}$, and C is a closed subset of $\mathbb{R}$.*

Proposition 3 *The problem $(\mathcal{P})$ has a solution if f is continuous on C, and C is a closed bounded subset of $\mathbb{R}$.*

Exercise 5 1) Prove that the function $x\to ax^2+bx+c$ with $a>0$ is coercive.
2) Prove that the function $x\to ax^3+bx^2+cx+d$ with $a\neq 0$ is not coercive.

Exercise 6 Let f be a coercive function from $\mathbb{R}$ to $\mathbb{R}$. Let g be a function from $\mathbb{R}$ to $\mathbb{R}$. We assume that there exists $r > 0$ such that for all $x \in]-\infty, -r] \cup [r, +\infty[$, $f(x) \leq g(x)$. Show that g is coercive.

Exercise 7 Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. We consider the above minimisation problem $(\mathcal{P})$ above and we assume that C is closed. Show that the problem $(\mathcal{P})$ has a solution if there exists $\bar{c} \in C$ such that the set $\{c \in C \mid f(c) \leq f(\bar{c})\}$ is bounded.

1.3.2 Necessary conditions of optimality

First order necessary condition

Proposition 4 *Let I be an interval of $\mathbb{R}$. Let $a = \inf I$ and $b = \sup I$. Let f be a differentiable function on I*[1]*. If $\overline{x} \in I$ is a local solution of $(P) \max_{x \in I} f(x)$ (resp. $(\mathcal{Q}) \min_{x \in I} f(x)$), then*

1. *if $\overline{x} \in]a, b[$, then $f'(\overline{x}) = 0$,*
2. *if $\overline{x} = a$, then $f'(\overline{x}) \leq 0$, (resp. $f'(\overline{x}) \geq 0$);*
3. *if $\overline{x} = b$, then $f'(\overline{x}) \geq 0$, (resp. $f'(\overline{x}) \leq 0$).*

Second order necessary condition

Proposition 5 *Let I be an interval of $\mathbb{R}$. Let $a = \inf I$ and $b = \sup I$. Let f be twice differentiable function on I*[2]*. If $\overline{x} \in I$ is a local solution of $(\mathcal{P}) \max_{x \in I} f(x)$ (resp. $(\mathcal{Q}) \min_{x \in I} f(x)$), then*

1. *if $\overline{x} \in]a, b[$, then $f''(\overline{x}) \leq 0$ (resp. $f''(\overline{x}) \geq 0$);*
2. *if $\overline{x} = a$, then $f'(\overline{x}) < 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) \leq 0$, (resp. $f'(\overline{x}) > 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) \geq 0$);*
3. *if $\overline{x} = b$, then $f'(\overline{x}) > 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) \leq 0$, (resp. $f'(\overline{x}) < 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) \geq 0$).*

Second order sufficient condition

Proposition 6 *Let I be an interval of $\mathbb{R}$. Let $a = \inf I$ and $b = \sup I$. Let f be two-times differentiable function on I. Let $\overline{x} \in I$. If one of the following conditions is satisfied, then $\overline{x}$ is a local solution of $(\mathcal{P}) \max_{x \in I} f(x)$ (resp. $(\mathcal{Q}) \min_{x \in I} f(x)$).*

---

[1] If a (resp. b) belongs to I, then we assume that f has a right (resp. left) derivative at a (resp. b), that is, $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ $\left(\text{resp. } \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}\right)$ exists. The right (resp. left) derivative is denoted $f'(a)$ (resp. $f'(b)$).

[2] As above, we assume that the derivative of f has a right derivative at a if $a \in I$ and a left derivative at b if $b \in I$.

1. $\overline{x} \in]a, b[$, $f'(\overline{x}) = 0$ and $f''(\overline{x}) < 0$ (resp. $f'(\overline{x}) = 0$ and $f''(\overline{x}) > 0$);

2. $\overline{x} = a$, $f'(\overline{x}) < 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) < 0$, (resp. $f'(\overline{x}) > 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) > 0$);

3. $\overline{x} = b$, $f'(\overline{x}) > 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) < 0$, (resp. $f'(\overline{x}) < 0$ or $f'(\overline{x}) = 0$ and $f''(\overline{x}) > 0$).

1.4 Convex and concave functions

1.4.1 Definition

Definition 7 Let f be a function from I an interval of $\mathbb{R}$ to $\mathbb{R}$. f is convex (resp. concave) if for all $(x, y) \in I \times I$ and for all $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

$$(\text{resp.} \quad f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).)$$

The function f is strictly convex (resp. strictly concave) if for all $(x, y) \in I \times I$ such that $x \neq y$ and for all $t \in]0, 1[$, $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ (resp. $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$).

Remark 1 A function f is convex if and only if $-f$ is concave. Consequently, the results obtained for convex functions can be translated in terms of concave functions.

Theorem 1 *Let f be a function from I an interval of $\mathbb{R}$ to $\mathbb{R}$. f is convex if and only if for all $k \geq 2$, $(x_i) \in I^k$ and $\lambda \in \mathbb{R}^k_+$ such that $\sum_{i=1}^k \lambda_i = 1$,*

$$f(\sum_{i=1}^k \lambda_i x_i) \leq \sum_{i=1}^k \lambda_i f(x_i)$$

Examples: An affine function ($x \to ax + b$) is convex and concave. A quadratic function ($ax^2 + bx + c$) is convex if $a > 0$ and concave if $a < 0$. The exponential function is convex, the logarithmic function is concave on $\mathbb{R}^*_+$. The absolute value is convex.

Proposition 7 *(i) A finite sum of convex (resp. concave) functions defined on I is convex (resp. concave);*

(ii) if f is convex (resp. concave) and $\lambda > 0$, λf is convex (resp. concave);

(iii) The supremum (resp. infimum) of a family of convex functions (resp. concave) defined on I is convex (resp. concave) on its domain (that is when the supremum (resp. infimum) is finite);

(iv) If f is a convex function (resp. concave) from I to J, intervals of $\mathbb{R}$, and if φ is a convex function (resp. concave) non-decreasing from J to $\mathbb{R}$ then $\varphi \circ f$ is convex (resp. concave).

(v) if g is an affine function from $\mathbb{R}$ to $\mathbb{R}$ and f a convex function on the interval $I \subset \mathbb{R}$, then $f \circ g$ is a convex function on $g^{-1}(I)$.

1.4.2 Properties of convex and concave functions

Proposition 8 *Let f be a convex (resp. concave) function on I an open interval of $\mathbb{R}$. Then f is continuous on I. Moreover, for all $\overline{x} \in I$, there exists $r > 0$ and $k \geq 0$ such that $]\overline{x} - r, \overline{x} + r[\subset I$ and for all $(x, \xi) \in]\overline{x} - r, \overline{x} + r[^2$, $|f(x) - f(\xi)| \leq k|x - \xi|$.*

Proposition 9 *Let f be a differentiable function defined on the interval I. The three following properties are equivalent:*

1. *f is convex (resp. concave);*
2. *for all $(x, \xi) \in I^2$, $f(\xi) - f(x) \geq (\text{resp.} \leq) f'(x)(\xi - x)$;*
3. *f' is increasing (resp. decreasing) on I.*

Proposition 10 *Let f be a twice differentiable function defined on the interval I. f is convex (resp. concave) if and only if f'' is non negative (resp. non positive) on I.*

1.4.3 Optimization of convex or concave functions

Proposition 11 *Let f be a convex (resp. concave) function defined on the interval I. Then:*

1. *the set of solutions of $(\mathcal{Q}) \min_{x \in I} f(x)$ (resp. $(\mathcal{P}) \max_{x \in I} f(x)$) is convex;*
2. *if $\overline{x}$ is a local solution of $(\mathcal{Q})$ (resp. $(\mathcal{P})$) then $\overline{x}$ is a solution of $(\mathcal{Q})$ (resp. $(\mathcal{P})$);*
3. *if I is open and J is a bounded closed interval of $\mathbb{R}$ included in I, then the problem $\min_{x \in J} f(x)$ and $\max_{x \in J} f(x)$ has a solution.*

Proposition 12 *Let f be a differentiable function defined on the interval I. Let $a = \inf I$ and $b = \sup I$. Let $\overline{x} \in I$. If f is convex (resp. concave) on I, then $\overline{x}$ is a solution of $(\mathcal{Q}) \min_{x \in I} f(x)$ (resp. $(\mathcal{P}) \max_{x \in I} f(x)$) if and only if*

1. *$\overline{x} \in]a, b[$ and $f'(\overline{x}) = 0$,*
2. *$\overline{x} = a$ and $f'(\overline{x}) \geq 0$, (resp. $f'(\overline{x}) \leq 0$);*
3. *$\overline{x} = b$ and $f'(\overline{x}) \leq 0$, (resp. $f'(\overline{x}) \geq 0$).*

1.5 Introduction to sensitivity analysis

Let f be a strictly concave continuously differentiable function on $\mathbb{R}$ satisfying $\lim_{x\to-\infty} f(x) = -\infty$. For all real parameter β, we consider the "parameterized" optimization problem

$$(\mathcal{P}_\beta)\begin{cases}\max f(x)\\ x \le \beta\end{cases}$$

We denote by $v(\beta)$ the value of the problem $(\mathcal{P}_\beta)$. We study the behavior of the function v and of the solution of the problem $(\mathcal{P}_\beta)$ with respect to the parameter β.

Theorem 2 *1. For all $\beta \in \mathbb{R}$, there exists a unique solution denoted by $\xi(\beta)$;*

2. the function ξ is continuous;

3. the function v is non-decreasing, concave and continuously differentiable on $\mathbb{R}$, and, $v'(\beta) = f'(\xi(\beta))$.

1.6 Quasi-convex and quasi-concave functions

Definition 8 Let f be a real-valued function defined on an interval I of $\mathbb{R}$. f is quasi-concave (resp. quasi-convex) if for all $\alpha \in \mathbb{R}$, the set $\{x \in I \mid f(x) \ge \alpha\}$ (resp. $\{x \in I \mid f(x) \le \alpha\}$) is convex.

Proposition 13 *Let f be a real-valued function defined on an interval I of $\mathbb{R}$. f is quasi-concave (resp. quasi-convex) if and only if for all (x, ξ) of I^2 and all $\lambda \in [0, 1]$,*

$$f(\lambda x + (1-\lambda)\xi) \ge \min\{f(x), f(\xi)\}(\textit{resp.} \le \max\{f(x), f(\xi)\})$$

Proposition 14 *Let f be a real-valued function defined on an interval,*
1) if f is convex, then f is quasi-convex.
2) The function f is quasi-convex if and only if $(-f)$ is quasi-concave.
3) if f is weakly monotone, then f is both quasi-concave and quasi-convex.

Proposition 15 *Let I be an interval I of $\mathbb{R}$ and f be a continuously differentiable function from I to $\mathbb{R}$. f is quasi-convex if and only if for all $(x, \xi) \in I^2$,*

$$f(\xi) \le f(x) \Rightarrow f'(x)(\xi - x) \le 0.$$

Definition 9 Let f be a real-valued function defined on an interval I, We say that f is strictly quasi-convex if for all $(x, \xi) \in I^2$ satisfying $x \ne \xi$ and for all $\lambda \in]0, 1[$,

$$f(\lambda x + (1-\lambda)\xi) < \max\{f(x), f(\xi)\}$$

Proposition 16 *Let f be a twice continuously differentiable function from I to $\mathbb{R}$. If for all $x \in I$, $f'(x) = 0 \Rightarrow f''(x) > 0$, then f is strictly quasi-convex.*

Proposition 17 *Let f be a quasi-convex function on an interval I. Then the set of solutions of $(\mathcal{Q})\min_{x\in I} f(x)$ is convex. If, furthermore, f is strictly quasi-convex, then*

1. *the set of solutions of $(\mathcal{Q})$ contains at most one element;*

2. *If $\overline{x}$ is a local solution of $(\mathcal{Q})$, then it is also a global solution.*

Proposition 18 *Let f be a twice continuously differentiable function from I to $\mathbb{R}$. Let $a = \inf I$ and $b = \sup I$. We assume that for all $x \in I$, $f'(x) = 0 \Rightarrow f''(x) > 0$. Let $\overline{x} \in I$. $\overline{x}$ is a solution of $(\mathcal{Q})\min_{x\in I} f(x)$ if and only if*

1. *$\overline{x} \in]a, b[$ and $f'(\overline{x}) = 0$,*

2. *$\overline{x} = a$ and $f'(\overline{x}) \geq 0$;*

3. *$\overline{x} = b$ and $f'(\overline{x}) \leq 0$.*

Exercise 8 We consider the following maximisation problem:

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } ax^2 + bx + c\\ x \geq \alpha\end{array}\right.$$

For which values of (a, b, c) this problem has a solution? For which values of (a, b, c) this problem has a finite value?

When a solution exists, compute the solution and give the value of the problem.

Exercise 9 We consider the following maximisation problem:

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } ax^2 + bx + c\\ \alpha \leq x \leq \beta\end{array}\right.$$

where (a, b, c, α, β) are real numbers with $\alpha < \beta$.

For which values of (a, b, c, α, β) this problem has a solution? For which values of (a, b, c) this problem has a finite value?

When a solution exists, compute the solution(s) and give the value of the problem.

Exercise 10 Find the solution(s) of the following maximisation problem when it exists and compute the value of the problem:

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } \sqrt{x} + 2\sqrt{c-x}\\ x \in [0, c]\end{array}\right.$$

where c is a positive real number.

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } x^2 + 2(c-x)\\ x \in [0, c]\end{array}\right.$$

where c is a positive real number.

$$(\mathcal{P}(\alpha)) \begin{cases} \text{Maximise } ax - e^x \\ x \in \mathbb{R} \end{cases}$$

where a is a positive real number.

Chapter 2

Norm, convergence and continuity in Euclidean spaces

2.1 Euclidean space

Euclidean space: $\mathbb{R}^n$, or a linear subspace of $\mathbb{R}^n$, or a finite dimensional linear space with an inner product and the associated norm.

$\mathbb{R}^n$ usual inner product:

$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \cdot y = \sum_{i=1}^n x_i y_i \in \mathbb{R}$

Check that $\forall (x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $\forall t \in \mathbb{R}$,

- $x \cdot y = y \cdot x$
- $(x+z) \cdot y = x \cdot y + z \cdot y$
- $x \cdot (y+z) = x \cdot y + x \cdot z$
- $(tx) \cdot y = x \cdot (ty) = tx \cdot y$
- $x \cdot x > 0$ for all $x \neq 0$

Note that if the vector x satisfies $x \cdot y = 0$ for all $y \in \mathbb{R}^n$, then $x = 0$.

Exercise 11 Let $x \in \mathbb{R}^n$.

1) Show that if for all $y \in \mathbb{R}^n$, $x \cdot y \leq 0$, then $x = 0$.

2) Show that if for all $y \in \mathbb{R}^n$, $x \cdot y \geq 0$, then $x = 0$.

3) Show that there exists a real number a such that for all $y \in \mathbb{R}^n$, $x \cdot y \geq a$, then $x = 0$.

4) Show that there exists a real number a such that for all $y \in \mathbb{R}^n$, $x \cdot y \leq a$, then $x = 0$.

Euclidean norm $\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$

Properties of the norm:

$\forall x \in \mathbb{R}^n$, $\|x\| \geq 0$;

$\|x\| = 0$ if and only if $x = 0$;

$\forall x \in \mathbb{R}^n$, for all $t \in \mathbb{R}$, $\|tx\| = |t|\|x\|$;

$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\|x+y\| \leq \|x\| + \|y\|$.

Furthermore, for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x \cdot y$$

Distance associated to the norm: for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $d(x,y) = \|x-y\|$.

Definition 10 Let r be a non-negative real number and $\bar{x}$ an element of $\mathbb{R}^n$.

the closed ball of center $\bar{x}$ and radius r is the set $\bar{B}(\bar{x},r) = \{x \in \mathbb{R}^n \mid \|x-\bar{x}\| \leq r\}$;

the open ball of center $\bar{x}$ and radius r is the set $B(\bar{x},r) = \{x \in \mathbb{R}^n \mid \|x-\bar{x}\| < r\}$;

Exercise 12 Let r be a non-negative real number and $\bar{x}$ an element of $\mathbb{R}^n$. Let $x \in B(\bar{x},r)$. Show that for all $\rho \leq r - \|x-\bar{x}\|$, $B(x,\rho) \subset B(\bar{x},r)$.

Theorem 3 *Cauchy-Schwartz Inequality:* $\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $|x \cdot y| \leq \|x\|\|y\|$ *and the equality happens when x and y are collinear.*

Exercise 13 Let $\bar{y} \in \mathbb{R}^n$, $\bar{y} \neq 0$. Using the Cauchy-Schwartz inequality, show that the solution of the following problem

$$\begin{cases} \text{Minimise } \bar{y} \cdot x \\ \|x\| \leq 1 \end{cases}$$

is $-\frac{1}{\|\bar{y}\|}\bar{y}$ and that the solution of the following problem

$$\begin{cases} \text{Maximise } \bar{y} \cdot x \\ \|x\| \leq 1 \end{cases}$$

is $\frac{1}{\|\bar{y}\|}\bar{y}$.

x and y are orthogonal if $x \cdot y = 0$.

Theorem 4 *Pythagore's Theorem:* $\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \cdot y = 0$ *if and only if* $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Proposition 19 *Parallelogram's rule:* $\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Exercise 14 Let $x \in \mathbb{R}^n$. Let $r > 0$ and $B(0,r) = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ the open ball of center 0 and radius r.

1) Show that if for all $y \in B(0,r)$, $x \cdot y \leq 0$, then $x = 0$.
2) Show that if for all $y \in B(0,r)$, $x \cdot y \geq 0$, then $x = 0$.
3) Show that there exists a real number a such that for all $y \in B(0,r)$, $x \cdot y \geq a$.
4) Show that there exists a real number b such that for all $y \in B(0,r)$, $x \cdot y \leq b$.

2.2 Sequences

Definition 11 A sequence is a mapping from $\mathbb{N}$ to $\mathbb{R}^n$.

A sequence is often denoted (u_ν) where u_ν is the image of $n \in \mathbb{N}$. If we consider the n components of $u_\nu = (u_\nu^1, u_\nu^2, \ldots, u_\nu^n)$, then a sequence in $\mathbb{R}^n$ generates n real sequences (u_ν^i).
Examples

Definition 12 A sequence (u_ν) is bounded if there exists $r > 0$ such that for all $\nu \in \mathbb{N}$, $u_\nu \in \bar{B}(0, r)$.

Definition 13 Let (u_ν) and (v_ν) be two sequences and $t \in \mathbb{R}$.

a) The sequence (w_ν) defined by for all $n \in \mathbb{N}$, $w_\nu = u_\nu + v_\nu$ is called the sum of (u_ν) and (v_ν).

b) The sequence (w_ν) defined by for all $n \in \mathbb{N}$, $w_\nu = tu_\nu$ is called the product of t and (u_ν).

Definition 14 The sequence (u_ν) converges to a limit $\ell \in \mathbb{R}^n$ if for all $r > 0$, there exists an integer $\nu_r \in \mathbb{N}$ such that for all $\nu \geq \nu_r$, $u_\nu \in B(\ell, r)$ or equivalently $\|u_\nu - \ell\| < r$.

If a sequence converges to a limit, we say that it is convergent and the limit is denoted by $\lim_{\nu \to \infty} u_\nu$.

Proposition 20 *(i) If a sequence is convergent, it has unique limit.*

(ii) The sequence (u_ν) converges to the limit ℓ if and only if the real sequence $(\|u_\nu - \ell\|)$ converges to 0.

(iii) The sequence $(u_\nu = (u_\nu^i)$ converges to the limit ℓ if and only if the n real sequences (u_ν^i) converge to ℓ^i the corresponding component of ℓ.

(iv) If the sequence (u_ν) is convergent, then it is bounded.

Exercise 15 Let (u_ν) and (v_ν) be two sequences. We assume that (u_ν) is convergent. Show that if the set $\{n \in \mathbb{N} \mid u_\nu \neq v_\nu\}$ is finite, then, (v_ν) is convergent and has the same limit as (u_ν).

We assume that (u_ν) is not convergent. Show that if the set $\{\nu \in \mathbb{N} \mid u_\nu \neq v_\nu\}$ is finite, then, (v_ν) is not convergent.

Proposition 21 *Let (u_ν) and (v_ν) be two sequences and $t \in \mathbb{R}$. We assume that (u_ν) converges to ℓ and (v_ν) converges to ℓ'. Then*

a) The sequence $(u_\nu + v_\nu)$ converges to $\ell + \ell'$.

b) The sequence $(u_\nu \cdot v_\nu)$ converges to $\ell\ell'$.

c) The sequence (tu_ν) converges to $t\ell$.

d) The sequence $(\|u_\nu\|)$ converges to $\|\ell\|$.

Cauchy Criterion: From the result for real sequences, we can derive the following very important criterion for convergence.

Proposition 22 *A sequence (u_ν) is convergent if and only if it satisfies the following Cauchy criterion:*

$$\forall r > 0, \exists \nu_r \in \mathbb{N}, \forall \nu, \mu \geq \nu_r, \|u_\nu - u_\mu\| \leq r$$

From a given sequence (u_ν), we can build many others by picking only some terms of it.

Definition 15 Let (u_ν) be a real sequence. A subsequence of (u_ν) is a sequence (v_ν) defined by a strictly increasing mapping φ from $\mathbb{N}$ to itself and for all $\nu \in \mathbb{N}$, $v_\nu = u_{\varphi(\nu)}$.

Proposition 23 *If (u_ν) is a converging sequence, then all subsequences of (u_ν) are convergent and they are converging to the same limit.*

We can now extend to $\mathbb{R}^n$ the fundamental Bolzano-Weierstrass Theorem

Theorem 5 *All bounded sequences in $\mathbb{R}^n$ have a converging subsequence.*

From this result, we deduce a new convergence criterion for bounded sequences.

Proposition 24 *Let (u_ν) be a bounded sequence. (u_ν) is convergent if and only if all convergent subsequences of (u_ν) have the same limit.*

Definition 16 Let (u_ν) be a sequence in $\mathbb{R}^n$. $c \in \mathbb{R}$ is a cluster point of (u_ν) if for all $r > 0$, the set $\{\nu \in \mathbb{N} \mid u_\nu \in B(c, r)\}$ is infinite.

Proposition 25 *Let (u_ν) be a sequence. $c \in \mathbb{R}$ is a cluster point of (u_ν) if and only if there exists a convergent subsequence (v_ν) of (u_ν) such that c is the limit of (v_ν).*

Note that the Bolzano-Weierstrass Theorem can be equivalently stated as all bounded real sequences have a cluster point. We also deduce from the previous results that a bounded sequence is convergent if and only if it has a unique cluster point. In that case, the limit is the unique cluster point.

2.3 Series

Let (u_ν) be a sequence in $\mathbb{R}^n$. The series associated to (u_ν) is the sequence (σ_ν) defined by $\sigma_\nu = \sum_{k=0} \nu u_k$.

Definition 17 The series associated to (u_ν) (or, in short, the series (u_ν)) is convergent if the sequence (σ_ν) defined by $\sigma_\nu = \sum_{k=0}^{\nu} u_k$ is convergent.

The series associated to (u_ν) is absolutely convergent if the real sequence $(\sum_{k=0}^{\nu} \|u_k\|)$ is convergent.

Remark 2 One easily shows (exercise) that if the series (u_ν) is convergent, then the sequence (u_ν) converges to 0. The converse is not true.

Using the Cauchy criterion of convergence, one has the fundamental following result.

Proposition 26 *If the series associated to (u_ν) is absolutely convergent, then the series associated to (u_ν) is convergent.*

Since the series associated to a non-negative sequence is increasing, we get the simple convergence criteria.

Proposition 27 *The series associated to the sequence (u_ν) is absolutely convergent if and only if the sequence $(\sum_{k=0}^{\nu} \|u_k\|)$ is bounded above.*

Exercise 16 Let (u_ν) and (v_ν) be two sequences such that the associated series are convergent.

Show that the series associated to $(u_\nu + v_\nu)$ is also convergent and that its limit is the sum of the limits of the series associated to (u_ν) and (v_ν).

Let $t \in \mathbb{R}$. Show that the series associated to (tu_ν) is also convergent and that its limit is t times the limit of the series associated to (u_ν).

Exercise 17 Let (u_ν) be a sequence such that the associated series is absolutely convergent. Let (t_ν) be a bounded real sequence.

Show that the series associated to $(t_\nu u_\nu)$ is also absolutely convergent.

2.4 Basic topology on $\mathbb{R}^n$

Definition 18 A subset F of $\mathbb{R}$ is closed if for all convergent sequences (u_ν) such that $u_\nu \in F$ for all $\nu \in \mathbb{N}$, then the limit of (u_ν) belongs to F.

A subset U of $\mathbb{R}$ is open if for all convergent sequences (u_ν) such that the limit belongs to U, then there exists $\nu_0 \in \mathbb{N}$ such that $u_\nu \in U$ for all $\nu \geq \nu_0$.

Remark 3 A closed ball is closed. An open ball is open. If I_1, I_2, ... , I_n are closed (resp. open) subsets of $\mathbb{R}$, then the set $\{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x^i \in I_i\}$ is closed (resp. open). The set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x^i \geq 0\}$ is closed. The set $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x^i > 0\}$ is open. All linear subspaces of $\mathbb{R}^n$ are closed. All affine subspaces of $\mathbb{R}^n$ are closed.

Proposition 28

A subset F of $\mathbb{R}^n$ is closed if and only if F^c, its complement in $\mathbb{R}^n$, is open.

A subset U of $\mathbb{R}^n$ is open if and only if U^c, its complement in $\mathbb{R}^n$, is closed.

A subset U of $\mathbb{R}^n$ is open if and only if for all $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$.

Proposition 29

A finite union of closed sets is closed.

An intersection of finitely many or infinitely many closed sets is closed.

A finite intersection of open sets is open.

A union of finitely many or infinitely many open sets is open.

Definition 19 Let A be a subset of $\mathbb{R}^n$.

The closure of A is the set of vectors $x \in \mathbb{R}^n$ such that there exists a sequence (u_ν) converging to x and satisfying $u_\nu \in A$ for all $\nu \in \mathbb{N}$. The closure of A is denoted cl$A$ or $\overline{A}$.

The interior of A is the set $a \in A$ for which there exists $r > 0$ such that $B(a,r) \subset A$. The interior of A is denoted int$A$ or $\mathring{A}$.

Proposition 30 *Let A be a subset of $\mathbb{R}^n$.*

$A \subset \overline{A}$;

$\overline{A}$ is a closed subset of $\mathbb{R}^n$;

$\overline{A}$ is the smallest closed subset of $\mathbb{R}^n$ containing A, that is, if F is closed and $A \subset F$, then $\overline{A} \subset F$;

$\overline{A}$ is the intersection of all closed subsets of $\mathbb{R}^n$ containing A.

Proposition 31 *Let A be a subset of $\mathbb{R}^n$.*

int$A \subset A$;

int$A$ *is an open subset of $\mathbb{R}^n$;*

int$A$ *is the largest open subset of $\mathbb{R}^n$ included in A, that is, if U is open and $U \subset A$, then $U \subset$ int$A$;*

$\overline{A}$ is the union of all open subsets of $\mathbb{R}$ included in A.

Exercise 18 Give the closure and the interior of the following subsets of $\mathbb{R}^n$.

$\mathbb{R}^n$;

$\mathbb{R}^n_+$;

$\mathbb{R}^n_{++}$;

a linear subspace of $\mathbb{R}^n$ different from $\mathbb{R}^n$;

$\bar{B}(x,r)$ for $x \in \mathbb{R}^n$ and $r > 0$;

$B(x,r)$ for $x \in \mathbb{R}^n$ and $r > 0$;

in $\mathbb{R}^2$, $]0,1]^2 \cup ([1,2] \times \{0\})$;

in $\mathbb{R}^2$, $\{(x,y) \in \mathbb{R}^2 \mid x + y \geq 0, x^2 + y^2 \leq 1\}$;

in $\mathbb{R}^2$, $\{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0, xy \geq 1\}$.

Definition 20 Let A be a subset of $\mathbb{R}^n$. The boundary of A denoted bd$A$ is the set $\overline{A} \cap \overline{A^c}$, that is the intersection of the closure of A with the closure of the complement of A in $\mathbb{R}^n$.

Remark 4 An element b belongs to the boundary of A if and only if it is a limit of a sequence of elements of A and a limit of a sequence of elements not in A.

Proposition 32 *Let A be a subset of $\mathbb{R}^n$.*

The boundary of A is a closed set.

A is closed if and only if the boundary of A is included in A.

A is open if and only if the intersection of the boundary of A and A is empty, $\mathrm{bd}A \cap A = \emptyset$.

Exercise 19 Give the boundary of the following subsets of $\mathbb{R}$.

$\mathbb{R}^n$;

$\mathbb{R}^n_+$;

$\mathbb{R}^n_{++}$;

a linear subspace of $\mathbb{R}^n$ different from $\mathbb{R}^n$;

$\bar{B}(x,r)$ for $x \in \mathbb{R}^n$ and $r > 0$;

$B(x,r)$ for $x \in \mathbb{R}^n$ and $r > 0$.

Definition 21 Let A be a subset of $\mathbb{R}^n$. The set A is compact if it is closed and bounded.

Proposition 33 *Let A be a subset of $\mathbb{R}^n$. A is compact if one of the following equivalent conditions is satisfied:*

If (u_ν) is a sequence such that $u_\nu \in A$ for all n, then it has a converging subsequence with a limit in A.

If $(U_i)_{i \in I}$ is a family of open subsets of $\mathbb{R}^n$ such that $A \subset \cup_{i \in I} U_i$, there exists a finite subset $J \subset I$ such that $A \subset \cup_{i \in J} U_i$.

If $(F_i)_{i\in I}$ is a family of closed subsets of $\mathbb{R}$ such that $A \cap (\cap_{i\in I} F_i) = \emptyset$, there exists a finite subset $J \subset I$ such that $A \cap (\cap_{i\in J} F_i) = \emptyset$.

Proposition 34 *Let A be a compact subset of $\mathbb{R}^n$ and F be a closed subset of $\mathbb{R}^n$. Then $A \cap F$ is a compact subset of $\mathbb{R}^n$.*

2.5 Mappings

In this section, U denotes a subset of $\mathbb{R}^n$. We will consider mappings from U to $\mathbb{R}^p$. To avoid confusion, we denote with the subscript n, the objects related to $\mathbb{R}^n$ like the ball $B_n(x,r)$ or the norm $\|\cdot\|_n$ and with the subscript p, the ones related to $\mathbb{R}^p$.

Definition 22 Let f be a mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$.

f is bounded if there exists $r > 0$ such that for all $x \in U$, $f(x) \in \bar{B}_p(0,r)$.

The image of U by f is the set $\{y \in \mathbb{R}^p \mid \exists x \in U, y = f(x)\}$.

Limit of a mapping

Definition 23 Let f be a mappint from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Let x_0 an element of the closure of U. The function f has a limit y_0 at x_0 if for all sequences (u_ν) satisfying $u_\nu \in U$ for all ν and $\lim_{\nu\to\infty} u_n = x_0$, then the sequence $(f(u_n))$ is convergent in $\mathbb{R}^p$ and its limit is y_0.

Proposition 35 *Let f be a mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Let x_0 an element of the closure of U.*

The function f has at most one limit at x_0.

The function f has a limit y_0 at x_0 if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in B_n(x_0,\rho) \cap U$, $f(x) \in B_p(y_0,r)$.

Cauchy criterion: the function f has a limit at x_0 if and only if for all $r > 0$, there exists $\rho > 0$ such that for all pair (x,x') in $B_n(x_0,\rho) \cap U$, $\|f(x) - f(x')\|_p < r$.

Limits and closed sets

Proposition 36 *Let f be a mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Let $x_0 \in \overline{U}$. We assume that f has a limit y_0 at x_0. If there exists $r > 0$ and F a closed subset of $\mathbb{R}^p$ such that for all $x \in U \cap B_n(x_0,r)$, $f(x) \in F$, then $y_0 \in F$.*

Basic calculus with limits

Proposition 37 *Let f and g be two mappings from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Let $x_0 \in \overline{U}$. We assume that f and g have a finite limit at x_0 denoted y_0 and z_0. Then*

a) The mapping $f + g$ has a limit at x_0 which is $y_0 + z_0$.

b) The mapping $f \cdot g$ has a limit at x_0 which is $y_0 \cdot z_0$.

c) For all $t \in \mathbb{R}$, the mapping tf has a limit at x_0 which is ty_0. In particular, $\lim_{x \to x_0} -f(x) = -\lim_{x \to x_0} f(x)$.

d) The mapping $\|f\|_p$ has a limit at x_0 which is $\|y_0\|_p$

Limit of the composition of two mappings

Proposition 38 *Let f be a mapping on $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$ and $x_0 \in \overline{U}$. Let g be a mapping on $V \subset \mathbb{R}^p$ to $\mathbb{R}^k$. We assume that for all $x \in U$, $f(x) \in V$. Let $y_0 = \lim_{x \to x_0} f(x)$. One easily checks that $y_0 \in \overline{V}$. Let $z_0 = \lim_{y \to y_0} g(y)$. Then the limit of $g \circ f$ at x_0 exists and is equal to z_0.*

Continuous mappings

Definition 24 Let f be a mapping on $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. f is continuous at a point $x_0 \in U$, if the limit of f at x_0 exists and is equal to $f(x_0)$. f is continuous on U if f is continuous at every point of U.

Remark 5 All the usual mappings: norm, inner product from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, linear mappings, bilinear mapppings are continuous on their domain of definition.

A particular class of continuous mapping is the class of **Lipschitzian** mappings, that is the function f from U to $\mathbb{R}^p$ such that there exists $k \geq 0$, for all $(x, x') \in U \times U$, $\|f(x) - f(x')\|_p \leq k\|x - x'\|_n$.

Proposition 39 *Let f be a mapping on $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. f is continuous on U if one of the two equivalent following conditions is satisfied:*

For all open set V of $\mathbb{R}^p$, the set $f^{-1}(V) = \{x \in U \mid f(x) \in V\} = W \cap U$ where W is an open set of $\mathbb{R}^n$.

For all closed set F of $\mathbb{R}^p$, the set $f^{-1}(F) = \{x \in U \mid f(x) \in F\} = G \cap U$ where G is a closed set of $\mathbb{R}^n$.

Proposition 40 *Let f be a continuous mapping on $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Then $\|f\|$ is a continuous function from U to $\mathbb{R}$.*

Proposition 41 *Let f and g be two continuous mappings from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Then*

$f + g$ is a continuous mapping from U to $\mathbb{R}^p$.

$f \cdot g$ is a continuous function from U to $\mathbb{R}$.

for all $t \in \mathbb{R}$, tf is a continuous function from U to $\mathbb{R}^p$.

Proposition 42 *Let f be a continuous mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Let g be a continuous mapping from $V \subset \mathbb{R}^p$ to $\mathbb{R}^k$. We assume that for all $x \in U$, $f(x) \in V$. Then $g \circ f$ is continuous on U.*

With these basic operations, we are able to show almost always that the usual functions are continuous.

Exercise 20 Show that the mapping f from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ to $\mathbb{R}$ defined by $f(x, y) = x/y$ is continuous.

The following criterion of continuity is very useful when the function f is defined as a solution of an optimisation problem.

Proposition 43 *Let f a* **bounded** *mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^p$. Then f is continuous on U if and only if the graph of U is closed, that is, for all sequences (x_ν) of elements of U converging to $x_0 \in U$ and such that the sequence $(f(x_\nu))$ converges in $\mathbb{R}^p$, then $\lim_{\nu \to \infty} f(x_\nu) = f(x_0)$.*

Exercise 21 Let f be a bounded continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Show that the function $g(y, r) = \sup_{x \in \bar{B}(y,r)}\{f(x)\}$ is continuous on $\mathbb{R}^n \times \mathbb{R}_{++}$.

Exercise 22 Let f be a bounded continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Let A be a subset of $\mathbb{R}^n$. Show that $\sup_{x \in A}\{f(x)\} = \sup_{x \in \bar{A}}\{f(x)\}$.

Exercise 23 Let A and B be two subsets of $\mathbb{R}^n$ and f linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$. We denote by $A + B$ the set $\{a + b \mid (a, b) \in A \times B\}$. We consider the three following optimisation problems:

$$(\mathcal{P}_A) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ x \in A \end{array} \right. \quad (\mathcal{P}_B) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ x \in B \end{array} \right. \quad (\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ x \in A + B \end{array} \right.$$

1) Let $\bar{a}$ be a solution of $(\mathcal{P}_A)$ and $\bar{b}$ be a solution of $(\mathcal{P}_B)$. Show that $\bar{a} + \bar{b}$ is a solution of $(\mathcal{P})$.
2) Let $\bar{x}$ be a solution of $(\mathcal{P})$. Let $(\bar{\alpha}, \bar{\beta}) \in A \times B$ such that $\bar{x} = \bar{\alpha} + \bar{\beta}$. Show that $\bar{\alpha}$ is a solution of $(\mathcal{P}_A)$ and $\bar{\beta}$ a solution of $(\mathcal{P}_B)$.
3) Show that $\sup_{a \in A}\{f(a)\} + \sup_{b \in B}\{f(b)\} = \sup_{x \in A+B}\{f(x)\}$.

Exercise 24 Let f be a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Let φ be a function from $\mathbb{R}_+$ to $\mathbb{R}$ such that $\lim_{t \to +\infty} \varphi(t) = +\infty$. We assume that there exists $\underline{a} \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $f(x) \geq \underline{a} + \varphi(\|x\|)$. Show that f is coercive.

2.6 Continuous function on a compact set

Theorem 6 *Let $K \subset \mathbb{R}^n$ be compact and f be a continuous mapping from K to $\mathbb{R}^p$. Then $f(K)$ is a compact subset of $\mathbb{R}^p$.*

Corollary 1 *Weierstrass Theorem. Let $K \subset \mathbb{R}^n$ be compact and f be a continuous mapping from K to $\mathbb{R}$. Then there exists $\overline{x} \in K$ and $\underline{x} \in K$ such that for all $x \in K$, $f(\underline{x}) \leq f(x) \leq f(\overline{x})$.*

Theorem 7 *Heine's Theorem Let $K \subset \mathbb{R}^n$ be compact and f be a continuous mapping from K to $\mathbb{R}^p$. Then f is uniformly continuous on K, which means that for all $r > 0$, there exists $\rho > 0$, such that for all $(x, x') \in K \times K$ such that $\|x - x'\|_n \leq \rho$, then $\|f(x) - f(x')\|_p \leq r$.*

2.7 Banach fixed point theorem

Theorem 8 *Let f be a mapping from $U \subset \mathbb{R}^n$ to U. We assume that U is closed and f is a contraction, that is, there exists $k \in [0, 1[$ such that for all $(x, x') \in U \times U$, $\|f(x) - f(x')\|_n \leq k\|x - x'\|_n$. Then there exists a unique element (fixed point) $\bar{x} \in U$ such that $f(\bar{x}) = \bar{x}$ and for all $x_0 \in U$, the sequence (u_ν) defined by $u_0 = x_0$ and for all $\nu \in \mathbb{N}$, $u_{\nu+1} = f(u_\nu)$ converges to $\bar{x}$.*

2.8 Sequence of continuous bounded mappings

Let U be a subset of $\mathbb{R}^n$. Let (f_ν) be a sequence of bounded continuous mappings from U to $\mathbb{R}^p$. We assume that for all $x \in U$, the real sequence $(f_\nu(x))$ is convergent. So we can define a function f on U by $f(x) = \lim_{\nu \to \infty} f_\nu(x)$. The question is to find a sufficient condition to obtain the continuity of f as a function from U to $\mathbb{R}$.

Theorem 9 *If f is bounded and the real sequence $(\sup_{x \in U}\{\|f_\nu(x) - f(x)\|_p\})$ converges to 0, then f is continuous on U. In this case, we say that the sequence (f_ν) converges uniformly to f.*

Like for the real sequences, we have a Cauchy criterion for the uniform convergence of limit of continuous functions.

Theorem 10 *Let U be a subset of $\mathbb{R}^n$. Let (f_ν) be a sequence of bounded continuous function from U to $\mathbb{R}^p$. If for all $r > 0$, there exists $\nu \in \mathbb{N}$ such that for all $p, q \geq \nu$, $\sup_{x \in U}\{\|f_p(x) - f_q(x)\|_p\} \leq r$, then there exists a continuous function f on U to $\mathbb{R}^p$ such that $\lim_{\nu \to \infty} \sup_{x \in U}\{\|f_\nu(x) - f(x)\|_p\} = 0$, which implies that for all $x \in U$, $f(x) = \lim_{\nu \to \infty} f_\nu(x)$.*

2.9 Norms on $\mathbb{R}^n$

A norm on $\mathbb{R}^n$ is a mapping N from $\mathbb{R}^n$ to $\mathbb{R}_+$ satisfying the following conditions:

$N(x) = 0$ if and only if $x = 0$;

$\forall x \in \mathbb{R}^n$, for all $t \in \mathbb{R}$, $N(tx) = |t|N(x)$;

$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $N(x + y) \leq N(x) + N(y)$.

We have consider above the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. But we can check that N_1 defined by $N_1(x) = \sum_{i=1}^n |x_i|$ is also a norm as well as N_∞ defined by $N_\infty(x) = \max\{|x_i| \mid i = 1, \ldots, n\}$. We have an infinity of norms on $\mathbb{R}^n$.

We will prove later that all norms are equivalent in the following sense: if N and $\tilde{N}$ are two norms on $\mathbb{R}^n$, there exists $k > 0$ and $k' > 0$ such that for all $x \in \mathbb{R}^n$,

$$kN(x) \leq \tilde{N}(x) \leq k'N(x)$$

In particular:

$$N_\infty(x) \leq \|x\| \leq N_1(x) \leq nN_\infty(x)$$

So, this means that the definition and results stated in the above sections can be expressed equivalently with any norm. In particular a closed set, an open set, a compact set, a bounded set is independent of the norm chosen for the definition. As well, the convergence of a sequence or the continuity of a mapping can be proved using any norm. So, we can choose the norm who is the most convenient for the computation.

Exercise 25 Let E be a linear subspace of $\mathbb{R}^n$ and F a linear complement of E, that is a linear subspace of $\mathbb{R}^n$ such that $E + F = \mathbb{R}^n$ and $E \cap F = \{0\}$. So, for all $x \in \mathbb{R}^n$, there exists a unique pair $(y, z) \in E \times F$ such that $y + z = x$. We consider the mapping N from $\mathbb{R}^n$ to $\mathbb{R}_+$ defined by $N(x) = \|y\| + \|z\|$ where $(y, z) \in E \times F$ and $x = y + z$.

1) In the case $n = 2$, $E = \{(x, 0) \mid x \in \mathbb{R}\}$ and F a linear complement of E, show that N is not equal to the Euclidean norm of $\mathbb{R}^2$.

2) Show that N is a norm on $\mathbb{R}^n$.

3) Show that N is equivalent to the Euclidean norm.

Exercise 26 We consider the linear space $\mathbb{R}^n \times \mathbb{R}^p$. We define the mapping N from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}_+$ by $N(x, y) = \max\{\|x\|_n, \|y\|_p\}$. Show that N is a norm on $\mathbb{R}^n \times \mathbb{R}^p$ and that it is equivalent to the Euclidean norm $\|(x, y)\| = \sqrt{\sum_{i=1}^n (x_i)^2 + \sum_{j=1}^p (y_j)^2}$.

2.10 Space of linear mappings

We consider the space of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$ denoted $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. We know that the sum of two linear mappings is a linear mapping and the multiplication of a linear mapping by a real number is a linear mapping. So, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ has a structure of linear space.

Taken the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^p$ ($(e^1, \dots, e^n)$ and $(\epsilon^1, \dots, \epsilon^p)$) we can identify the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ with the space $\mathcal{M}(n,p)$ of matrices with p rows and n column, which is a linear space of dimension pn. A linear mapping is represented by its matrix in the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^p$. More precisely, if $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, the term m_{ij} of its matrix in the canonical basis on the jth column and the ith row is the ith component in the canonical basis of $\mathbb{R}^p$ of the image $\varphi(e^j)$ of the jth vector of the canonical basis of $\mathbb{R}^n$. We know that the matrix of the sum of two linear mappings is the sum of their matrices and the matrix of $t\varphi$ for a real number t is t times the matrix of φ.

Since we are considering a finite dimensional space, we can work on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ as we did with $\mathbb{R}^n$ in the previous sections. But, it is convenient to choose a particular norm, which is not the Euclidean norm. Let us define it as follows for $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$:

$$N_{\mathcal{L}}(\varphi) = \sup\{\|\varphi(x)\|_p \mid x \in \bar{B}_n(0,1)\}$$

$N_{\mathcal{L}}$ is well defined since φ is continuous as a linear mapping and $\bar{B}_n(0,1)$ is compact. The supremum is actually a maximum.

We leave the reader checks that $N_{\mathcal{L}}$ is a norm. We just provides two useful properties of this norm.

Proposition 44 *For all $x \in \mathbb{R}^n$, $\|\varphi(x)\|_p \leq N_{\mathcal{L}}(\varphi)\|x\|_n$.*

In other words, φ is $N_{\mathcal{L}}(\varphi)$ Lipschitz continuous.

Proposition 45 *Let φ be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$ and ψ be a linear mapping from $\mathbb{R}^p$ to $\mathbb{R}^k$. Then, $N_{\mathcal{L}}(\psi \circ \varphi) \leq N_{\mathcal{L}}(\psi)N_{\mathcal{L}}(\varphi)$.*

Exercise 27 We consider the linear space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ with the norm $N_{\mathcal{L}}$ and f an element of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. We consider the mapping Φ from $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ to itself defined by $\Phi(g) = g \circ f$.

1) Show that Φ is a linear mapping. Show that it is Lipschitz continuous with a coefficient $N_{\mathcal{L}}(f)$.

Same question with Ψ defined by $\Psi(g) = f \circ g$.

2.11 Existence of a solution for optimisation problems

Definition 25 Let f be a function from $\mathbb{R}^N$ to $\mathbb{R}$. f is coercive if $\lim_{\|x\| \to \infty} f(x) = +\infty$, which means that for all $r \in \mathbb{R}$, there exists $\rho > 0$, for all $x \in \mathbb{R}^n$, if $\|x\| \geq \rho$, then $f(x) \geq r$.

We consider the following minimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ x \in C \end{array} \right.$$

Proposition 46 *The problem $(\mathcal{P})$ has a solution if f is continuous, coercive on $\mathbb{R}^n$, and C is a closed subset of $\mathbb{R}$.*

Exercise 28 Let C be a closed subset of $\mathbb{R}^n$ and $\bar{x}$ an element of $\mathbb{R}^n$. Show that the following minimisation problem has a solution:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } \|x - \bar{x}\| \\ x \in C \end{array} \right.$$

Proposition 47 *The problem $(\mathcal{P})$ has a solution if f is continuous on C, and C is a closed bounded subset of $\mathbb{R}$.*

Exercise 29 Let f be a coercive function from $\mathbb{R}^n$ to $\mathbb{R}$. Let g be a function from $\mathbb{R}^n$ to $\mathbb{R}$. We assume that there exists $r > 0$ such that for all x satisfying $\|x\| \geq r$, $f(x) \leq g(x)$. Show that g is coercive.

Exercise 30 Let f be a continuous function from U an open subset of $\mathbb{R}^n$ to $\mathbb{R}$. We consider the minimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ x \in C \cap U \end{array} \right.$$

We assume that C is closed. Show that the problem $(\mathcal{P})$ has a solution if there exists $\bar{c} \in C \cap U$ such that the set $\{c \in C \cap U \mid f(c) \geq f(\bar{c})\}$ is bounded and closed in $\mathbb{R}^n$.

Exercise 31 Let f be a continuous function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$. We assume that for all $x \in \mathbb{R}^n_{++}$, the set $A = \{x' \in \mathbb{R}^n_{++} \mid f(x') \geq f(x)\}$ is closed in $\mathbb{R}^n$. Show that for all closed subset C of $\mathbb{R}^n$ such that $C \cap \mathbb{R}^n_{++}$ is nonempty and bounded, the problem

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ x \in C \cap \mathbb{R}^n_{++} \end{array} \right.$$

has a solution.

Chapter 3

Multivariable Calculus

3.1 Derivative of $f : \mathbb{R} \to \mathbb{R}^p$

Definition 26 Let f be a mapping of an interval J into $\mathbb{R}^p$. We assume that the interval has more than one point, but the interval may contain its end points. We say that f is differentiable at a number t in its interval of definition if $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists, in which case this limit is called the derivative of f at t and is denoted by $f'(t)$.
We say that f is differentiable (on J) if it is differentiable at every $t \in J$, and in that case, f' is a mapping of J into $\mathbb{R}^p$.
If f has p continuous derivatives, we say f is of class $\mathcal{C}^p$.
If f is infinitely differentiable, we say that f is $\mathcal{C}^\infty$.

Remark 6 $f : J \to \mathbb{R}^p$ can be represented by coordinate functions,
$f(t) = (f_1(t), \ldots, f_p(t))$ and $\frac{f(t+h)-f(t)}{h} = \left(\frac{f_1(t+h)-f_1(t)}{h}, \ldots, \frac{f_p(t+h)-f_p(t)}{h}\right)$.
The limit can be taken componentwise, and consequently f is differentiable if and only if each coordinate function is differentiable, and then
$f'(t) = (f_1'(t), \ldots, f_p'(t))$.
One usually views a map f such as above as a parametrized curve in $\mathbb{R}^p$.

Examples: Let $f(t) = (cos(t),\ sin(t))$ parametrizes the circle. We have $f'(t) = (-sin(t),\ cos(t))$.
Let $f(t) = (cos(t),\ sin(t),\ t)$. Then $f(t)$ describes a spiral. Its projection in the plane of the first two coordinates is of course the circle.

The examples give a curve in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

To distinguish such curves from those given by an equation like $x^2 + y^2 = 1$ we also call them parametrized curves. If f is a differentiable curve, then the derivative f' is called the velocity of the curve. The second derivative f'', if it exists, is called the acceleration of the curve.

Proposition 48 *Let f and g from $J \to \mathbb{R}^p$, if f and g are differentiable at t,*

then so is $f + g$ *and* $(f + g)'(t) = f'(t) + g'(t)$.
If $f : J \to \mathbb{R}^p$ *and* $g : J \to \mathbb{R}^p$ *are differentiable at* t*, let* $f \cdot g$ *be defined by* $(f \cdot g)(t) = f(t) \cdot g(t)$*. Then:*
$(f \cdot g)'(t) = f(t) \cdot g'(t) + f'(t) \cdot g(t)$.

Proposition 49 (Chain rule) *Let* J_1, J_2 *be intervals. Let* $f : J_1 \to J_2$ *and* $g : J_2 \to \mathbb{R}^p$ *be maps. Let* $t \in J_1$*. If* f *is differentiable at* t *and* g *is differentiable at* $f(t)$*, then* $g \circ f$ *is differentiable at* t *and*
$(g \circ f)'(t) = g'(f(t))f'(t)$.

3.2 Derivative of $f : \mathbb{R}^n \to \mathbb{R}$

3.2.1 Partial Derivatives

Definition 27 Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a function. We define its partial derivative at a point $x = (x_1, \ldots, x_n) \in U$ by
$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0, h \neq 0} \frac{f(x+he^i)-f(x)}{h} = \lim_{h \to 0, h \neq 0} \frac{f(x_1,\ldots,x_i+h,\ldots,x_n)-f(x_1,\ldots,x_n)}{h}$ if the limit exists.
$ei = (0, \ldots, 1, \ldots, 0)$ is the unit vector with the i-th component being equal to 1 and all others equal to 0. Note that $f(x + he^i)$ is well defined if h is small enough since U, the domain of f, is open and x belongs to U.

Remark 7 We see that $\frac{\partial f}{\partial x_i}$ is an ordinary derivative which keeps all variables fixed but not the i-th variable. In particular, we know that the derivative of a sum, and the derivative of a constant times a function follow the usual rules, that is $\frac{\partial f+g}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$ and $\frac{\partial cf}{\partial x_i} = c\frac{\partial f}{\partial x_i}$ for any constant c.

Example: If $f(x, y) = 3x^3y^2$ then $\frac{\partial f}{\partial x}(x, y) = 9x^2y^2$ and $\frac{\partial f}{\partial y}(x, y) = 6x^3y$. Of course we may iterate partial derivatives. In this example, we have $\frac{\partial^2 f}{\partial x^2}(x, y) = 18xy^2$, $\frac{\partial^2 f}{\partial y^2}(x, y) = 6x^3$ and $\frac{\partial^2 f}{\partial x \partial y}(x, y) = 18x^2y$,
$\frac{\partial^2 f}{\partial y \partial x}(x, y) = 18x^2y$.
Observe that the two last iterated partials are equal. This is not an accident, and is a special case of the following general theorem.

Theorem 11 (Schwarz) *Let* f *be a function on an open set* $U \in \mathbb{R}^2$*. Assume that the partial derivatives* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial x}, \frac{\partial^2 f}{\partial y \partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ *exist and are continuous. Then* $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Definition 28 We define the gradient of f at any point x at which all partial derivatives exist to be the vector $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$.

Definition 29 We define the Hessian matrix of f at any point

$x = (x_1, \ldots, x_n)$ by

$$H(f, x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \ldots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \ldots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

Remark on higher order partial derivatives

Let f be a function on an open set U of $\mathbb{R}^n$. We may take iterated partial derivatives (if they exist) of the form $\left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n} f$ where $i_1, \ldots, i_n$ are integers ≥ 0. It does not matter in which order we take the partials (provided they exist and are continuous) according to the Schwarz's theorem. If $c_{i_0 \ldots i_n}$ are numbers, we may form finite sums $\sum c_{i_1 \ldots i_n} \left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$ which we view as applicable to functions which have enough partial derivatives. More precisely, we say that a function f on U is of class $\mathcal{C}^p$, for some integer $p \geq 0$, if all partial derivatives $\left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n} f$ exist for $i_1, \ldots, i_n \leq p$ and are continuous. It is clear that the functions of class $\mathcal{C}^p$ form a vector space, that is the sum of two functions of class $\mathcal{C}^p$ is of class $\mathcal{C}^p$ and the product of a function of class $\mathcal{C}^p$ by a real number is a function of class $\mathcal{C}^p$.. Let $i_1, \ldots, i_n$ be integers ≥ 0 such that $i_1 + \ldots + i_n = r \leq p$.
Let F_p be the vector space of functions of class $\mathcal{C}^p$. (For $p = 0$, this is the vector space of continuous functions on U.) Then any monomial $\left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$ may be viewed as a linear map $F_p \to F_{p-r}$ given by $f \mapsto \left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n} f$.
We say that f is of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^p$ for every positive integer p.
If f is of class $\mathcal{C}^\infty$, then $\left(\frac{\partial}{\partial x_1}\right)^{i_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{i_n} f$ is also of class $\mathcal{C}^\infty$.

3.2.2 Reminder on Euclidean algebra

Orthogonal spaces

Definition 30 $(u_1, \ldots, u_n)$ is an orthogonal basis of $\mathbb{R}^n$ if $(u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ and $u_i \cdot u_j = 0$ for all (i, j), $i \neq j$.

$(u_1, \ldots, u_n)$ os an orthonormal basis of $\mathbb{R}^n$ if $(u_1, \ldots, u_n)$ is an orthogonal basis of $\mathbb{R}^n$ and $\|u_i\| = 1$ for all i.

The canonical basis of $\mathbb{R}^n$ is orthonormal.

Proposition 50 *Let $\mathcal{B} = (u_1, \ldots, u_n)$ be an orthonormal basis of $\mathbb{R}^n$ and let x and y be two vectors of $\mathbb{R}^n$. Let $(\xi_1, \ldots, \xi_n)$ be the coordinates of x is the basis $\mathcal{B}$ and $(\zeta_1, \ldots, \zeta_n)$ be the coordinates of y in the basis $\mathcal{B}$. Then, for all i, $\xi_i = x \cdot u_i$,*

$\zeta_i = y \cdot u_i$ *and*

$$x \cdot y = \sum_{i=1}^{n} \xi_i \zeta_i \text{ and } \|x\| = \sqrt{\sum_{i=1}^{n} \xi_i^2}$$

Let E be a linear subspace of $\mathbb{R}^n$ and $(u_1, \ldots, u_p)$ a basis of E. One can built an orthogonal basis of E, $(v_1, \ldots, v_p)$, starting from $(u_1, \ldots, u_p)$ using the Gram-Schmidt orthogonalisation method as follows:

$v_1 = u_1$;

$v_2 = u_2 - \frac{v_1 \cdot u_2}{\|v_1\|^2} v_1$

$\vdots$

$v_k = u_k - \frac{v_1 \cdot u_k}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_k}{\|v_2\|^2} v_2 - \ldots - \frac{v_{k-1} \cdot u_k}{\|v_{k-1}\|^2} v_{k-1}$

$\vdots$

$v_p = u_p - \frac{v_1 \cdot u_p}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_p}{\|v_2\|^2} v_2 - \ldots - \frac{v_{p-1} \cdot u_p}{\|v_{p-1}\|^2} v_{p-1}$

From which, we deduces that all linear subspace of $\mathbb{R}^n$ has an orthogonal basis.

Exercise 32 Let $(u = (1, 0, 1), v = (2, -1, 1), w = (-1, -1, 2))$ be three vectors of $\mathbb{R}^3$. Show that this is a basis of $\mathbb{R}^3$. Apply the Gram-Schmidt orthogonalisation method to find an orthogonal basis of $\mathbb{R}^3$.

Same question with $((1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1))$ in $\mathbb{R}^4$.

Let E be a linear subspace of $\mathbb{R}^n$. The orthogonal complement of E denoted $E^\perp$is the set defined by:

$$E^\perp = \{v \in \mathbb{R}^n \mid \forall u \in E, u \cdot v = 0\}$$

Proposition 51 *$E^\perp$ is a linear subspace of $\mathbb{R}^n$. $E \cap E^\perp = \{0\}$.*

Let $(u_1, \ldots, u_p)$ be a basis of E, then

$$E^\perp = \{v \in \mathbb{R}^n \mid \forall i = 1, \ldots, p \in E, u_i \cdot v = 0\}$$

In other words, $E^\perp$ is the kernel of the linear mapping f from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by $f(v) = (u_1 \cdot v, \ldots, u_p \cdot v)$.

Let E be a linear subspace of $\mathbb{R}^n$ and $(u_1, \ldots, u_p)$ be an orthogonal basis of E. we know that there exists $(u_{p+1}, \ldots, u_n) \in (\mathbb{R}^n)_{n-p}$ such that $(u_1, \ldots, u_p, u_{p+1}, \ldots, u_n)$ is a basis of $\mathbb{R}^n$. Using the Gram-Schmidt orthogonalisation method, we build orthogonal basis of $\mathbb{R}^n$ $(v_1, \ldots, v_p, v_{p+1}, \ldots, v_n)$ from $(u_1, \ldots, u_p, u_{p+1}, \ldots, u_n)$. Since $(u_1, \ldots, u_p)$ is an orthogonal basis of E, we remark that$v_1 = u_1$, $v_2 = u_2$, $\ldots$, $v_p = u_p$. So $(v_{p+1}, \ldots, v_n)$ are linearly independent vectors of $E^\perp$ and they are a basis of $E^\perp$. So we conclude that

Proposition 52 *1) E are $E^\perp$ are complements in $\mathbb{R}^n$, $\mathbb{R}^n = E \oplus E^\perp$ and $\dim E^\perp = n - \dim E$.*
2) THe orthogonal complement of $E^\perp$ is E: $(E^\perp)^\perp = E$

For all $x \in \mathbb{R}^n$, there exists a unique pair $(y, z) \in E \times E^\perp$ such that $x = y + z$. y is the orthogonal projection of x on E, z is the orthogonal projection of x on $E^\perp$. They are denoted $\mathrm{proj}^\perp_E(x)$ and $\mathrm{proj}^\perp_{E^\perp}(x)$.

Remark that $\mathrm{proj}^\perp_E(x) \cdot \mathrm{proj}^\perp_{E^\perp}(x) = 0$.

Proposition 53 *1) The mappings* $\mathrm{proj}^\perp_E$ *are* $\mathrm{proj}^\perp_{E^\perp}$ *are linear;*

2) The kernel of $\mathrm{proj}^\perp_E$ *(resp. the range* $\mathrm{proj}^\perp_{E^\perp}$*) is* $E^\perp$*, the range of* $\mathrm{proj}^\perp_E$ *(resp. the kernel of* $\mathrm{proj}^\perp_{E^\perp}$*) is* E.

3) $\mathrm{proj}^\perp_E \circ \mathrm{proj}^\perp_E = \mathrm{proj}^\perp_E$.

4) $\mathrm{proj}^\perp_E + \mathrm{proj}^\perp_{E^\perp} = \mathrm{Id}$.

We remark that all linear subspaces of $\mathbb{R}^n$ is the kernel of a linear mapping. We can also represent a linear subspace E of $\mathbb{R}^n$ of dimension p by $n-p$ independent linear equations. Indeed, if $(v_1, \ldots, v_{n-p})$ is a basis of $E^\perp$, then

$$E = \{x \in \mathbb{R}^n \mid \forall j = 1, \ldots, n-p, v_j \cdot x = 0\}$$

Proposition 54 *Let* E *and* F *be two linear subspaces of* $\mathbb{R}^n$*. Then* $(E \cap F)^\perp = E^\perp + F^\perp$ *and* $(E + F)^\perp = E^\perp \cap F^\perp$*. If* $E \subset F$*, then* $F^\perp \subset E^\perp$.

Let u be a non zero vector in $\mathbb{R}^n$; we denote by $u^\perp$ the orthogonal complement of the line D generated by u: $D = \{tu \mid t \in \mathbb{R}\}$. We remark that $u^\perp$ is an hyperplan, that is, a linear subspace of dimension $n-1$ and the projections on $u^\perp$ and on D are defined as follows:

$$\mathrm{proj}^\perp_{u^\perp}(x) = x - \frac{x \cdot u}{\|u\|^2} u \text{ et } \mathrm{proj}^\perp_D(x) = \frac{x \cdot u}{\|u\|^2} u$$

Linear mappings and inner product

Let f be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$. Let M its $p \times n$ matrix. in the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^p$. we denote by $(\ell_j)_{j=1}^p$ the rows of the matrix M which are vectors in $\mathbb{R}^n$ and by $(c_i)_{i=1}^n$ the columns of M which are vectors in $\mathbb{R}^p$. The transpose of M denoted M^t is the $n \times p$ matrix whose column vectors are the row vectors of M.

For all $x \in \mathbb{R}^n$, we have two ways to compute the image of x by f.

$$f(x) = \sum_{i=1}^n x_i c_i = (\ell_j \cdot x)_{j=1}^p$$

The transpose of f is the unique linear mapping f^t from $\mathbb{R}^p$ to $\mathbb{R}^n$ satisfying for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$, $y \cdot f(x) = f^t(y) \cdot x$.

The matrix of f^t in the canonical basis is M^t. We remark that the transpose of the transpose of f is equal to f.

Proposition 55 *Let* f *be a linear mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^p$.

1) $\mathrm{Ker} f = (\mathrm{Im} f^t)^\perp$*,* $\mathrm{Im} f = (\mathrm{Ker} f^t)^\perp$*;*

2) $\mathrm{Ker} f^t = (\mathrm{Im} f)^\perp$*,* $\mathrm{Im} f^t = (\mathrm{Ker} f)^\perp$*;*

3) f *and* f^t *have the same rank, the dimension of their ranges are equal.*

Properties of the symmetric matrices Let M be a $n \times n$ symmetric matrix, that is, M is equal to its transpose, $M^t = M$. M is the matrix of a linear mapping f from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by the matrix-vector product $f(x) = Mx$. This linear mapping satisfies $y \cdot f(x) = y \cdot Mx = M^t y \cdot x = My \cdot x = f(y) \cdot x$ and it is called a symmetric linear mapping.

We recall the fundamental spectral theorem on the symmetric matrices. An orthonormal basis of $\mathbb{R}^n$ is a basis $\mathcal{B} = (u_1, \ldots, u_n)$ such that $u_i \cdot u_j = 0$ for all (i, j) with $i \neq j$ and $\|u_i\| = 1$ for all i.

Theorem 12 *Let* f *be a symmetric linear mapping on* $\mathbb{R}^n$ *and* M *its symmetric matrix in the canonical basis. Then it exists an orthonormal basis* $\mathcal{B} = (u_1, \ldots, u_n)$ *such that for all* i*, there exists a real number* λ_i *such that* $f(u_i) = \lambda_i u_i$*. In other words, the matrix of* f *in the basis* $\mathcal{B}$ *is diagonal and* λ_i *is the term on the diagonal and on the* i*th row. Equivalently, we can say that there exists a* $n \times n$ *matrix* P *such that* $P^{-1} = P^t$ *and* $P^{-1}MP$ *is a diagonal matrix.*

Definition 31 Let M be a $n \times n$ symmetric matrix. Then M is

positive definite if all its eigenvalues are positive;

positive semi-definite if all its eigenvalues are non negative;

negative definite if all its eigenvalues are negative;

negative semi-definite if all its eigenvalues are non positive;

Proposition 56 *Let* M *be a* $n \times n$ *symmetric matrix. If* M *is positive definite (resp. negative definite), then* M *is invertible and its inverse is positive definite (resp. negative definite).*

From a symmetric $n \times n$ matrix M, we define a quadratic form q from $\mathbb{R}^n$ to $\mathbb{R}$ and a bilinear symmetric form φ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ as follows:
$q(x) = x \cdot Mx$;
$\varphi(x, y) = x \cdot My$.

We note that $q(x) = \varphi(x, x)$ and $\forall (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $\forall t \in \mathbb{R}$,
- $\varphi(x, y) = \varphi(y, x)$
- $\varphi(x + z, y) = \varphi(x, y) + \varphi(z, y)$
- $\varphi(x, y + z) = \varphi(x, y) + \varphi(x, z)$
- $\varphi(tx, y) = \varphi(x, ty) = t\varphi(x, y)$
- $q(tx) = t^2 q(x)$
- $q(x + y) = q(x) + q(y) + 2\varphi(x, y)$
- $\varphi(x, y) = \frac{1}{4}(q(x + y) - q(x - y))$

Remark 8 M is positive definite (resp. positive semi-definite, negative semi-definite, negativedefinite) if and only if $q(x) > 0$ (resp. ≥ 0, ≤ 0, < 0) for all $x \neq 0$. More precisely, if $\underline{\lambda}$ is the smallest eigenvalue of M and $\overline{\lambda}$ the largest eigenvalue of M, then

$$\underline{\lambda}\|x\|^2 \leq q(x) \leq \overline{\lambda}\|x\|^2$$

Exercise 33 Let M be a $p \times n$ matrix. Let P be the $p \times p$ matrix defined by $P = MM^t$.
1) Show that P is a symmetric positive semi-definite matrix.
2) Show that if the rank of M is equal to p, then P is positive definite.

Let N be a $n \times n$ symmetric positive definite matrix. Same questions with $Q = MNM^t$.

Criterion for a positive definite symmetric matrix

If M is a 2×2 symmetric matrix. M is positive definite if both trace and the determinant are positive.

If M is a $n \times n$ symmetric matrix. We denote M^p the $p \times p$ submatrix containing the first p columns and the first p rows of the matrix M. M is postive definite if the determinant of the matrices M^p with $p = 1, \ldots, n$ are positive.

$$M = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{pmatrix} \quad M^p = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1p} \\ m_{21} & m_{22} & \ldots & m_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p1} & m_{p2} & \ldots & m_{pp} \end{pmatrix}$$

Exercise 34 Let $a \in \mathbb{R}$ and q be a quadratic function define on $\mathbb{R}^3$ as:

$$q(x, y, z) = x^2 + (1 + a)y^2 + (1 + a + a^2)z^2 + 2xy - 2ayz$$

1) Compute the bilinear form φ associated to q.
2) Give the matrix of q in the canonical basis of $\mathbb{R}^3$.
3) For which values of a, φ is positive definite?

Exercise 35 Let q be the quadratic form defined by its matrix in the canonical basis:

$$M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

1) Compute the bilinear form φ associated to q.
2) Show that φ is positive definite?

3.2.3 Differentiable Functions

Definition 32 A function $f : U \to \mathbb{R}$, where U is an open set of $\mathbb{R}^n$, is differentiable at a point x if there exists a vector $g \in \mathbb{R}^n$ and a mapping ϵ defined on an open set containing 0 such that $f(x + h) = f(x) + g \cdot h + \|h\|\epsilon(h)$ with $\lim_{h \to 0} \epsilon(h) = 0$.

Proposition 57 *Let f be a function $U \to \mathbb{R}$, where U is an open set of $\mathbb{R}^n$. If f is differentiable at a point x, then it is continuous at x.*

Theorem 13 *Let f be differentiable at a point x and let g be a vector such that $f(x+h) = f(x)+g\cdot h+\|h\|\epsilon(h)$ with $\lim_{h\to 0}\ \epsilon(h) = 0$. Then all partial derivatives of f at x exist, and $g = \nabla f(x)$.*
Conversely, assume that all partial derivatives of f exist in some open set containing x and are continuous functions. Then f is differentiable at x.

Remark 9 Note that a function may have partial derivatives everywhere and not being differentiable. For example:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

You can check that $\frac{\partial f}{\partial x_1}(x,y)$ and $\frac{\partial f}{\partial x_2}(x,y)$ are defined for all (x,y), included at $(0,0)$, but f is not differentiable at $(0,0)$. It is not even continuous at the origin.

Definition 33 A function f from U an open set of $\mathbb{R}^n$ to $\mathbb{R}$ is differentiable on U if it is differentiable at every point of U. It is continuously differentiable on U if all partial derivatives are continuous on U.

Proposition 58 *Let f and g be two differentiable functions from U an open set of $\mathbb{R}^n$ to $\mathbb{R}$. Then, for all $x \in U$, $\nabla(f+g)(x) = \nabla f(x) + \nabla g(x)$, $\nabla(fg)(x) = f(x)\nabla g(x) + g(x)\nabla f(x)$ and for all $c \in \mathbb{R}$, $\nabla(cf)(x) = c\nabla f(x)$.*

Remark 10 Suppose f is defined on an open set U, and let $\varphi : [a,b] \to U$ be a differentiable curve. Then we may form the composite function $f \circ \varphi$ given by $(f \circ \varphi)(t) = f(\varphi(t))$. We may think of φ as parametrization of a curve, or we may think of $\varphi(t)$ as representing the position on a curve at time t. If $f(x)$ represents, say, the value of the basket of commodities x, then $f(\varphi(t))$ is the value of the commodities at time t of $x = \varphi(t)$. The rate of change of the value along the curve is then given by the derivative $\frac{\partial f(\varphi(t))}{\partial t}$. The chain rule which follows gives an expression for this derivative in terms of the gradient, and generalises the usual chain rule to n variables.

Theorem 14 *Let $\varphi : J \to U$ be a differentiable function defined on some interval, and with values in an open set U of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a differentiable function. Then $f \circ \varphi : J \to \mathbb{R}$ is differentiable, and $(f \circ \varphi)'(t) = \nabla f(\varphi(t)) \cdot \varphi'(t)$.*

3.2.4 Geometric properties of the gradient

From the chain rule, we deduce a geometric interpretation for the gradient.

Definition 34 Let x be a point of U and let v be a fixed vector. We define the directional derivative of f at x in the direction of v to be $f'(x,v) = \lim_{t\to 0, t\neq 0}\ \frac{1}{t}(f(x+tv) - f(x))$.

Remark 11 This means that if we let $g(t) = f(x + tv)$ then $f'(x, v) = g'(0)$. By the chain rule, $g'(t) = \nabla f(x + tv) \cdot v$ whence $f'(x, v) = \nabla f(x) \cdot v$.

From this formula we obtain an interpretation for the gradient. We use the standard expression for the dot product, namely $f'(x, v) = \|f(x)\| \|v\| cos(\theta)$ where θ is the angle between v and $\nabla f(x)$. Depending on the direction of the vector v, the number $cos(\theta)$ ranges from -1 to 1. The maximal value occurs when v has the same direction as $\nabla f(x)$, in which case for such unit vector v we obtain $f'(x, v) = \|\nabla f(x)\| \|v\|$. Therefore we get an interpretation for the direction and norm of the gradient:
The direction of $\nabla f(x)$ is the direction of maximal increase of the function f at x. The norm $\|\nabla f(x)\|$ is equal to the rate of change of f in its normalized direction of maximal increase.

Example: Find the directional derivative of the function $f(x, y) = x^2y^3$ at $(1, -2)$ for $v = \frac{1}{\sqrt{10}}(3, 1)$.
We have $\nabla f(x, y) = (2xy^3, 3x^2y^2)$ and $\nabla f(1, -2) = (-16, 12)$. Hence the desired directional derivative is $f'((1, -2), v) = (-16, 12).\frac{1}{\sqrt{10}}(3, 1) = \frac{1}{\sqrt{10}}(-36)$.

3.2.5 Tangent plane to a surface

Consider the set of all $x \in U$ such that $f(x) = 0$; or given a number c, the set of all $x \in U$ such that $f(x) = c$. This set, denoted by S_c, is called the level hypersurface at c. Let $x \in S_c$ and assume again that $\nabla f(x) \neq 0$.

It will be shown later as a consequence of the implicit function theorem that given any direction v perpendicular to the gradient, there exists a differentiable curve $\alpha : J \to U$ defined on some interval J containing 0 such that $\alpha(0) = x$ and $\alpha'(0) = v$ and $f(\alpha(t)) = c$ for all $t \in J$. In other words, the curve is contained in the level hypersurface. Conversely, we see from the chain rule that if we have a curve α lying in the hypersurface such that $\alpha(0) = x$, then $0 = \frac{\partial f}{\partial t}(\alpha(t)) = \nabla f(\alpha(t)).\alpha'(t)$.
In particular, for $t = 0$, $0 = \nabla f(\alpha(0)).\alpha'(0) = \nabla f(x).\alpha'(0)$. Hence the velocity vector $\alpha'(0)$ of the curve at $t = 0$ is perpendicular to $\nabla f(x)$. From this result, we make the geometric conclusion that $\nabla f(x)$ is perpendicular to the level hypersurface at x.

So we get the formal definition of the tangent plane to the level surface as follows:

Definition 35 Let f be a differentiable mapping for U, an open subset of $\mathbb{R}^n$, to $\mathbb{R}$. Let $x \in U$ such that $\nabla f(x) \neq 0$. Let $c = f(x)$. The set $S_c = \{x' \in U \mid f(x') = c\}$ is the level surface of f at the level c. The tangent hyperplane of S_c at x denoted $T_{S_c}(x)$ is defined by:

$$T_{S_c}(x) = \{u \in \mathbb{R}^n \mid u \cdot \nabla f(x) = 0\}$$

or, in other words, $T_{S_c}(x)$ is the orthogonal space to $\nabla f(x)$.

Note that we often consider a translation of the tangent plan which contains the point x and which is defined as $\{u \in \mathbb{R}^n \mid u \cdot \nabla f(x) = x \cdot \nabla f(x)\}$. Sometimes, there is a confusion between the two plans.

Example: Let $f(x, y, z) = x^2 + y^2 + z^2$. The surface S of points $X = (x, y, z)$ such that $f(X) = 4$ is the sphere of radius 2 centered at the origin. Let $P = (1, 1, \sqrt{2})$. We have $\nabla f(x, y, z) = (2x, 2y, 2z)$ and so $\nabla f(P) = (2, 2, 2\sqrt{2})$. Hence the tangent plane at P is given by the equation $2x + 2y + 2\sqrt{2}z = 0$.

Exercise 36 Let f be a differentiable function on $\mathbb{R}^n \setminus \{0\}$, depending only on the distance from the origin, that is, there exists a differentiable function g on $\mathbb{R}_{++}$ such that $f(x) = g(\|x\|)$ where $\|x\|$ is the Euclidean norm. Show that $\nabla f(x) = \dfrac{g'(\|x\|)}{\|x\|}x$.

3.2.6 Taylor Formula

By applying the result on the directional derivatives to the first order partial derivatives, we obtain the following result:

Proposition 59 *Let f be a $\mathcal{C}^2$ function from U, an open subset of $\mathbb{R}^n$, to $\mathbb{R}$. Let $\bar{x} \in U$ and $u \in \mathbb{R}^n$. Let φ be the function from the open interval I containing* 0 *in $\mathbb{R}$ defined by $\varphi(t) = f(\bar{x} + tu)$. then,*

$$\varphi''(t) = u^t H_f(\bar{x} + tu)u = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x} + tu)u_i u_j$$

Using the Taylor-Lagrange development of φ, we obtain the following result for f:

Proposition 60 *Let f be a $\mathcal{C}^2$ function from U, an open subset of $\mathbb{R}^n$, to $\mathbb{R}$. Let x and x' be two elements of U such that the segment $[x, x'] \subset U$. Then, it exists $\xi \in]x, x'[$ such that:*

$$f(x') = f(x) + Df(x)(x' - x) + \frac{1}{2}(x' - x)^t H_f(\xi)(x' - x)$$

Using the continuity of the second order partial derivatives, we obtain the following Taylor development:

Proposition 61 *Let f be a $\mathcal{C}^2$ function from U, an open subset of $\mathbb{R}^n$, to $\mathbb{R}$. Then, it exists a continuous function η from $U \times U$ to $\mathbb{R}$ such that:*

$$f(x') = f(x) + Df(x)(x' - x) + \frac{1}{2}(x' - x)^t H_f(x)(x' - x) + \|x' - x\|^2 \eta(x', x)$$

and $\eta(x, x) = 0$ for all $x \in U$.

Note that the continuity of η implies that for all $x \in U$, $\lim_{x' \to x} \eta(x', x) = 0$.

3.2.7 Euler's formula

Definition 36 For any real number k, a real-valued function f defined on a cone K[1] of $\mathbb{R}^n$ is homogeneous of degree k if $f(tx) = t^k f(x_1, \ldots, x_n)$ for all $x \in K$ and all $t > 0$.

Theorem 15 *Let $f(x)$ be a C^1 function on an open cone K of $\mathbb{R}^n$. If f is homogeneous of degree k, its first order partial derivatives are homogeneous of degree $k-1$.*

Theorem 16 (Euler's formula) *Let f be a C^1 homogeneous function of degree k on an open cone K of $\mathbb{R}^n$. Then for all x,*

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \ldots + x_n \frac{\partial f}{\partial x_n}(x) = kf(x)$$

or using the gradient

$$x \cdot \nabla f(x) = kf(x)$$

3.3 Derivative of $f : \mathbb{R}^n \to \mathbb{R}^p$

3.3.1 The Frechet derivative as a linear map

Definition 37 Let U be an open subset of $\mathbb{R}^n$ and let $x \in U$. Let f be a mapping from U to $\mathbb{R}^p$. We shall say that f is (Frechet)-differentiable at x if there exists a continuous linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ and a map η defined for all sufficiently small $h \in \mathbb{R}^n$, with values in $\mathbb{R}^p$, such that $\lim_{h\to 0} \ \eta(h) = 0$ and $f(x+h) = f(x) + \varphi(h) + \|h\|\eta(h)$.

Remark 12 Setting $h = 0$ shows that we may assume that η is defined at 0 and that $\eta(0) = 0$. The preceding formula still holds.
We view the definition of the derivative as stating that near x, the values of f can be approximated by a affine map $f(x) + \varphi(x')$ with an error term described by the limit property of η at 0.

Theorem 17 *If f is (Frechet)-differentiable at x, then f is continuous at x.*

Definition 38 If f is (Frechet)-differentiable at every point x of U, then we say that f is (Frechet)-differentiable on U. In that case, the derivative Df is a mapping from U to the space of continuous linear mappings $\mathcal{L}(\mathbb{R}^n,\ \mathbb{R}^p)$, and thus to each $x \in U$, we have associated the linear map $Df(x) \in L(\mathbb{R}^n, \mathbb{R}^p)$.

We shall now see systematically how the definition of the derivative as a linear map actually includes the cases which we have been studied previously. We have three cases:

[1]For all $x \in K$ and for all $t > 0$, $tx \in K$.

We consider a map $f : J \to \mathbb{R}$ from an open interval J into $\mathbb{R}$. Then $Df(x)$ is the linear mapping from $\mathbb{R}$ to $\mathbb{R}$ define by $Df(x)(t) = f'(x)t$.
Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a mapping, differentiable at a point $x \in U$. Then $Df(x)$ is the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$ define by $Df(x)(u) = \nabla f(x) \cdot u$.
Let J be an interval in $\mathbb{R}$, and let $f : J \to \mathbb{R}^p$ be a mapping. Then $Df(x)$ is the linear mapping from $\mathbb{R}$ to $\mathbb{R}^p$ define by $Df(x)(t) = tf'(x)$.

Theorem 18 (Maps with coordinates) *Let U be open in $\mathbb{R}^n$, let f be a mapping from U to $\mathbb{R}^{p_1} \times \ldots \times \mathbb{R}^{p_k}$. Let $(f_1, \ldots, f_k)$ be the coordinate mappings from U to $\mathbb{R}^{p_j}$, that is $f(x) = (f_1(x), \ldots, f_k(x))$. Then f is (Frechet)-differentiable at x if and only if each f_j is differentiable at x, and if this is the case, then $Df(x) = (Df_1(x), \ldots, Df_k(x))$.*

Theorem 19 *Let $\psi : \mathbb{R}^n \to \mathbb{R}^p$ be a linear mapping. Then ψ is (Frechet)-differentiable at every point of $\mathbb{R}^n$ and $D\psi(x) = \psi$ for every $x \in \mathbb{R}^n$.*

Let ϕ from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^k$ be a bilinear mapping, that is the partial mapping $\phi(x, \cdot)$ is linear for all x in $\mathbb{R}^n$ and $\phi(\cdot, y)$ is linear for all y in $\mathbb{R}^p$. Then ϕ is (Frechet)-differentiable at every point of $\mathbb{R}^n \times \mathbb{R}^p$ and for all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^p$, $D\phi(x)(u, v) = \phi(u, y) + \phi(x, v)$ for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$.

3.3.2 The Jacobian matrix of a differentiable map

Theorem 20 *Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^p$ be a mapping which is (Frechet)-differentiable at x. Then the continuous linear map $Df(x)$ is represented by the Jacobian matrix*

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x) & \ldots & \frac{\partial f_p}{\partial x_n}(x) \end{pmatrix}$$

where f_i is the i-th coordinate function of f.

We see that if f is (Frechet)-differentiable at every point of U, then $x \mapsto J_f(x)$ is a mapping from U into the space of $p \times n$ matrices, which is a space of dimension pn.

Definition 39 We shall say that f is of class $\mathcal{C}^1$ on U, or is a $\mathcal{C}^1$ mapping, if f is (Frechet)-differentiable on U and if in addition the derivative $Df : U \to L(\mathbb{R}^n, \mathbb{R}^p)$ is continuous, which is equivalent to assume that pn partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous.

Theorem 21 *Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^p$ be a mapping. If the pn partial derivatives $\frac{\partial f_i}{\partial x_j}$ are defined on U and are continuous, then f is $\mathcal{C}^1$ on U.*

3.3.3 Basic Properties of the Derivative

Proposition 62 *Let U be open in $\mathbb{R}^n$. Let $f, g : U \to \mathbb{R}^p$ be two mappings which are (Frechet)-differentiable at $x \in U$. Then $f + g$ is (Frechet)-differentiable at x and $D(f+g)(x) = Df(x) + Dg(x)$. If c is real number, then cf is (Frechet)-differentiable at x and $D(cf)(x) = cDf(x)$.*

We recall that a bilinear mapping ψ from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^k$ is a mapping such that the partial mapping $\psi(x, \cdot)$ is linear for all x in $\mathbb{R}^n$ and $\psi(\cdot, y)$ is linear for all y in $\mathbb{R}^p$.

Proposition 63 *Let ψ be a bilinear mapping ψ from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^k$. Let U be an open subset of $\mathbb{R}^\ell$ and let $f : U \to \mathbb{R}^n$ and $g : U \to \mathbb{R}^p$ be two (Frechet)-differentiable mappings at $x \in U$. Then $\psi(f, g)$ is differentiable at x and for all $v \in \mathbb{R}^\ell$,*

$$D\psi(f, g)(x)(v) = \psi(Df(x)(v), g(x)) + \psi(f(x), Dg(x)(v))$$

Remark 13 If ψ is the inner product on $\mathbb{R}^n \times \mathbb{R}^n$, we have the following formula for all $v \in \mathbb{R}^\ell$,

$$D(f \cdot g)(x)(v) = Df(x)(v) \cdot g(x) + f(x) \cdot Dg(x)(v)$$

or

$$\nabla(f \cdot g)(x) = Df(x)^t(g(x)) + Dg(x)^t(f(x))$$

where $Df(x)^t$ is the transpose of the linear mapping $Df(x)$ and $Dg(x)^t$ is the transpose of the linear mapping $Dg(x)$.

Example: Let J be an open interval in $\mathbb{R}$ and let $t \mapsto A(t) = (a_{ij}(t))$ and $t \mapsto X(t)$ be two differentiable maps from J into the space of $p \times n$ matrices, and into $\mathbb{R}^n$ respectively. Thus for each t, $A(t)$ is an $p \times n$ matrix, and $X(t)$ is a column vector of dimension n. We can form the product $A(t)X(t)$, and thus the product map $t \mapsto A(t)X(t)$, which is differentiable. Our rule in this special case asserts that $\frac{\partial}{\partial t}A(t)X(t) = A'(t)X(t) + A(t)X'(t)$ where differentiation with respect to t is taken componentwise both on the matrix $A(t)$ and the vector $X(t)$. The product here is the product of a matrix and a vector.

3.3.4 Chain Rule

Proposition 64 (Chain Rule) *Let U be an open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^p$. Let $f : U \to V$ and $g : V \to \mathbb{R}^k$ be two mappings. Let $x \in U$. Assume that f is (Frechet)-differentiable at x and g is (Frechet)-differentiable at $f(x)$. Then $g \circ f$ is (Frechet)-differentiable at x and $D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$.*

Remark 14 $Df(x) : \mathbb{R}^n \to \mathbb{R}^p$ is a linear map, and $Dg(f(x)) : \mathbb{R}^p \to \mathbb{R}^s$ is a linear map, and so these linear maps can be composed, and the composite is a linear map, which is continuous because both $Dg(f(x))$ and $Df(x)$ are continuous. The composed linear map goes from $\mathbb{R}^n$ into $\mathbb{R}^s$, as it should.

Remark 15 In terms of the Jacobian matrix we have: $J_{g\circ f}(x) = J_g(f(x))J_f(x)$, the multiplication being that of matrices.

Corollary 2 *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a (Frechet)-differentiable mapping. Let J be an open interval of $\mathbb{R}$ and let $\varphi : J \to \mathbb{R}^n$ be a differentiable mapping such that $\varphi(t) \in U$ for $t \in J$. Then $f \circ \varphi$ is differentiable on J and $(f \circ \varphi)'(t) = \sum_{i=1}^n \varphi_i'(t)\frac{\partial f}{\partial x_i}(\varphi(t))$ for all $t \in J$.*
In particular if $\varphi(t) = \bar{x} + tu$ for some $\bar{x} \in U$ and $u \in \mathbb{R}^n$, we get $(f \circ \varphi)'(t) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}(\varphi(t)) = \nabla f(\bar{x} + tu) \cdot u$ for all $t \in J$.

3.3.5 The Mean Value Theorem

Theorem 22 *Let U be an open subset of $\mathbb{R}^n$ and f be a differentiable mapping from U to $\mathbb{R}$. Let $\underline{x}$ and $\bar{x}$ two elements of U such that the segment $[\underline{x}, \bar{x}] = \{(1-t)\underline{x} + t\bar{x} \mid t \in [0,1]\} \subset U$. Then, it exists $\xi \in]\underline{x}, \bar{x}[$ such that $f(\bar{x}) - f(\underline{x}) = Df(\xi)(\bar{x} - \underline{x})$.*

Remark 16 This theorem cannot be generalised to a differentiable mapping f taken its value in $\mathbb{R}^p$ with $p > 1$. Indeed, let f from $\mathbb{R}$ to $\mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$. f is $\mathcal{C}^1$ on $\mathbb{R}$. We remark that $f(0) = f(2\pi)$. It does not exists $t \in]0, 2\pi[$ such that $f(2\pi) - f(0) = Df(t)(2\pi)$. Indeed, $Df(t) \neq 0$ for all $t \in \mathbb{R}$.

Nevertheless, we can obtain an upper bound of the norm $f(\bar{x}) - f(\underline{x})$ by using the norm as a linear mapping of $Df(\xi)$ for $\xi \in [\underline{x}, \bar{x}]$.

Theorem 23 *Let U be an open subset of $\mathbb{R}^n$ and f be a differentiable mapping from U to $\mathbb{R}^p$. Let $\underline{x}$ and $\bar{x}$ two elements of U such that the segment $[\underline{x}, \bar{x}] = \{(1-t)\underline{x} + t\bar{x} \mid t \in [0,1]\}$ is included in U. Then, it exists $\xi \in]\underline{x}, \bar{x}[$ such that $\|f(\bar{x}) - f(\underline{x})\|_p \leq \|Df(\xi)\|_{\mathcal{L}} \|\bar{x} - \underline{x}\|_n$.*

Corollary 3 *Let U be an open subset of $\mathbb{R}^n$ and f be a differentiable mapping from U to $\mathbb{R}^p$. Let $\underline{x}$ and $\bar{x}$ two elements of U such that the segment $[\underline{x}, \bar{x}] = \{(1-t)\underline{x} + t\bar{x} \mid t \in [0,1]\}$ is included in U. Then, $\|f(\bar{x}) - f(\underline{x})\|_p \leq \max\{\|Df(\xi)\|_{\mathcal{L}} \mid \xi \in [\underline{x}, \bar{x}]\} \|\bar{x} - \underline{x}\|_n$.*

A first consequence of this theorem is the fact that a differentiable mapping f such that $Df(x)$ is the nul linear mapping for every x is locally constant.

Corollary 4 *Let U be an open subset of $\mathbb{R}^n$ and f be a differentiable mapping from U to $\mathbb{R}^p$. If $Df(x) = 0_{\mathcal{L}}$ for all $x \in U$, the, for all $\bar{x} \in U$, f is constant on the ball $B(\bar{x}, r)$ such that $B(\bar{x}, r) \subset U$.*

Another consequence is the fact that a $\mathcal{C}^1$ mapping is locally Lipschitz continuous.

Corollary 5 *Let U be an open subset of $\mathbb{R}^n$ and f be a continuously differentiable mapping from U to $\mathbb{R}^p$. Let $\bar{x} \in U$ and $r > 0$ such that the closed ball $\bar{B}(\bar{x}, r)$*

is included in U. Then it exists $k \geq 0$ such that for all $(x, x') \in B(\bar{x}, r)^2$, $\|f(x') - f(x)\|_p \leq k\|x' - x\|_n$.

Exercise 37 Compute the partial derivatives of the following mappings

1) $f(x,y) = x(2\ln(x+1) + y + 1) + e^{-y} + 2\ln(x+1) + y$;
2) $f(x,y,z) = x^\alpha y^\beta z^\gamma$; $\alpha > 0$, $\beta > 0$, $\gamma > 0$;
3) $f(x,y,z) = \sqrt{\alpha x + \beta y + \gamma z}$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$;
4) $f(x,y,z) = y(x + x^{\frac{1}{2}} z^{\frac{1}{2}} + z)$;
5) $f(x,y,z) = (\alpha x^\rho + \beta y^\rho + \gamma z^\rho)^{\frac{1}{\rho}}$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$ $\rho > 0$;
6) $f(x,y,z) = \frac{xyz}{x+y+z}$;
7) $f(x,y,z) = e^{\alpha x} e^{\beta y} e^{\gamma z}$;
8) $f(x,y,z) = \frac{z}{(\sqrt{x}+\sqrt{y})^2}$;
9) $f(x,y,z) = \ln(z) - \alpha\ln(x) - \beta\ln(y)$;
10) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$;
11) $(x,y) \in \mathbb{R}^2 \mapsto f(x,y) = x^2 + (x+y-1)^2 + y^2$;
12) $(x,y) \in \mathbb{R}^2 \mapsto f(x,y) = (x+y)^2 + x^4 + y^4$;
13) $(x,y) \in \mathbb{R}^2 \mapsto f(x,y) = 2x^4 - 3x^2y + y^2$;
14) $f(x,y) = x^2 - xy + y^2 + x + y$, where $X = \{(x,y) \in \mathbb{R}^2, x \leq 0, y \leq 0, x + y \geq -3\}$;
15) $f(x,y) = x^2(1+y)^3 + y^4$;
16) $f(x,y) = x^2 - y^2 + y^4/4$;
17) $f(x,y) = x^3 - 3x(1+y^2)$;
18) $f(x,y) = \begin{cases} \frac{xy(1-x)(1-y)}{1-xy} & \text{if} \quad (x,y) \neq (1,1) \\ 0 \quad \text{if} \quad (x,y) = (1,1) \end{cases}$

Exercise 38 Let N be a norm on $\mathbb{R}^n$. Show that N is not differentiable at 0.

Exercise 39 Let f be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Show that f is differentiable on $\mathbb{R}^n$ and $Df(x) = f$ for all $x \in \mathbb{R}^n$.

Exercise 40 Let M be a $n \times p$ matrix. Let f be the mapping from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}$ defined by:

$$f(x, y) = x \cdot My$$

1) Show that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$, $f(x, y) \leq \|M\|_{\mathcal{L}} \|x\| \|y\|$.
2) Using the definition of the derivative show that f is differentiable on $\mathbb{R}^n \times \mathbb{R}^p$ and that the derivative is defined by :

$$Df(x, y)(h, k) = h \cdot My + x \cdot Mk$$

3) Deduce the derivative of the standard inner product on $\mathbb{R}^n$ as a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.

Exercise 41 Let A be a $n \times n$ matrix, b, a vector in $\mathbb{R}^n$ and c a real number. Let f be the mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by:

$$f(x) = x \cdot Ax + b \cdot x + c$$

1) Compute the partial derivatives of f on $\mathbb{R}^n$.
2) Show that the derivatives are continuous.
3) Provide the formula for the derivative of f at each point $\bar{x}$ of $\mathbb{R}^n$.

Exercise 42 Let f be the mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by:

$$f(x) = \|x\|^2 = \sum_{i=1}^{n} x_i^2$$

1) Compute the partial derivatives of f at each point $\bar{x}$.
2) Show that f is differentiable at each point $\bar{x} \in \mathbb{R}^n$ and show that $Df(\bar{x})$ is defined by $Df(\bar{x})(h) = 2\bar{x} \cdot h$.

Exercise 43 Let f be a mapping from $\mathbb{R}^n$ to $\mathbb{R}$. We assume that there exists $c \in \mathbb{R}_+$ and $\alpha > 0$ such that for all $(x, y) \in (\mathbb{R}^n)^2$,

$$|f(y) - f(x)| \leq c\|y - x\|^{1+\alpha}$$

1) Show that the partial derivatives of f at each point of $\mathbb{R}^n$ are vanishing.
2) Deduce that f is constant.

Exercise 44 Let f be a differentiable mapping from $\mathbb{R}^3$ to $\mathbb{R}$. We assume that for all $(x, y, z) \in \mathbb{R}^3$, the three partial derivatives of f at (x, y, z) are non negative. Show that if (x', y', z') satisfies $x' \geq x$, $y' \geq y$ and $z' \geq z$, then $f(x', y', z') \geq f(x, y, z)$.

We now assume that for all $(x, y, z) \in \mathbb{R}^3$ the three partial derivatives of f at (x, y, z) are positive. Show that if (x', y', z') satisfies $x' \geq x$, $y' \geq y$ and $z' \geq z$ with one strict inequality among the three, then $f(x', y', z') > f(x, y, z)$.

Exercise 45 Let $\mathcal{M}_2$ be the space of dimension 4 of the 2×2 matrices. We consider the mapping "determinant" from $\mathcal{M}_2$ to $\mathbb{R}$.

For all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det M = ad - bc$.

1) Compute the partial derivative of the mapping det.
2) Show that the mapping det is differentiable and give its derivative at $M \in \mathcal{M}_2$.
3) Show that $D\det(M) = 0_{\mathcal{L}}$ if and only if $M = 0$.

Exercise 46 Let f and g two differentiable mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$. Let $\bar{x} \in \mathbb{R}^n$. We assume that $f(x) = g(x) + \|x - \bar{x}\|\varepsilon(x)$ where ε is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$ satisfying $\lim_{x \to \bar{x}} \varepsilon(x) = 0_p$. Show that $f(\bar{x}) = g(\bar{x})$ and $Df(\bar{x}) = Dg(\bar{x})$.

Exercise 47 Let f be a differentiable mapping from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^p$. We assume that f is k Lipschitz continuous on U, i.e., $\exists k > 0$, $\forall x, y \in U^2$, $\|f(x) - f(y)\|_p \leq k\|x - y\|_n$. Show that for all x in U, $\|Df(x)\|_{\mathcal{L}} \leq k$.

3.4 Unconstrained Optimisation

Note that the title of this section is somehow misleading since it does not mean that we are maximising the objective function on the whole space $\mathbb{R}^n$ but an open space, for example of vectors with positive coordinates. So, we may have some constraints to define the set of feasible points, but no feasible point lies on the boundary of the feasible set.

3.4.1 First order necessary optimality condition

Let U be an open subset of $\mathbb{R}^n$ and f be a continuously differentiable mapping from U to $\mathbb{R}$. We consider the two following problems:

$$(\mathcal{P}) \max_{x \in U} f(x) \qquad \textit{resp.}\ (\mathcal{Q}) \min_{x \in U} f(x)$$

The first order necessary optimality condition is then:

Theorem 24 *If $\bar{x}$ is a (local) solution of $(\mathcal{P})$ of $(\mathcal{Q})$, then $\nabla f(\bar{x}) = 0$ or, in terms of partial derivative $\frac{\partial f}{\partial x_i}(\bar{x}) = 0$ for all $i = 1, \ldots, n$.*

Proof. We give the proof for a solution of $(\mathcal{P})$. For the problem $(\mathcal{Q})$, it suffices to reverse the inequalities. Let $u \in \mathbb{R}^n \setminus \{0\}$. Since U is open, for $r > 0$ small enough, $B(\bar{x}, r) \subset U$. So, with $\rho = r/\|u\|$, for all $t \in]-\rho, \rho[$, $\bar{x} + tu \in U$. Hence, since $\bar{x}$ is a local solution of $(\mathcal{P})$, $f(\bar{x} + tu) \leq f(\bar{x})$ for all $t \in]-\rho, \rho[$. Hence, $\lim_{t \to 0^+} \frac{f(\bar{x}+tu) - f(\bar{x})}{t} = \nabla f(\bar{x}) \cdot u \leq 0$. For $-u$, we get the same inequality $\nabla f(\bar{x}) \cdot (-u) \leq 0$, hence $\nabla f(\bar{x}) \cdot u = 0$. Since these equality is true for all $u \in \mathbb{R}^n \setminus \{0\}$, we can conclude that $\nabla f(\bar{x}) = 0$. $\square$

Exercise 48 For the following functions, find the critical points where the gradient vanish.

1) $f(x,y) = \ln(1+xy)$, $(x,y) \in \{(x',y') \in \mathbb{R}^2 \mid xy > -1\}$

2) $f(x,y) = xy^2 + xy - 2x - 12y$

3) $f(x,y,z) = -2x^2 - 2xy - xz - \frac{1}{2}y^2 + 2xz - 2z^2 + x - 2y - z$

4) $f(x,y) = x^2y^2 - 4x^2 - y^2$

5) $f(x,y) = 2x^4 + 2x^2y + y^2 - 2x^2 + 1$

6) $f(x,y) = \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{(x-1)^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0),(1,0)\}$

7) $f(x,y) = x(2\ln(x) - y - 1) + e^y$ on $\mathbb{R}_+^* \times \mathbb{R}$

8) $f(x,y) = (x^2 + (x-1)^2)(y-3)^2 + y$

9) $f(x,y) = x^4 - 2x^2y + x^2 + 3y^2 - 2xy - 2y + 3$

10) $f(x,y) = x^4 - x^2 + y^4 + 2xy^2 - 2y^2 - 2x + 2$

11) $f(x,y) = 14x^2 - 6xy + 6y^2$

Exercise 49 Let φ be a non zero linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Let U be an open subset of $\mathbb{R}^n$. Show that the following optimisation problem

$$\left\{ \begin{array}{l} \text{Minimise } \varphi(x) \\ x \in U \end{array} \right. \quad \left\{ \begin{array}{l} \text{Maximise } \varphi(x) \\ x \in U \end{array} \right.$$

have no solution

The following exercise show that the solution of the equation $y = \nabla f(x)$ where y is given and x the unknown is also the solution of an optimisation problem.

Exercise 50 Let f be a continuously differentiable mapping from U an open subset of $\mathbb{R}^n$ to $\mathbb{R}$. Let $y \in \mathbb{R}^n$. Show that if the following optimisation problem

$$\left\{ \begin{array}{l} \text{Maximise } x \cdot y - f(x) \\ x \in U \end{array} \right.$$

has a solution $\bar{x}$ then $y = \nabla f(\bar{x})$.

3.4.2 Second order necessary optimality condition

We are now considering that f is $\mathcal{C}^2$ on U. Then we also get an information on the second derivative of f or the Hessian matrix of f.

Theorem 25 *If $\bar{x}$ is a solution of $(\mathcal{P})$ (resp. $(\mathcal{Q})$ then $\nabla f(\bar{x}) = 0$ and the Hessian matrix of f at $\bar{x}$, $H_f(\bar{x})$, is negative semi-definite (resp. positive semi-definite).*

Proof. We are doing the proof for $(\mathcal{P})$ the maximisation problem. The one for $(\mathcal{Q})$ is the same with the reverse inequalities. Let $u \in \mathbb{R}^n \setminus \{0\}$. As in the previous proof, for $t \in]-\rho, \rho[$, $\bar{x} + tu \in U$. So $f(\bar{x} + tu) \leq f(\bar{x})$. The Taylor development of f at $\bar{x}$ is

$$f(\bar{x} + tu) = f(\bar{x}) + \nabla f(\bar{x}) \cdot u + \frac{t^2}{2} u \cdot H_f(\bar{x})(u) + t^2 \|u\|^2 \eta(\bar{x} + tu)$$

where η is a function from U to $\mathbb{R}$ satisfying $\lim_{x \to \bar{x}} \eta(x) = 0$. SInce $\nabla f(\bar{x}) = 0$ from the first order necessary optimality condition, dividing by t^2, we obtain:

$$0 \geq \frac{1}{2} u \cdot H_f(\bar{x})(u) + \|u\|^2 \eta(\bar{x} + tu)$$

and at the limit at 0^+, since $\lim_{t \to 0^+} \eta(\bar{x} + tu) = 0$, $0 \geq \frac{1}{2} u \cdot H_f(\bar{x})(u)$. This shows that $H_f(\bar{x})$ is negative semi-definite.

Exercise 51 Check for the critical points of Exercise 48 if they satisfy the second order necessary optimality condition.

3.5 Optimisation with linear equality constraints

We now add to the previous problems several linear equality constraints. This means that the set of feasible points is defined as the elements x of the open set U satisfying the following equality:

$$\forall j = 1, \ldots, p, a^j \cdot x = \sum_{i=1}^{n} a_i^j x_i = b^j$$

where a^1, ... ,a^p are p given vectors of $\mathbb{R}^n$ and $b^1, \ldots, b^p$, p given real numbers. In a matrix form, we can define the $p \times n$ matrix A with the rows a^j and the vector $b = (b^1, \ldots, b^p)$ of $\mathbb{R}^p$ and the constraints are now summarised in the compact form $Ax = b$.

So, we are considering the following optimisation problems:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ Ax = b \text{ or } a^j j \cdot x = b^j \text{ for all } j = 1, \ldots, p \\ x \in U \end{array} \right.$$

$$(\mathcal{Q}) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ Ax = b \text{ or } a^j j \cdot x = b^j \text{ for all } j = 1, \ldots, p \\ x \in U \end{array} \right.$$

The first order necessary optimality condition is then:

Theorem 26 *If $\bar{x}$ is a (local) solution of $(\mathcal{P})$ or $(\mathcal{Q})$, then there exists a vector of Lagrange multipliers λ in $\mathbb{R}^p$ such that $\nabla f(\bar{x}) = \sum_{j=1}^{p} \lambda^j a^j$ or, in terms of partial derivative $\frac{\partial f}{\partial x_i}(\bar{x}) = \sum_{j=1}^{p} \lambda^j a_i^j$ for all $i = 1, \ldots, n$.*

Note that $\bar{x}$ satisfies also the constraints $a^j \cdot \bar{x} = b^j$ for all j or $A\bar{x} = b$.

Proof. Let $\mathrm{Ker}A = \{u \in \mathbb{R}^n \mid Au = 0\}$. Let $\bar{x}$ a solution of $(\mathcal{P})$. The proof is similar for a solution of $(\mathcal{Q})$.

For all $u \in \mathrm{Ker}A \setminus \{0\}$, for all $t \in \mathbb{R}$, $A(\bar{x} + tu) = b$ and for all t in a small enough interval around 0, $\bar{x} + tu \in U$. So, $f(\bar{x} + tu) \geq f(\bar{x})$. So the directional derivative of f at $\bar{x}$ in the direction u

$$f'(\bar{x}, u) = \lim_{t \to 0^+} \frac{f(\bar{x} + tu) - f(\bar{x})}{t} = \nabla f(\bar{x}) \cdot u$$

is non positive. Since this is also true for $-u$ thanks to the fact that $\mathrm{Ker}A$ is a linear subspace, $\nabla f(\bar{x}) \cdot u = 0$. Since, this is true for all $u \in \mathrm{Ker}A$, $\nabla f(\bar{x})$ belongs to $(\mathrm{Ker}A)^\perp$. Hence, since $(\mathrm{Ker}A)^\perp = \mathrm{Im}A^t$, there exists a vector $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) = A^t \lambda = \sum_{j=1}^p \lambda^j a^j$ since the vectors a^j are the column vectors of A^t. □

With the same proof as above, we can obtain the following second-order necessary optimality condition. It suffices to remark that $\nabla f(\bar{x}) \cdot u = 0$ for all $u \in \mathrm{Ker}A$.

Theorem 27 *If $\bar{x}$ is a solution of $(\mathcal{P})$ (resp. $(\mathcal{Q})$ then the Hessian matrix of f at $\bar{x}$, $H_f(\bar{x})$, is negative semi-definite (resp. positive semi-definite) on* Ker$A$*, that is, for all $u \in$* Ker$A$*, $u \cdot H_f(\bar{x})u \leq$ (resp. $\geq$)0.*

Remark 17 It is an exercise of Euclidean algebra to prove that the matrix $H_f(\bar{x})$, is negative semi-definite (resp. positive semi-definite) on Ker$A$ if and only if the $p \times p$ matrix $AH_f(\bar{x})A^t$ is negative semi-definite.

Exercise 52 Let A be a $p \times n$ matrix, b a vector of $\mathbb{R}^p$. We assume that $F = \{x \in \mathbb{R}^n \mid Ax = b\}$ is non empty. Let $\bar{x} \in \mathbb{R}^n$ a given vector. We consider the following problem consisting of finding the closest point in F to $\bar{x}$.

$$\begin{cases} \text{Minimise } \|y - \bar{x}\|^2 \\ Ay = b \end{cases}$$

1) Show that this problem has a solution.
1) Let$\bar{y}$ be a solution. Show that $\bar{y} - x$ is orthogonal to the kernel of A.
2) Let $a \in \mathbb{R}^n \setminus \{0\}$. The matrix A is the $1 \times n$ matrix whose unique row is a, i.e. $Ax = v \cdot x$. Let b be a real number.

a) Compute explicitly the unique solution of the problem and the associated multiplier.

b) Compute the value of the problem, which depends on b. Show that the value function is differentiable with respect to b and show that the derivative of the value function with respect to b is equal to the multiplier.

Exercise 53 1) Using the previous exercise, show that if $D = \{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0\}$ with $(a, b) \neq (0, 0)$, then the distance of (x, y) to the line D is equal to:

$$\frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

2) Using the previous exercise, show that if $P = \{(x, y, z) \in \mathbb{R}^3 \mid ax+by+cz+d = 0\}$ with $(a, b, c) \neq (0, 0, 0)$, then the distance of (x, y, z) to the plan P is equal to:

$$\frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercise 54 Let $\alpha \in \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid x_i > 0, \forall i = 1, \dots, n\}$.The function f from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ is defined by

$$f(x) = \sum_{i=1}^{n} \alpha_i \ln(x_i)$$

where $\ln(x_i)$ is the standard logarithm function of x_i. Let $\beta \in \mathbb{R}^n_{++}$. We consider the following optimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ \sum_{i=1}^{n} \beta_i x_i = 1 \\ x \in \mathbb{R}^n_{++} \end{array} \right.$$

Compute the unique point satisfying the first order necessary condition. Are the second order necessary condition satisfied at this point?

Exercise 55 Let us consider the following optimisation problem:

$$(P) \left\{ \begin{array}{l} \text{Minimise } 5x^2 + 4xy + y^2 \\ 3x + 2y = 5 \end{array} \right.$$

1) First method: solve the problem by reducing it to a one dimensional optimisation problem.
2) Second method: write the first order necessary condition and find the solutions and the multipliers.

Chapter 4

Optimization with equality contraints and sensitiviy analysis

We consider an objective function f from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and p constraints, which are represented by p functions g_i from U to $\mathbb{R}$. We consider the following optimisation problems:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \\ x \in U \end{array} \right. \qquad (\mathcal{Q}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \\ x \in U \end{array} \right.$$

To find the first order necessary conditions of optimality, we need to have a convenient description of the set $\{x \in U \mid g_i(x) = 0, \forall i = 1, \ldots, p\}$. For this, we will use the implicit function theorem.

4.1 Introduction to the implicit function theorem

Let us first consider the case where we have a unique linear constraint: $g_1(x) = a_1x_1 + a_2x_2 + \ldots + a_nx_n + b_1$. Then, if $a_n \neq 0$, for all $x_1, x_2, \ldots, x_{n-1} \in \mathbb{R}^{n-1}$, we have a unique x_n such that $g_1(x_1, x_2, \ldots, x_n) = 0$, which is given by the simple formula: $x_n = \varphi(x_1, x_2, , \ldots, x_{n-1}) = b_1 - (1/a_n)(a_1x_1 + a_2x_2 + \ldots + a_{n-1}x_{n-1})$. So, the set $S = \{x \in \mathbb{R}^n \mid g_1(x) = 0\}$, implicitly described by g_1, is explicitly described by the function φ, as $S = \{x \in \mathbb{R}^n \mid x_n = \varphi(x_1, x_2, \ldots, x_{n-1})\}$.

We remark that φ is a differentiable mapping and

$$\frac{\partial \varphi}{\partial x_i}(x_1, x_2, \ldots, x_{n-1}) = -(a_i/a_n) = -\left(\frac{\partial g_1}{\partial x_i}(x_1, x_2, , \ldots, x_n) / \frac{\partial g_1}{\partial x_n}(x_1, x_2, , \ldots, x_n)\right)$$

The implicit function theorem tells us that we can generalise this result when g_1 is $\mathcal{C}^1$ locally around a point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ such that $g_1(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = 0$ and $\frac{\partial g_1}{\partial x_n}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \neq 0$. For example, let $g_1(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$. Then let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ such that $g_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 0$ and $x_3 \neq 0$. We remark that $\frac{\partial g_1}{\partial x_3}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 2\bar{x}_3 \neq 0$. In a neighborhood of $(\bar{x}_1, \bar{x}_2)$ in $\mathbb{R}^2$, we have

$g_1(x_1, x_2, \varphi(x_1, x_2)) = 0$ and $\varphi(\bar{x}_1, \bar{x}_2) = \bar{x}_3$ with $\varphi(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$ if $x_3 > 0$ or $\varphi(x_1, x_2) = -\sqrt{1 - x_1^2 - x_2^2}$ if $x_3 < 0$. So, the unit sphere $S = \{x \in \mathbb{R}^3 \mid g_1(x) = 0\}$, implicitly described by g_1, is explicitly described by the function φ around the point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, as $S \cap [B(\bar{x}_1, \bar{x}_2, r) \times]\bar{x}_3 - r, \bar{x}_3 + r[] = \{x \in \mathbb{R}^n \mid (x_1, x_2) \in B(\bar{x}_1, \bar{x}_2, r), x_3 = \sqrt{1 - x_1^2 - x_2^2}\}$ if $\bar{x}_3 > 0$.

We also easily checks that

$$\frac{\partial \varphi}{\partial x_i}(\bar{x}_1, \bar{x}_2) = \frac{-\bar{x}_i}{\sqrt{1 - \bar{x}_1^2 - \bar{x}_2^2}} = -\frac{\bar{x}_i}{\bar{x}_3} = -\left(\frac{\partial g_1}{\partial x_i}(\bar{x}_1, \bar{x}_2, \bar{x}_3) / \frac{\partial g_1}{\partial x_3}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\right).$$

We also remark that if $\bar{x}_3 = 0$, then $\frac{\partial g_1}{\partial x_3}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 2\bar{x}_3 = 0$ and $\bar{x}_1^2 + \bar{x}_2^2 = 1$. So, it does not exist a function φ on a neighborhoud of $(\bar{x}_1, \bar{x}_2)$ such that $g_1(x_1, x_2, \varphi(x_1, x_2)) = 0$ since it does not exists a real number x_3 satisfying $g_1(x_1, x_2, x_3) = 0$ if we increase even a little bit x_1 and x_2 starting from $(\bar{x}_1, \bar{x}_2)$. Formally if $t > 0$, there is no solution in x_3 to $g_1(\bar{x}_1 + t, \bar{x}_2 + t, x_3) = 0$. So, in that case, we can not describe the set using the variables (x_1, x_2).

Now, we remark the following consequence of this explicit description. Let $u \in \mathbb{R}^n \setminus \{0\}$ such that $\nabla g_1(\bar{x}) \cdot u = \sum_{i=1}^n \frac{\partial g_1}{\partial x_i}(\bar{x}) u_i = 0$. Then,

$$u_n = -\left(1 / \frac{\partial g_1}{\partial x_n}(\bar{x})\right) \sum_{i=1}^{n-1} \frac{\partial g_1}{\partial x_i}(\bar{x}) u_i$$

Let us consider the mapping ψ from an open interval around 0 in $\mathbb{R}$ to $\mathbb{R}^n$ defined by $\psi(t) = (\bar{x}_1 + tu_1, \bar{x}_2 + tu_2, \ldots, \bar{x}_{n-1} + tu_{n-1}, \varphi(\bar{x}_1 + tu_1, \bar{x}_2 + tu_2, \ldots, \bar{x}_{n-1} + tu_{n-1}))$. Clearly $g_1(\psi(t)) = 0$ for all t from the property of the mapping φ. Now, we remark that

$$\psi'(0) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ \sum_{i=1}^{n-1} \frac{\partial \varphi}{\partial x_i}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}) u_i \end{pmatrix}$$

From the above computation of the partial derivatives of φ, we get that:

$$\sum_{i=1}^{n-1} \frac{\partial \varphi}{\partial x_i}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}) u_i = -\left(1 / \frac{\partial g_1}{\partial x_n}(\bar{x})\right) \sum_{i=1}^{n-1} \frac{\partial g_1}{\partial x_i}(\bar{x}) u_i = u_n$$

So, $\psi'(0) = u$. Hence, for all vectors u orthogonal to $\nabla g_1(\bar{x})$, we can draw a differentiable path ψ on the set $S = \{x \in \mathbb{R}^n \mid g_1(x) = 0\}$ such that $\psi'(0) = u$. This property will allow us to derive a first order necessary condition of optimality. Indeed, if $\bar{x}$ is a (local) solution of the above maximisation problem $(\mathcal{Q})$. Then, $f(\psi(t)) \leq f(\bar{x}) = f(\psi(0))$. So 0 is a maximum of the real function $f(\psi(t))$. Hence the derivative of the function is 0 at 0. From the theorem on the composition of the derivatives, we deduces that $\nabla f(\psi(0)) \cdot \psi'(0) = \nabla f(\bar{x}) \cdot u = 0$. So, $\nabla f(\bar{x})$ is orthogonal to all vectors u, which are orthogonal to $\nabla g_1(\bar{x})$. In other words,

$\nabla f(\bar{x})$ is orthogonal to all vectors u in the kernel of the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $v \to \sum_{i=1}^n \frac{\partial g_1}{\partial x_i}(\bar{x}) v_i = \nabla g_1(\bar{x}) \cdot v$. Hence $\nabla f(\bar{x})$ belongs to the image of the transpose of this linear mapping, which is the mapping from $\mathbb{R}$ to $\mathbb{R}^n$ defined by $t \to t\nabla g_1(\bar{x})$. Consequently, there exists a "Lagrange multiplier" λ such that $\nabla f(\bar{x}) = \lambda \nabla f(\bar{x})$, or, in other words, $\nabla f(\bar{x})$ is proportional to $\nabla g_1(\bar{x})$.

Now let us come to the case of several linear constraints. Let us consider the case where we have p linear constraints: $g_i(x) = a_1^i x_1 + a_2^i x_2 + \ldots + a_n^i x_n + b_i$. Let us call G the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by $G(x) = (a_1^i x_1 + a_2^i x_2 + \ldots + a_n^i x_n)_{i=1}^p$. The entry on the ith row and the j column of the matrix A of this mapping is a_j^i. So, the set $S = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \forall i = 1, \ldots, p\}$ is equal to $\{x \in \mathbb{R}^n \mid G(x) = -b\}$, that is the set of inverse images of $-b$ by G.

Let us assume that the $p \times p$ matrix D extracted from A by choosing the last p columns is a regular matrix. We call B the the $(n-p) \times p$ matrix extracted from A by choosing the first $(n-p)$ columns. So $A = (B \vdots D)$.

If we choose arbitrarily $(x_1, x_2, \ldots, x_{n-p}) \in \mathbb{R}^{n-p}$, then there exists a unique vector $(x_{n-p+1}, \ldots, x_n)$ in $\mathbb{R}^p$ such that

$$G(x_1, x_2, \ldots, x_{n-p}, x_{n-p+1}, \ldots, x_n) = B(x_1, x_2, \ldots, x_{n-p}) + D(x_{n-p+1}, \ldots, x_n) = b$$

which is given by the following formula:

$$(x_{n-p+1}, \ldots, x_n) = -D^{-1}(b + B(x_1, x_2, \ldots, x_{n-p}))$$

So the mapping φ from $\mathbb{R}^{n-p}$ to $\mathbb{R}^p$ defined by $-D^{-1}(b + B(x_1, x_2, \ldots, x_{n-p}))$ describes explicitely the set S in the sense that

$$S = \{x \in \mathbb{R}^n \mid (x_{n-p+1}, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_{n-p})\}$$

We remark that the function φ is differentiable and the derivative $D^{-1}B$ is computable from the partial derivatives of G.

The implicit function theorem is a generalisation of this properties for non-linear mapping. Let us illustrate it by an example. Let f from $U = \{x \in \mathbb{R}^4 \mid x_3 \neq 0\}$ to $\mathbb{R}^2$ defined by:

$$f(x_1, x_2, x_3, x_4) = \left(x_1 + \frac{x_4}{x_3^2}, x_2 - \frac{1}{x_3} \right)$$

We remark that $f(0, -1, -1, 0) = 0$. The derivative of f is:

$$J_f(x) = \begin{pmatrix} 1 & 0 & -\frac{2x_4}{x_3^3} & \frac{1}{x_3^2} \\ 0 & 1 & \frac{1}{x_3^2} & 0 \end{pmatrix}$$

We remark that the 2×2 matrix $D(x)$ extracted from $J_f(x)$ by choosing the two last columns

$$D(x) = \begin{pmatrix} -\frac{2x_4}{x_3^3} & \frac{1}{x_3^2} \\ \frac{1}{x_3^2} & 0 \end{pmatrix}$$

is regular. In particular:

$$D(0,-1,-1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D^{-1}(0,-1,-1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So, we can find a mapping φ from a neighborhoud of $(0,-1)$ to $\mathbb{R}^2$ such that $\varphi(0,-1) = (-1,0)$ and $f(x_1, x_2, \varphi(x_1, x_2)) = 0$. We can do the computation and we find

$$\varphi(x_1, x_2) = \left(\frac{1}{x_2}, -\frac{x_1}{x_2^2} \right)$$

We can compute the derivative of φ.

$$J_\varphi(x_1, x_2) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix} \qquad J_\varphi(0,-1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Let $B(x)$ be the matrix extracted from $J_f(x)$ by choosing the two first columns

$$B(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As in the linear case, we remark that

$$J_\varphi(0,-1) = -D^{-1}(0,-1,-1,0)B(0,-1,-1,0)$$

So, even if, in the general case, we are not able to find a closed form for the mapping φ, the theorem states that this mapping exists, it is continuously differentiable and the Jacobian matrix can be computed explicitly starting from the Jacobian matrix of f. We also remark the following property of the kernel of the Jacobian matrix $J_f(0,-1,-1,0)$. If u belongs to $\mathrm{Ker}J_f(0,-1,-1,0)$, then,

$$B(0,-1,-1,0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D(0,-1,-1,0) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = -D^{-1}(0,-1,-1,0)B(0,-1,-1,0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = J_\varphi(0,-1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So, the kernel of $J_f(0,-1,-1,0)$ is the graph of $J_\varphi(0,-1)$, that is:

$$\mathrm{Ker}J_f(0,-1,-1,0) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = J_\varphi(0,-1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

4.2 The Implicit Function Theorem

We formally state the implicit function theorem. We consider a $\mathcal{C}^1$ mapping g from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^p$. We denote g_i, $i = 1, \ldots, p$ the component mappings of g. We recall that the row of the Jacobian matrix of g, $Jg(x)$ are the gradient vectors of the component mappings $\nabla g_i(x)$. So, $Jg(x)$ is of rank p if and only if the gradients vectors $(\nabla g_i(x))_{i=1}^p$ are linearly independent. For the sake of simpler notation, we will assume that the p last columns of the matrix $Jg(x)$ are linearly independent. We denote $Jg^{1;n-p}(x)$ the matrix extracted from $Jg(x)$ by choosing the first $n-p$ columns and by $Jg^{n-p+1;n}(x)$ the square matrix extracted from $Jg(x)$ by choosing the last p columns. So $Jg(x) = (Jg^{1;n-p}(x) \vdots Jg^{n-p+1;n}(x))$. So for all u in $\mathbb{R}^n$, $Jg(x)u = Jg^{1;n-p}(x)u^{1;n-p} + Jg^{n-p+1,n}(x)u^{n-p+1;n}$, where $u^{1;n-p}$ is the $n-p$ dimensional vector with the first $n-p$ components of u and $u^{n-p+1;n}$ is the p dimensional vector with the last p components of u.

Theorem 28 *(Implicit Function Theorem) Let g be a $\mathcal{C}^1$ mapping from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^p$. Let $\bar{x} \in U$ and $\bar{z} = g(\bar{x})$. We assume that the p square matrix $Jg^{n-p+1,n}(\bar{x})$ is regular. Then, it exists an open subset V_1 of $\mathbb{R}^{n-p}$ containing $\bar{x}^{1;n-p}$, an open subset V_2 of $\mathbb{R}^p$ containing $\bar{z}$, an open subset W of $\mathbb{R}^p$ containing $\bar{x}^{n-p+1;n}$ and a $\mathcal{C}^1$ mapping φ from $V_1 \times V_2$ to W such that:*

1) $\varphi(\bar{x}^{1;n-p}, \bar{z}) = \bar{x}^{n-p+1;n}$;

2) for all $(\xi, z) \in V_1 \times V_2$, $g(\xi, \varphi(\xi, z)) = z$;

3) for all $(\xi, \zeta, z) \in V_1 \times W \times V_2$ such that $z = g(\xi, \zeta)$, then $\zeta = \varphi(\xi, z)$.

Furthermore, the Jacobian matrix of φ at $(\bar{x}^{1;n-p}, \bar{z})$ is defined as follows:

$$J\varphi(\bar{x}^{1;n-p}, \bar{z}) = \left[Jg^{n-p+1;n}(\bar{x})\right]^{-1} \left(-Jg^{1;n-p}(\bar{x}) \vdots \mathrm{Id}_p\right)$$

In most of the cases, we do not use the dependency with respect to z of φ and we just consider the partial function $\varphi(\cdot, \bar{z})$.

Example. Let us consider the mapping $g(x, y) = e^{x-y} - x - y - 1$ from $\mathbb{R}^2$ to $\mathbb{R}$. We remark that $g(0, 0) = 0$. Furthermore, $\frac{\partial g}{\partial y}(0, 0) = -2 \neq 0$. So, it exists an open interval W of 0 in $\mathbb{R}$, an open interval V of 0 in $\mathbb{R}$ and a $\mathcal{C}^1$ function ψ from W to V such that $\psi(0) = 0$, $e^{x-\psi(x)} - x - \psi(x) - 1 = 0$ for all $x \in W$ and $\psi(x)$ is the unique solution in V of the equation $e^{x-y} - x - y - 1 = 0$. We can compute the derivative of ψ at 0 which is $-\frac{-1}{2} \times 0 = 0$.

For our application in optimisation, we will use the following corollary of the Implicit Function Theorem.

Corollary 6 *Let g be a $\mathcal{C}^1$ mapping from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^p$. Let $\bar{x} \in U$ and $\bar{z} = g(\bar{x})$. We assume that the Jacobian square matrix $Jg(x)$ is of rank p. Then, for all u in the kernel of $Jg(\bar{x})$, that is satisfying $Jg(\bar{x})(u) = 0$, there exists a $\mathcal{C}^1$ mapping ψ from an open interval I in $\mathbb{R}$ containing 0 to $\mathbb{R}^n$ such that*

1) $\psi(0) = \bar{x}$;

2) $\psi'(0) = u$*;*

3) for all $t \in I$*,* $g(\psi(t)) = \bar{z}$*;*

Proof. We will assume that the p square matrix $Jg^{n-p+1,n}(x)$ is regular. If it is not the case, we just do a permutation of the components, that is a permutation of the columns of the matrix $Jg(\bar{x})$ so that the last p columns be linearly independent. This is always possible since $Jg(x)$ is of rank p so it has p linearly independent columns among its n columns. Now, we use the mapping φ given by the Implicit Function Theorem and we define ψ as follows:

$$\psi(t) = (\bar{x}^{1;n-p} + tu^{1;n-p}, \varphi(\bar{x}^{1;n-p} + tu^{1;n-p}, \bar{z}))$$

The mapping ψ is defined on an open interval I of 0 in $\mathbb{R}$ since φ is defined on an open set containing $(\bar{x}^{1;n-p}, \bar{z})$. $\psi(0) = (\bar{x}^{1;n-p}, \varphi(\bar{x}^{1;n-p}, \bar{z})) = \bar{x}$ since $\varphi(\bar{x}^{1;n-p}, \bar{z}) = \bar{x}^{n-p+1;n}$. For all $t \in I$, $g(\psi(t)) = g((\bar{x}^{1;n-p} + tu^{1;n-p}, \varphi(\bar{x}^{1;n-p} + tu^{1;n-p}, \bar{z})) = \bar{z}$ from the second property of φ in the Implicit Function Theorem.

Let us compute now $\psi'(0)$. From the composition of the derivatives and the formula defining ψ,

$$\psi'(0) = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-p} \\ J\varphi(\bar{x}^{1;n-p}, \bar{z}) \begin{pmatrix} u^{1;n-p} \\ 0_p \end{pmatrix} \end{pmatrix}$$

From the implicit function Theorem,

$$J\varphi(\bar{x}^{1;n-p}, \bar{z}) \begin{pmatrix} u^{1;n-p} \\ 0_p \end{pmatrix} = -\left[Jg^{n-p+1;n}(\bar{x})\right]^{-1} Jg^{1;n-p}(\bar{x})u^{1;n-p}$$

Since u is in the kernel of $Jg(\bar{x})$, we have:

$$Jg(\bar{x})u = Jg^{1;n-p}(\bar{x})u^{1;n-p} + Jg^{n-p+1;n}(\bar{x})u^{n-p+1;n} = 0$$

so

$$-\left[Jg^{n-p+1;n}(\bar{x})\right]^{-1} Jg^{1;n-p}(\bar{x})u^{1;n-p} = u^{n-p+1;n}$$

and finally,

$$\psi'(0) = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-p} \\ u^{n-p+1;n} \end{pmatrix} = u$$

□

4.3 First order necessary optimality condition

Let us come back to our optimisation problem with equality constraints.

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ g_i(x) = 0, \ i = 1, \ldots, p \\ x \in U \end{cases} \qquad (\mathcal{Q}) \begin{cases} \text{Maximise } f(x) \\ g_i(x) = 0, \ i = 1, \ldots, p \\ x \in U \end{cases}$$

Using the results presented above, we can state the following first order necessary optimality conditions.

Proposition 65 *Let us assume that the mappings f and g_i are $\mathcal{C}^1$ and that $\bar{x}$ is a (local) solution of the problem $(\mathcal{P})$ (resp. $(\mathcal{Q})$). So if the gradient vectors $(\nabla g_i(\bar{x}))_{i=1}^p$ are linearly independent, there exists a vector of Lagrange multipliers $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) + \sum_{i=1}^p \lambda_i \nabla g_i(\bar{x}) = 0$.*

In other words, the gradient vector of the objective function is a linear combination of the gradient vectors of the constraints.

Example. Let $f(x, y) = x^2 - y$ and $g(x, y) = x^2 + y^2 - 1$. Let us consider the following optimisation problem:

$$\begin{cases} \text{Minimise } f(x, y) \\ g(x, y) = 0 \\ (x, y) \in \mathbb{R}^2 \end{cases}$$

We remark that this problem has a solution since the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is closed and bounded and the mapping f is continuous. We remark that for all (x, y) such that $g(x, y) = 0$, $\nabla g(x, y) \neq 0$. We look for the points $(\bar{x}, \bar{y})$ such that $g(\bar{x}, \bar{y}) = 0$ and it exists a multiplier $\lambda \in \mathbb{R}$ such that $\nabla f(\bar{x}, \bar{y}) = \lambda \nabla g(\bar{x}, \bar{y})$. We find four points: $(0, 1)$ with $\lambda = -\frac{1}{2}$; $(0, -1)$ with $\lambda = \frac{1}{2}$; $(\frac{\sqrt{3}}{2}, \frac{-1}{2})$ with $\lambda = 1$; $(\frac{-\sqrt{3}}{2}, \frac{-1}{2})$ with $\lambda = 1$. From the first order necessary condition, we know that the solution(s) are among these four points. We compute $f(0, 1) = -1$, $f(0, -1) = 1$, $f(\frac{\sqrt{3}}{2}, \frac{-1}{2}) = \frac{5}{4}$, $f(\frac{-\sqrt{3}}{2}, \frac{-1}{2}) = \frac{5}{4}$ and we deduce that the solution is $(0, 1)$.

Proof. The proof is just the precise argument sketched above in the introduction. We do it for the minimisation problem $(\mathcal{P})$. Let $g(x) = (g_i(x))_{i=1}^p$ the mapping from U to $\mathbb{R}^n$. Since the gradient vectors $(\nabla g_i(\bar{x}))_{i=1}^p$, which are the rows of the matrix $Jg(\bar{x})$, are linearly independent, $Jg(\bar{x})$ is of rank p. We apply the corollary of the Implicit function Theorem and, for all $u \in \mathrm{Ker} Jg(\bar{x})$, there exists a $\mathcal{C}^1$ mapping ψ from an open interval I in $\mathbb{R}$ containing 0 to $\mathbb{R}^n$ such that

1) $\psi(0) = \bar{x}$;

2) $\psi'(0) = u$;

3) for all $t \in I$, $g(\psi(t)) = 0$;

So, $f(\psi(t)) \geq f(\bar{x}) = f(\psi(0))$. Hence, 0 is a minimum of $f \circ \psi$, hence its derivative is equal to 0. From the composition of the derivatives, we deduce that $\nabla f(\bar{x}) \cdot u = 0$. Consequently, $\nabla f(\bar{x})$ is orthogonal to all vectors of $\mathrm{Ker} Jg(\bar{x})$, hence it belongs to the range of the transpose of $Jg(\bar{x})$. So, there exists a vector $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) + Jg^t(\bar{x})\lambda = 0$. If we develop this formula, we get $\nabla f(\bar{x}) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\bar{x}) = 0$. □

Remark 18 Note that the result does not hold if we do not assume the linear independence of the gradient vectors $(\nabla g_i(\bar{x}))_{i=1}^{p}$. Indeed, let us consider the following minimisation problem:

$$\begin{cases} \text{Minimise } f(x,y) = x + y \\ g_1(x,y) = (x-1)^2 + y^2 - 1 = 0 \\ g_2(x,y) = (x+1)^2 + y^2 - 1 = 0 \\ (x,y) \in \mathbb{R}^2 \end{cases}$$

We remark that the set $\{(x,y) \in \mathbb{R}^2 \mid g_1(x,y) = g_2(x,y) = 0\}$ is the singleton $(0,0)$. So the solution of the problem is $(0,0)$. Now, $\nabla f(0,0) = (1,1)$, $\nabla g_1(0,0) = (-2,0)$ and $\nabla g_2(0,0) = (2,0)$. So it does not exist a vector $\lambda \in \mathbb{R}^2$ such that $\nabla f(0,0) = \lambda_1 \nabla g_1(0,0) + \lambda_2 \nabla g_2(0,0)$. This is due to the fact that $\nabla g_1(0,0)$ is colinear to $\nabla g_2(0,0)$, so the two vectors are not linearly independent.

4.4 Lagrangian function and second order necessary condition

In this section, we consider the problem $(\mathcal{P})$

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ g_i(x) = 0, \ i = 1, \ldots, p \\ x \in U \end{cases}$$

and we left the reader adapt the following result to the problem $(\mathcal{Q})$.

Definition 40 The Lagrangian function $\mathcal{L}$ associated to the problem $(\mathcal{P})$ is the function from $U \times \mathbb{R}^p$ to $\mathbb{R}$ defined by:

$$\mathcal{L}(x,\lambda) = f(x) + \sum_{i=1}^{p} \lambda_i g_i(x)$$

We remark that the first order necessary optimality conditions can be written as follows: $\nabla_x \mathcal{L}(\bar{x},\bar{\lambda}) = \nabla f(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$ and for all $i = 1, \ldots, p$, $\frac{\partial \mathcal{L}}{\partial \lambda_i}(\bar{x},\bar{\lambda}) = g_i(\bar{x}) = 0$. In other words, all partial derivatives of the Lagrangian function vanishes at $(\bar{x},\bar{\lambda})$.

We also remark that for all x satisfying the constraints $g_i(x) = 0$ for all i, $\mathcal{L}(x,\lambda) = f(x)$. So if $\bar{x}$ is a (local) minimum of the function $\mathcal{L}(\cdot,\bar{\lambda})$ on U, then $\bar{x}$ is a (local) solution of the problem $(\mathcal{P})$. The converse is not true.

We now state the second order necessary condition of optimality. We denote by $A(x)$, the tangent space to the feasibility set $\{x \in U \mid g_i(x) = 0, \forall i = 1, \ldots, p\}$, that is:

$$A(x) = \{u \in \mathbb{R}^n \mid \nabla g_i(x) \cdot u = 0, \forall i = 1, \ldots, p\}$$

Proposition 66 *We assume that the functions f and g_i, $i = 1, \ldots, p$, are $\mathcal{C}^2$ on U. Let $\bar{x}$ be a local solution of the problem $(\mathcal{P})$. We assume that the vectors $(\nabla g_i(\bar{x}))_{i=1}^p$ are linearly independent. Let $\bar{\lambda} \in \mathbb{R}^p$ be the multiplier associated to $\bar{x}$. Then, for all $u \in A(\bar{x})$,*

$$u \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda})(u) = u \cdot \left(Hf(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i H g_i(\bar{x}) \right)(u) \geq 0$$

Proof. Let $u \in A(\bar{x})$, $u \neq 0$. From the corollary of the Implicit Function Theorem, it exists a $\mathcal{C}^1$ mapping ψ from an open interval of $\mathbb{R}$ containing 0 to $\mathbb{R}^n$ such that $\psi(0) = \bar{x}$, $\psi'(0) = u$, $g_i(\psi(t)) = 0$ for all $i = 1, \ldots, p$. So, for all t, $\mathcal{L}(\psi(t), \bar{\lambda}) = f(\psi(t)) \geq f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$. Let us use a second order Taylor expansion of the mapping $\mathcal{L}(\cdot, \bar{\lambda})$ around $\bar{x}$:

$$\begin{array}{ll} 0 & \leq \nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) \cdot (\xi(t) - \bar{x}) + \frac{1}{2}(\xi(t) - \bar{x}) \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda})(\xi(t) - \bar{x}) \\ & +\|\xi(t) - \bar{x}\|^2 \bar{\eta}(\xi(t) - \bar{x}) \\ & = \frac{1}{2}(\xi(t) - \bar{x}) \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(\xi(t) - \bar{x}) + \|\xi(t) - \bar{x}\|^2 \bar{\eta}(\xi(t) - \bar{x}) \end{array}$$

with $\lim_{x \to 0} \bar{\eta}(x) = 0$. Dividing by t^2 and taking the limit at 0^+, noticing that $\lim_{t \to 0^+} \frac{\xi(t) - \bar{x}}{t} = u$, we obtain

$$0 \leq u \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(u)$$

which is the desired result. $\square$

Exercise 56 Let $U = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x_i > -1\}$. The function f from U to $\mathbb{R}$ is defined by

$$f(x) = \sum_{i=1}^{n} \ln(x_i + 1)$$

where $\ln(x_i + 1)$ is the natural logarithm of $x_i + 1$. we consider the following optimisation problem :

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ \sum_{i=1}^n x_i = 0 \\ x \in U \end{array} \right.$$

Show that there exists a unique point satisfying the first order necessary condition.

Exercise 57 Let (f^i) be n differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let E be the linear subspace of $\mathbb{R}^n$ defined by:

$$E = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0\}$$

Let $\bar{x}$ be a solution of the following optimisation problem:

$$\begin{cases} \text{Minimise } \sum_{i=1}^{n} f^i(x_i) \\ x \in E \end{cases}$$

Show that for all $i = 2, \ldots, n$, $(f^i)'(\bar{x}_i) = (f^1)'(\bar{x}_1)$.

Exercise 58 Let $\alpha \in \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid x_i > 0, \forall i = 1, \ldots, n\}$. The function f from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ is defined by:

$$f(x) = \sum_{i=1}^{n} \alpha_i \ln(x_i)$$

where $\ln(x_i)$ is the natural logarithm of x_i. Let $\beta \in \mathbb{R}^n_{++}$. We consider the following optimisation problem:

$$(\mathcal{P}) \begin{cases} \text{Maximise } f(x) \\ \sum_{i=1}^{n} \beta_i x_i = 1 \\ x \in \mathbb{R}^n_{++} \end{cases}$$

Compute the unique point satisfying the first order necessary condition.

Exercise 59 Let us consider the following problem:

$$(P) \begin{cases} \text{Minimise } 5x^2 + 4xy + y^2 \\ 3x + 2y = 5 \end{cases}$$

1) Solve this problem by reducing it to a one variable problem using the equality constraint.
2) Solve the first order necessary condition and find the associated multipliers.

Exercise 60 For the following problem, find the points satisfying the first order necessary conditions (minimum or maximum):

$$\begin{cases} \text{Optimise } \frac{1}{3}x - \frac{1}{4}y \\ x^2 - 2x + y^2 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } \ln x + \ln y + \ln z \\ x^2 + y^2 + z^2 = 3 \\ x > 0, y > 0, z > 0 \end{cases}$$

$$\begin{cases} \text{Optimise } 4x^2 + y^2 \\ xy + 2 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } xy \\ x^2 + 4y^2 - 8 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } 2y^4 - 2xy^2 + x^2 - 4y^2 + 2x + 2 \\ -x + y^2 - 2 = 0 \end{cases}$$
$$\begin{cases} \text{Optimise } x + 3y - z \\ x^2 + 3y^2 + z^2 - 2\sqrt{x^2 + 3y^2} - 4 = 0 \end{cases}$$
$$\begin{cases} \text{Optimise } x^2 - \frac{3}{2}x + y^2 - \frac{3}{2}y \\ x^2 + y^2 - 2xy - x - y = 0 \end{cases}$$
$$\begin{cases} \text{Optimise } 4x + y + 2 \\ \ln x + 2\ln y = 0 \\ x > 0, y > 0 \end{cases}$$
$$\begin{cases} \text{Optimise } -\frac{2}{3}xy + \frac{5}{2}y + \frac{8}{3}x - \frac{11}{6} \\ x^2 + y - 1 = 0 \end{cases}$$

Exercise 61 For the above optimisation problems, write explicitly the associated Lagrangian mapping and check if the second order necessary condition is satisfied or not at the points satisfying the first order necessary condition.

Exercise 62 Let M be a $n \times n$ symmetric matrix $n \times n$. We consider the following optimisation problem:

$$\begin{cases} \text{Minimise } x \cdot Mx \\ \|x\| = 1, \\ x \in \mathbb{R}^n \end{cases}$$

1) Show that this problem has at least one solution $\bar{x}$.
2) Show that there exists $\lambda \in \mathbb{R}$ such that $M\bar{x} = \lambda\bar{x}$.

4.5 Multipliers and derivative of the value function

In this section, we consider that the optimisation problem depends on a parameter y in $\mathbb{R}^q$, which appears in the objective function as well as in the constraint functions. So, for a given y, we have the following problem:

$$(\mathcal{P}_y) \begin{cases} \text{Minimiser } f(x,y) \\ g_i(x,y) = 0,\ i = 1,\dots,p \\ x \in U \end{cases}$$

We assume that the objective function and the constraint functions are $\mathcal{C}^2$. We start from a point $\bar{x}$ and a vector of multipliers $\bar{\lambda}$ which satisfies the first order necessary condition for the problem $(\mathcal{P}_{\bar{y}})$, that is:

$$\nabla_x f(\bar{x},\bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i \nabla_x g_i(\bar{x},\bar{y}) = 0$$

We are looking for the existence of points satisfying the first order necessary condition when the parameter y is closed to $\bar{y}$. For this, we need a strong second order condition and we use the Implicit Function Theorem. Then we study the value of the objective function, the derivative of this value function and the link with the multipliers. Later, using some sufficient second order optimality conditions, we will show that these points are actually local solutions of the perturbed problem.

Proposition 67 *We assume that the objective function f and the constraint functions g_i are $\mathcal{C}^2$. Let $\bar{x} \in U$ such that the gradient vectors $(\nabla g_i(\bar{x}, \bar{y}))_{i=1}^p$ are linearly independent and a vector of multipliers $\bar{\lambda} \in \mathbb{R}^p$ satisfying*

$$\nabla_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i \nabla_x g_i(\bar{x}, \bar{y}) = 0$$

We also assume that for all $u \in A(\bar{x}, \bar{y}) = \{u \in \mathbb{R}^n \mid \nabla g_i(\bar{x}, \bar{y}) \cdot u = 0, \forall i = 1, \ldots, p\} \setminus \{0\}$,

$$u \cdot H_x\mathcal{L}(\bar{x}, \bar{\lambda})(u) = u \cdot (H_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i H_x g_i(\bar{x}, \bar{y}))(u) > 0$$

Then, there exists $\mathcal{C}^1$ functions ξ and Λ from an open neighborhood W of $\bar{y}$ to $U \times \mathbb{R}^p$ such that for all $y \in W$, $g_i(\xi(y), y) = 0$ for all $i = 1, \ldots, p$ and

$$\nabla_x f(\xi(y), y) + \sum_{i=1}^{p} \Lambda_i(y) \nabla_x g_i(\xi(y), y) = 0$$

Proof. Let us consider the mapping Γ from $U \times \mathbb{R}^p \times \mathbb{R}^q$ to $\mathbb{R}^n \times \mathbb{R}^p$ defined by:

$$\Gamma(x, \lambda, y) = \left(\nabla_x f(x, y) + \sum_{i=1}^{p} \lambda_i \nabla_x g_i(x, y), (g_i(x, y))_{i=1}^p \right)$$

Note that $\Gamma(\bar{x}, \bar{\lambda}, \bar{y}) = 0$. We now show that the partial Jacobian matrix of Γ with respect to (x, λ) at $(\bar{x}, \bar{\lambda}, \bar{y})$, $J\Gamma_{x,\lambda}(\bar{x}, \bar{\lambda}, \bar{y})$ is regular

$$J\Gamma_{x,\lambda}(\bar{x}, \bar{\lambda}, \bar{y}) = \begin{pmatrix} H_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i H_x g_i(\bar{x}, \bar{y}) & \nabla_x g_1(\bar{x}, \bar{y}) & \ldots & \nabla_x g_p(\bar{x}, \bar{y}) \\ \nabla_x g_1(\bar{x}, \bar{y})^t & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \nabla_x g_p(\bar{x}, \bar{y})^t & 0 & \ldots & 0 \end{pmatrix}$$

To show that $J\Gamma_{x,\lambda}(\bar{x}, \bar{\lambda}, \bar{y})$ is regular, we consider an element (z, μ) in its kernel and we show that it is equal to $(0, 0)$.

$$\left\{ \begin{array}{ll} [H_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i H_x g_i(\bar{x}, \bar{y})](z) + \sum_{i=1}^{p} \mu_i \nabla_x g_i(\bar{x}, \bar{y}) & = 0 \\ \nabla_x g_1(\bar{x}, \bar{y}) \cdot z & = 0 \\ \vdots & \vdots \\ \nabla_x g_p(\bar{x}, \bar{y}) \cdot z & = 0 \end{array} \right.$$

By doing the inner product of the first line by z, we obtain

$$z \cdot [H_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i H_x g_i(\bar{x}, \bar{y})](z) + \sum_{i=1}^{p} \mu_i \nabla_x g_i(\bar{x}, \bar{y}) \cdot z = 0$$

Using the fact that $\nabla_x g_i(\bar{x}, \bar{y}) \cdot z = 0$ for all i, we get

$$z \cdot [H_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \bar{\lambda}_i H_x g_i(\bar{x}, \bar{y})](z) = 0$$

Noticing that z belongs to $A(\bar{x}, \bar{y})$, we conclude from our assumption that $z = 0$. Using the first equation, we deduce that $\mu = 0$ since the gradient vectors $(\nabla_x g_i(\bar{x}, \bar{y}))$ are linearly independent.

We can now apply the Implicit Function Theorem to Γ and we deduce that there exists an open neighborhood W of $\bar{y}$ and a $\mathcal{C}^1$ mapping (ξ, Λ) from W to $U \times \mathbb{R}^p$ such that for all $y \in W$, $\Gamma(\xi(y), \Lambda(y), y) = 0$, $\xi(\bar{y}) = \bar{x}$ and $\Lambda(\bar{y}) = \bar{\lambda}$, which is the desired result.□

With the same assumption as in the previous proposition, we now study the value function v from W to $\mathbb{R}$ defined by $v(y) = f(\xi(y), y)$. We know that this mapping is $\mathcal{C}^1$. If there is no constraint in the problem, we get the so called Envelop Theorem:

$\nabla v(\bar{y}) = \nabla f_y(\bar{x}, \bar{y})$

Indeed, $\nabla v(\bar{y}) = J\xi(\bar{y})^t \nabla f_x(\bar{x}, \bar{y}) + \nabla f_y(\bar{x}, \bar{y})$ but $\nabla f_x(\bar{x}, \bar{y}) = 0$ since the first order necessary condition are satisfied at $\bar{x}$. So, the effect of a variation of the parameter y on the value function is equal to the effect of this parameter on the objective function at the solution.

Let us now consider a left hand side perturbation. This means that $y \in \mathbb{R}^p$, f does not depend on y and $g_i(x, y) = \gamma_i(x) - y_i$. Note that $g_i(\xi(y), y) = \gamma_i(\xi(y)) - y_i = 0$ for all $y \in W$. So, $J\xi(\bar{y})^t \nabla_x \gamma_i(\bar{x}) - \epsilon^i = 0$, where ϵ^i is the ith vector of the canonical basis of $\mathbb{R}^p$. Since, $\nabla_x g_i(x, y) = \nabla_x \gamma_i(x)$ and $\nabla_x f(\bar{x}) = -\sum_{i=1}^{p} \bar{\lambda}_i \nabla_x g_i(\bar{x}, \bar{y})$, one deduces that

$$J\xi(\bar{y})^t \nabla_x f(\bar{x}) = -\sum_{i=1}^{p} \bar{\lambda}_i J\xi(\bar{y})^t \nabla_x \gamma_i(\bar{x}) = -\sum_{i=1}^{p} \bar{\lambda}_i \epsilon^i = -\bar{\lambda}.$$

Since $\nabla v(\bar{y}) = J\xi(\bar{y})^t \nabla_x f(\bar{x})$, one concludes that $\nabla v(\bar{y}) = -\bar{\lambda}$. So the multiplier is actually the vector of the partial derivative of the value function. In economy, we interpret it as the shadow price of the constraint. Indeed, if the economic agent tries to minimise the objective function f under the constraints g_i and she has the possibility to buy a quantity $t > 0$ of the commodities i at the price π_i in $\mathbb{R}$ to relax the ith constraint, then the first order approximation of the objective function $v(\bar{y} + t\epsilon^i)$ is $v(\bar{y}) - \bar{\lambda}_i t$. But the cost increases by $\pi_i t$. So, the economic agent has a gain if the price π_i is strictly smaller than the shadow price $\bar{\lambda}_i$ and a loss if the price π_i is strictly greater than the shadow price $\bar{\lambda}_i$. So the shadow price is the price for which the economic agent is indifferent and will not buy or sell on the commodity market.

Chapter 5

Convex functions and convex sets

5.1 Properties of convex functions

Let U be a convex open subset of $\mathbb{R}^n$, i.e. for all $(x, y) \in U \times U$, for all $t \in [0, 1]$, $tx + (1-t)y \in U$. Let f be a mapping from U to $\mathbb{R}$.

Definition 41 The mapping f is convex (resp. concave) if for all $(x, y) \in U \times U$ and for all $t \in [0, 1]$, $f(tx + (1-t)y) \leq$ (resp. $\geq$) $tf(x) + (1-t)f(y)$.

A function f is convex if and only if $-f$ is concave. So, the results obtained for convex functions are straightforwardly transposed for concave functions.

For some issues, it is interesting to consider strictly convex (resp. concave) function, i.e. the functions f satisfying for all $(x, y) \in U \times U$ and for all $t \in]0, 1[$, $f(tx + (1-t)y) <$ (resp. $>$) $tf(x) + (1-t)f(y)$.

In the following, S^k denotes the simplex of $\mathbb{R}^k$, i.e. $S^k = \{\lambda \in \mathbb{R}^k_+ \mid \sum_{j=1}^k \lambda_j = 1\}$.

Definition 42 The epigraph (resp. hypograph) of a function is the set defined by:

$$\text{épi (resp. hypo)}(f) = \{(x, t) \in U \times R \mid t \geq \text{ (resp. } \leq) f(x)\}.$$

It is denoted épi(f) (resp. hypo(f)).

Theorem 29 *The three following assertions are equivalent:*

(i) f is convex (resp. concave);

(ii) For all $k \geq 2$, $(x_i) \in (\mathbb{R}^n)^k$ and $\lambda \in S^k$,
$f(\sum_{i=1}^k \lambda_i x_i) \leq$ (resp. $\geq$) $\sum_{i=1}^k \lambda_i f(x_i)$;

(iii) The epigraph (resp. hypograph) of f is a convex subset of $\mathbb{R}^n \times R$.

Proof of Theorem 29. We are giving the proof only in the convex case. (ii) implies (i) is obvious. We now show that (i) implies (iii). Let (x, λ) and (y, μ) two elements of épi(f) and let $t \in [0, 1]$. So $f(x) \leq \lambda$ and $f(y) \leq \mu$. As f is convex, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. So, $f(tx + (1-t)y) \leq t\lambda + (1-t)\mu$.

This is equivalent to $t(x,\lambda)+(1-t)(y,\mu)=(tx+(1-t)y, t\lambda+(1-t)\mu)$ belongs to the epigraph of f. So this set is convex.

We end the proof by showing that (iii) implies (ii). Let $k \geq 2$, $(x_i) \in (\mathbb{R}^n)^k$ and $\lambda \in S^k$. Then, $(x_i, f(x_i))$ is an element of the epigraph of f. Since this set is convex,

$$\sum_{i=1}^k \lambda_i (x_i, f(x_i)) = \left(\sum_{i=1}^k \lambda_i x_i, \sum_{i=1}^k \lambda_i f(x_i) \right)$$

is an element of épi(f) (See Proposition 74). So, from the very definition of the epigraph, $f(\sum_{i=1}^k \lambda_i x_i) \leq \sum_{i=1}^k \lambda_i f(x_i)$. $\square$

Examples : All linear or affine functions are convex and concave. If a function is convex and concave, it is affine. A norm is convex. If C is a nonempty convex subset of $\mathbb{R}^n$, the distance function to C defined by $d_C(x) = \inf\{\|x-c\| \mid c \in C\}$ is convex.

Proposition 68 *(i) A finite sum of convex (resp. concave) functions defined on U is convex (resp. concave);*

(ii) If f is convex (resp. concave) and $\lambda > 0$, λf is convex (resp. concave);

(iii) The supremum (resp. infimum) of a family of convex (resp. concave) functions defined on U is convex (resp. concave) when it is finite;

(iv) If f is a convex (resp. concave) function from $\mathbb{R}^n$ to I, an inteval of $\mathbb{R}$, and if φ is an increasing convex (resp. concave) function from I to $\mathbb{R}$ then $\varphi \circ f$ is convex (resp. concave).

(v) If g is an affine function from $\mathbb{R}^n$ to $\mathbb{R}^p$ and f is a convex function from $U \subset \mathbb{R}^p$ to $\mathbb{R}$, then $f \circ g$ is a convex function on $g^{-1}(U)$.

The proof of this proposition is left to the reader. We now state an important result on the continuity of the convex functions.

Theorem 30 *Let f be a convex function from U a convex open subset of $\mathbb{R}^n$ to $\mathbb{R}$. Then f is locally Lipschitz continuous and then continuous on U.*

We recall that a function is locally Lipschitz continuous if for all $\bar{x} \in U$, there exist $r > 0$ and $k \geq 0$ such that for all $(x,x') \in B(\bar{x},r) \times B(\bar{x},r)$, $|f(x) - f(x')| \leq k\|x-x'\|$. One easily prove that a locally Lipschitz continuous function is continuous.

Proof of Theorem 30. We first show that f is locally upper bounded on U. Let $x_0 \in U$ and let $(u_1,\ldots,u_n)$, a basis of $\mathbb{R}^n$. So, it exists $r > 0$ such that for all $i = 1,\ldots,n$, $x_0 + ru_i \in U$ et $x_0 - ru_i \in U$. Let

$$m = \max\{f(x_0), f(x_0+ru_1),\ldots,f(x_0+ru_n), f(x_0-ru_1),\ldots,f(x_0-ru_n)\}$$

Let $\Lambda = \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n |\lambda_i| < 1\}$ and let φ, the affine isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\lambda \to \varphi(\lambda) = x_0 + r\sum_{i=1}^n \lambda_i u_i$. Clearly, Λ is an open subset of $\mathbb{R}^n$, so $\varphi(\Lambda)$ is an open subset of $\mathbb{R}^n$ containing x_0. We show now that $\varphi(\Lambda)$ is included in U and that f is upper bounded on this set, which ends the first part of the proof.

Let $x \in \varphi(\Lambda)$. It exists $\lambda \in \Lambda$ such that $x = x_0 + r\sum_{i=1}^{n} \lambda_i u_i = (1 - \sum_{i=1}^{n} |\lambda_i|)x_0 + \sum_{i=1}^{n} |\lambda_i|(x_0 + r\varepsilon_i u_i)$ with $\varepsilon_i = 1$ if $\lambda_i \geq 0$ and -1 otherwise. So x belongs to U since it is a convex combination of elements of U. Furthermore, as f is convex:

$$\begin{aligned} f(x) &\leq (1 - \textstyle\sum_{i=1}^{n} |\lambda_i|)f(x_0) + \sum_{i=1}^{n} |\lambda_i| f(x_0 + r\varepsilon_i u_i) \\ &\leq \max\{f(x_0), f(x_0 + r\varepsilon_1 u_1), \ldots, f(x_0 + r\varepsilon_n u_n)\} \\ &\leq m \end{aligned}$$

So f is upper bounded on $\varphi(\Lambda)$.

We now show that f is locally Lipschitz continuous around x_0. Let $r > 0$ and $a \in \mathbb{R}$ such that $f(x) \leq a$ for all $x \in B(x_0, r)$.

Let $x \in B(x_0, \frac{r}{2})$. For all $y \in B(x, \frac{r}{2})$ such that $y \neq x$, let $z^+ = x + \frac{r}{2\|y-x\|}(y-x)$ and $z^- = x - \frac{r}{2\|y-x\|}(y-x)$. It is clear that z^+ and z^- belong to $B(x_0, r)$. So, $f(z^+) \leq a$ and $f(z^-) \leq a$.

We remark that $y = \frac{2\|y-x\|}{r} z^+ + (1 - \frac{2\|y-x\|}{r})x$ and $x = \frac{2\|y-x\|}{r+2\|y-x\|} z^- + \frac{r}{r+2\|y-x\|} y$. Hence, using the convexity of f, one deduces that

$$f(y) \leq \frac{2\|y-x\|}{r} f(z^+) + (1 - \frac{2\|y-x\|}{r}) f(x)$$

and

$$f(x) \leq \frac{2\|y-x\|}{r+2\|y-x\|} f(z^-) + \frac{r}{r+2\|y-x\|} f(y)$$

So, one deduces that

$$f(y) - f(x) \leq \tfrac{2\|y-x\|}{r}(f(z^+) - f(x)) \leq \tfrac{2\|y-x\|}{r}(a - f(x)) \qquad (1)$$

$$\text{and} \quad f(x) - f(y) \leq \tfrac{2\|y-x\|}{r+2\|y-x\|}(f(z^-) - f(y)) \leq \tfrac{2\|y-x\|}{r+2\|y-x\|}(a - f(y)) \qquad (2)$$

Rewriting Inequality (2), we obtain

$$f(y) \geq \tfrac{2\|y-x\|+r}{r} f(x) - \tfrac{2\|y-x\|}{r} a$$

which is equivalent to

$$a - f(y) \leq -\tfrac{2\|y-x\|+r}{r} f(x) + \tfrac{r+2\|y-x\|}{r} a = \tfrac{2\|y-x\|+r}{r}(a - f(x)) \qquad (3)$$

Plug in Inequality (3) in Inequality (2), we get

$$f(x) - f(y) \leq \tfrac{2\|y-x\|}{r+2\|y-x\|} \tfrac{2\|y-x\|+r}{r}(a - f(x)) = \tfrac{2\|y-x\|}{r}(a - f(x)) \qquad (4)$$

One deduces from Inequalities (1) and (4),

$$|f(y) - f(x)| \leq \tfrac{2\|y-x\|}{r}(a - f(x)) \qquad (5)$$

Let us consider now and element $x \in B(x_0, \frac{r}{4})$. Applying Inequality (3) to x and x_0, we obtain

$$a - f(x) \leq \tfrac{2\|x-x_0\|+r}{r}(a - f(x_0)) \leq \tfrac{3}{2}(a - f(x_0)) \qquad (6)$$

For all $y \in B(x_0, \frac{r}{4})$, y belongs also to $B(x, \frac{r}{2})$. Combining Inequalities (5) and (6), we then get

$$|f(y) - f(x)| \leq \frac{3}{r}(a - f(x_0))\|y - x\|$$

So, f is Lipschitz continuous with a constant $\frac{3}{r}(a - f(x_0))$ on the ball $B(x_0, \frac{r}{4})$.□

We end this part by studying $\mathcal{C}^1$ or $\mathcal{C}^2$ convex functions. Let U be an open convex subset of $\mathbb{R}^n$. Let f be a differentiable function from U to $\mathbb{R}$. We denote by $Df(x)$ the derivative of f at x and by $\nabla f(x)$ its gradient vector.

Proposition 69 *f is convex if and only if for all $(x, y) \in U \times U$, $f(y) - f(x) \geq Df(x)(y - x) = \nabla f(x) \cdot (y - x)$.*

The above condition means that the gradient of f is everywhere a sub-gradient, that is: $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$ for all $(x, y) \in U \times U$.

Exercise 63 Let f be a $\mathcal{C}^1$ function on an open convex subset U of $\mathbb{R}^n$. At $\bar{x} \in U$, we assumet that f has a sub-gradient that is a vector $v \in \mathbb{R}^n$ such that for all $y \in U$, $f(y) - f(\bar{x}) \geq v \cdot (y - \bar{x})$. Show that $v = \nabla f(\bar{x})$.

Proof of Proposition 69. Let $(x, y) \in U \times U$. If f is convex, then the function φ defined on a neighbourhood of 0 in $\mathbb{R}$ by $\varphi(t) = f(x + t(y - x))$ satisfies $\varphi(t) \leq \psi(t) = f(x) + t(f(y) - f(x))$ and $\varphi(0) = \psi(0)$. So the right derivative of φ at 0 is equal to $Df(x)(y - x)$ and it is lower or equal to the one of ψ, which is equal to $f(y) - f(x)$, which provides the desired result.

Let us now assume that the property on the derivative of f is satisfied. Let $(x, y) \in U \times U$ and let $t \in]0, 1[$. We have $f(x) - f(x + t(y - x)) \geq Df(x + t(y - x))(-t(y - x))$ and $f(y) - f(x + t(y - x)) \geq Df(x + t(y - x))((1 - t)(y - x))$. Multiplying the first inequality by $(1 - t)$ and the second one by t, then by summing the two inequalities, we get $(1 - t)(f(x) - f(x + t(y - x))) + t(f(y) - f(x + t(y - x))) \geq (-t(1 - t) + t(1 - t))Df(x + t(y - x))(y - x) = 0$. So $f(x + t(y - x)) = f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$ thus f is convex. □

Proposition 70 *f is convex if and only if for all $(x, y) \in U \times U$, $(Df(y) - Df(x))(y - x) = (\nabla f(y) - \nabla f(x)) \cdot (y - x) \geq 0$.*

The condition above means that the gradient of f is, in a certain sense, monotone. In the one dimensional case, it means that the derivative f' is increasing.

Proof of Proposition 70. Let $(x, y) \in U \times U$. If f is convex, from the previous proposition, $f(x) - f(y) \geq Df(y)(x - y)$ and $f(y) - f(x) \geq Df(x)(y - x)$. By doing the sum of these two inequalities, we obtain the desired result.

For the converse, let us assume by contradiction that f is not convex. Then, from the previous proposition, it exists $(x, y) \in U \times U$ such that $f(y) - f(x) < Df(x)(y - x)$. Let us define the function φ defined on a neighbourhood of $[0, 1]$ in $\mathbb{R}$ by $\varphi(t) = f(x + t(y - x))$. As φ is differentiable on $[0, 1]$, there exists $\bar{t} \in]0, 1[$

such that $f(y) - f(x) = \varphi(1) - \varphi(0) = \varphi'(\bar{t}) = Df(x + \bar{t}(y - x))(y - x)$. So $(Df(x + \bar{t}(y - x)) - Df(x))(y - x) = f(y) - f(x) - Df(x)(y - x) < 0$ hence $(Df(\xi) - Df(x))(\xi - x) < 0$ with $\xi = x + \bar{t}(y - x)$ which contradicts our assumption and ends the proof. □

We now consider the case where f is $\mathcal{C}^2$ on U.

Proposition 71 *f is convex if and only if for all $x \in U$, $Hf(x)$ is positive semi-definite.*

The above condition means that for all $u \in \mathbb{R}^n$,

$$u \cdot Hf(x)(u) \geq 0.$$

If $Hf(x)$ is positive definite for all $x \in U$, then f is strictly convex.

Proof of Proposition Let us assume that f is convex. For $x \in U$ and $u \in \mathbb{R}^n$, let φ be the function from a neighbourhood of 0 in $\mathbb{R}$ to $\mathbb{R}$ defined by $\varphi(t) = f(x + tu)$. Using a Taylor expansion, we have $\varphi(t) - \varphi(0) = f(x + tu) - f(x) = t\varphi'(0) + \frac{t^2}{2}\varphi''(0) + t^2\epsilon(t)$ with $\lim_{t \to 0} \epsilon(t) = 0$. From the previous proposition, $f(x+tu) - f(x) \geq Df(x)(tu) = t\varphi'(0)$. So $\frac{t^2}{2}\varphi''(0) + t^2\epsilon(t) \geq 0$ and $\frac{1}{2}\varphi''(0) + \epsilon(t) \geq 0$. Passing to the limit, we obtain $\varphi''(0) \geq 0$. But $\varphi''(0) = u \cdot Hf(x)(u)$ hence $Hf(x)$ is positive semi-definite.

Conversely, if $(x, y) \in U \times U$, let φ defined an neighbourhood of $[0, 1]$ in $\mathbb{R}$ to $\mathbb{R}$ by $\varphi(t) = f(x + t(y - x))$. Then, using a Taylor expansion, it exists $\bar{t} \in]0, 1[$ such that $\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(\bar{t})$. But $\varphi''(\bar{t}) = (y - x) \cdot Hf(x + \bar{t}(y - x))(y - x)$ which is non negative by assumption. Thus $f(y) - f(x) = \varphi(1) - \varphi(0) \geq \varphi'(0) = Df(x)(y - x)$ which implies that f is convex from the previous proposition. □

We can deduce from the previous result a criterion for a local convexity of a $\mathcal{C}^2$ function.

Proposition 72 *Let f be a $\mathcal{C}^2$ function on an open subset U of $\mathbb{R}^n$. Let $\bar{x} \in U$ such that the Hessian matrix $Hf(\bar{x})$ is positive definite. Then, it exists $r > 0$ such that for all $x \in B(\bar{x}, r)$, the Hessian matrix $Hf(x)$ is positive definite, so the restriction of f to the ball $B(\bar{x}, r)$ is strictly convex.*

Proof. If for all $r > 0$, the property does not hold, then it exists a sequence(x^ν, u^ν) such that (x^ν) converges to $\bar{x}$, $\|u^\nu\| = 1$ and $u^\nu \cdot Hf(x^\nu)u^\nu \leq 0$ for all ν. It exists a converging subsequence of (u^ν), $(u^{\varphi(\nu)})$, whose limit $\bar{u}$ is of norm 1. As f is $\mathcal{C}^2$, the sequence $u^{\varphi(\nu)} \cdot Hf(x^{\varphi(\nu)})u^{\varphi(\nu)}$ converges to $\bar{u}Hf(\bar{x})\bar{u}$ and we conclude that $\bar{u}H_f(\bar{x})\bar{u} \leq 0$ which is in contradiction with the fact that $H_f(\bar{x})$ is positive definite. For the second part of the proposition, we just remark that the Hessian matrix of the restriction of f to the ball $B(\bar{x}, r)$ is equal to the Hessian matrix of f and that it is positive definite on the open convex set $B(\bar{x}, r)$, from which one concludes that it is strictly convex. □

Exercise 64 Let a be a real number and f be a function from $\mathbb{R}^3$ to $\mathbb{R}$ defined by:

$$f(x,y,z) = 3x^2 + 2y^2 + z^2 + axy + 2yz + 2xz$$

For which values of a, the function f is convex?

5.2 Necessary and sufficient condition of optimality for convex (concave) functions

A fundamental property of the convex (concave) functions in optimisation is the fact that the first order necessary conditions for a minimisation (maximisation) problem are sufficient. But another fundamental property is the fact that a local solution is a global solution.

We consider a $\mathcal{C}^1$ convex function on an open convex subset U of $\mathbb{R}^n$. Let A be a $p\times n$ matrix and b be a vector of $\mathbb{R}^p$. Let us consider the following minimisation problem:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ Ax = b \text{ or } a^j \cdot x = b^j \text{ for all } j = 1, \dots, p \\ x \in U \end{cases}$$

Let $\bar{x} \in U$ satisfying the constraint $A\bar{x} = b$ and the first order optimality conditions, that is, there exists $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) = A^t\lambda = \sum_{j=1}^p \lambda^j a^j$ where a^j is the j-th row of A. Then, $\bar{x}$ is a solution of the problem $(\mathcal{P})$. Indeed, for all $x \in U$ satisfying the constraints $Ax = b$, $A(x-\bar{x}) = 0$ and so, $\nabla f(\bar{x}) \cdot (x - \bar{x}) = A^t\lambda \cdot (x-\bar{x}) = \lambda \cdot A(x-\bar{x}) = \lambda \cdot 0 = 0$. As f is convex,

$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x}) = f(\bar{x})$$

which shows that $\bar{x}$ is a solution of the problem $(\mathcal{P})$.

Theorem 31 *Let f be a $\mathcal{C}^1$ convex function on an open convex subset U of $\mathbb{R}^n$. Let A be a $p \times n$ matrix and b be a vector of $\mathbb{R}^p$. Let $\bar{x} \in U$ satisfying the constraints $A\bar{x} = b$. Then $\bar{x}$ is a solution of the problem $(\mathcal{P})$ if and only if it exists a vector of multipliers $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) = A^t y = \sum_{j=1}^p \lambda^j a^j$ where a^j is the j-th row of A.*

We remark that the optimality condition is necessary and sufficient and that it involves only a local information concerning f, namely its gradient. So, we can deduce that a local solution on a neighbourhood of $\bar{x}$ is also a global solution. Indeed, if $\bar{x}$ is a local solution, the necessary condition is satisfied. But, as this condition is also a sufficient condition for a global minimum, we can conclude that $\bar{x}$ is a global solution.

Exercise 65 Let f be a convex function from an open convex subset U of $\mathbb{R}^n$ to $\mathbb{R}$. Let A be a $p\times n$ matrix. For all $b \in \mathbb{R}^p$, we consider the following minimisation problem:

$$(\mathcal{P}_b) \begin{cases} \text{Minimise } f(x) \\ Ax = b, \\ x \in U \end{cases}$$

We assume that there exists an open convex subset V of $\mathbb{R}^p$ such that for all $b \in V$, the value $v(b)$ of the problem $(\mathcal{P}_b)$ is finite. Show that the function v is convex.

Exercise 66 Let f be a $\mathcal{C}^2$ convex function on the open convex subset U of $\mathbb{R}^n$. We assume that for all $x \in U$, $Hf(x)$ is positive definite. Let V be an open convex subset of $\mathbb{R}^n$ such that for all $y \in V$, the problem:

$$(\mathcal{P}_y) \max\{y \cdot x - f(x) \mid x \in U\}$$

has a solution denoted $\xi(y)$.
1) Using the fact that f is strictly convex, show that the solution $\xi(y)$ is unique.
2) For all $y \in V$, show that $y = \nabla f(\xi(y))$.
3) Using the Implicit Function Theorem, show that $\xi(\cdot)$ is $\mathcal{C}^1$ mapping.
Let v be the value function of this problem, that is $v(y) = y \cdot \xi(y) - f(\xi(y))$.
4) Show that v is a $\mathcal{C}^1$ convex function on V.
5) Show that for all $(\bar{y}, y) \in V \times V$, $v(y) - v(\bar{y}) \geq \xi(\bar{y})(y - \bar{y})$ and deduce that $\nabla v(\bar{y}) = \xi(\bar{y})$.
6) Show that $\bar{y}$ is a solution of the problem $\max\{\xi(\bar{y}) \cdot y - v(y) \mid y \in V\}$ and that the value function of this problem is $f(\xi(\bar{y}))$.

Exercise 67 We consider the function f from $\mathbb{R}^2$ to $\mathbb{R}$ defined by:

$$f(x, y) = \sqrt{x^2 + y^2} - x$$

1) For all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, compute the gradient of the function f.
2) Show, without any computation, that the function f is convex.
3) Show that $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$.
4) Compute $f(0, 0)$ and give the minimum of f on $\mathbb{R}^2$.
5) Give all minima of f on $\mathbb{R}^2$.
We consider the sequence $(x^n, y^n) = (n, 1)$.
6) Show that the sequence $\nabla f(x^n, y^n)$ converges to a limit and compute this limit. Show that the sequence $f(x^n, y^n)$ converges to the minimal value of f on $\mathbb{R}^2$. Show that the sequence (x^n, y^n) does not converge to a minimum of f on $\mathbb{R}^2$.

5.3 Sufficient condition for local solutions

To get a sufficient condition for local solutions, we will put a condition on the Hessian matrix of the objective function, which is a reinforcement of the second order necessary condition. Then, the objective function being locally strictly convex, the first order necessary condition becomes a sufficient optimality for a local solution.

Let us consider the same problem as above:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ Ax = b \\ x \in U \end{cases}$$

Theorem 32 *Let f be a $\mathcal{C}^2$ function on an open subset U of $\mathbb{R}^n$. Let A be a $p \times n$ matrix and b be a vector of $\mathbb{R}^p$. Let $\bar{x} \in U$ satisfying the constraints $A\bar{x} = b$ and the first order necessary condition : it exists $\lambda \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) = A^t\lambda = \sum_{j=1}^p \lambda^j a^j$ where a^j is the j-th row of A. We assume that the Hessian matrix of f at $\bar{x}$, $Hf(\bar{x})$ is positive definite. Then it exists a real number $r > 0$ such that $B(\bar{x}, r) \subset U$ and $\bar{x}$ is the unique solution of $(\mathcal{P}^r)$*

$$(\mathcal{P}^r) \begin{cases} \textit{Minimise } f(x) \\ Ax = b \\ x \in B(\bar{x}, r) \end{cases}$$

that is a local solution of $(\mathcal{P})$.

Proof. From Proposition 71, f is strictly convex an on open ball $B(\bar{x}, r)$ for some $r > 0$. So, the first order necessary conditions of the problem $(\mathcal{P}^r)$ are sufficient and $\bar{x}$ satisfies this necessary conditions, since, actually, they are identical to the one of Problem $(\mathcal{P})$. So $\bar{x}$ is a solution of the problem $(\mathcal{P}^r)$ and it is the unique solution since the objective function is strictly convex.□

Exercise 68 Let U be an open convex subset of $\mathbb{R}^n$. Let f be a $\mathcal{C}^2$ function on U. We assume that for all $x \in U$,

1) $\nabla f(x) \neq 0$;

2) for all $u \in \nabla f(x)^\perp \setminus \{0\}$, $u \cdot H_f(x)(u) > 0$.

We show that for all $\bar{x} \in U$, the set $\{x \in U \mid f(x) \leq f(\bar{x})\}$ is convex. Let us assume that it does not hold, then it exists $(\bar{x}, x, x', \tau) \in U \times U \times U \times]0,1[$, such that $f(x) \leq f(\bar{x})$, $f(x') \leq f(\bar{x})$ and $f(\tau x + (1-\tau)x') > f(\bar{x})$. Let φ from $[0,1]$ to $\mathbb{R}$ defined by $\varphi(t) = f(tx + (1-t)x')$.
a) Show that the problem $\max\{\varphi(t) \mid t \in [0,1]\}$ has a solution $\bar{t}$ in the open interval $]0,1[$.
We note $\xi = \bar{t}x + (1-\bar{t})x'$.
b) Show that $\varphi'(\bar{t}) = 0$ and that $\varphi''(\bar{t}) \leq 0$ and deduce that $\nabla f(\xi) \cdot (x' - x) = 0$ and $(x' - x) \cdot H_f(\xi)(x' - x) \leq 0$. Conclude.

We now show that for all $\bar{x} \in U$, for all $x \in U$ such that $f(x) < f(\bar{x})$, $\nabla f(\bar{x}) \cdot (x - \bar{x}) < 0$.
c) Show that there exists $r > 0$ such that $x' = x + r\nabla f(\bar{x}) \in U$ and $f(x') < f(\bar{x})$.
d) Using the convexity of the set $\{x \in U \mid f(x) \leq f(\bar{x})\}$, show that:

$$\lim_{t \to 0^+} \frac{f(\bar{x} + t(x' - \bar{x})) - f(\bar{x})}{t} \leq 0$$

Deduce that $\nabla f(\bar{x}) \cdot (x' - \bar{x}) \leq 0$ and that $\nabla f(\bar{x}) \cdot (x - \bar{x}) < 0$.

Let A be a $p \times n$ matrix and b be a vector of $\mathbb{R}^p$. We consider the following problem:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ Ax = b \\ x \in U \end{cases}$$

Let $\bar{x} \in U$ satisfying $A\bar{x} = b$ and the first order necessary conditions, namely it exists $y \in \mathbb{R}^p$ such that $\nabla f(\bar{x}) = A^t y$. Show that $\bar{x}$ is a solution of the problem. Hint: use a similar argument than the one for convex functions and the properties of f proved above.

Exercise 69 Show that the function f define on $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x_i > 0\}$ by $f(x) = -x_1 x_2 \ldots x_n$ is not convex but it satisfies the assumption of the previous exercise. Deduce the solution of the following problem:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b \\ x \in \mathbb{R}^n_{++} \end{cases}$$

where $a_1, a_2, \ldots, a_n, b$ are $n+1$ non negative real numbers.

Exercise 70 Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by :

$$f(x, y) = x^2 y^2 - 4x^2 - y^2$$

We are looking for the extremum of this function.
1) Compute the gradient vector and Hessian matrix of f at any point (x, y) of $\mathbb{R}^2$.
2) Find the points for which the gradient vanishes.
3) By studying the sign of the Hessian matrix at the points found above, find the local maximum and minimum of f and the critical points which are neither a local minimum nor a local maximum.
4) Show that the function f has neither a global maximum nor a global minimum on $\mathbb{R}^2$.

Exercise 71 Let f be the function from $\mathbb{R}^3$ to $\mathbb{R}$ defined by :

$$f(x, y, z) = x^2 + xy + y^2 + 2z^4 - z^2$$

We consider the following optimisation problem:

$$(\mathcal{P}) \begin{cases} \min & f(x, y, z) \\ s.c. & (x, y, z) \in \mathbb{R}^3 \end{cases}$$

1) Compute the gradient vector and the Hessian matrix of f at any point (x, y, z) of $\mathbb{R}^3$.

2) Find the points where the gradient vanishes.
3) By studying the sign of the Hessian matrix at the points found above, find the local maximum and minimum of f and the critical points which are neither a local minimum nor a local maximum.
4) By studying $f(-x,-y,-z)$, what can we say about the uniqueness of a solution?

Exercise 72 Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x,y) = x^4 + y^4 - (x-y)^2$.
1) Compute the points where the gradient of f vanishes and study the sufficient second order conditions at these points.
2) Show that f is coercice.
3) Show that f has a minimum on $\mathbb{R}^2$ and give this minimum.

Exercise 73 Let f be the function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ defined by $f(x) = \sum_{i=1}^n x_i \ln\left(\frac{1}{x_i}\right)$. Show that this function has a maximum on $\mathbb{R}^n_{++}$.

5.4 Convex set

To go deeper in the analysis of optimisation problems and, in particular, to introduce inequality constraints, we need to study some properties of the convex sets, prove the fundamental theorem of convex analysis, which is the separation theorem, introduce the notion of polarity among convex cones to generalise the orthogonality among linear subspaces and finally the Farkas' Lemma, which allows us to describe the polar cone to a finitely generated cone.

5.4.1 Basic properties of convex sets

For a pair of elements (x,y) in $\mathbb{R}^n$, we denote by $[x,y]$ the segment joining x to y that this the set defined by:

$$[x,y] = \{tx + (1-t)y \mid t \in [0,1]\}$$

Definition 43 A subset C of $\mathbb{R}^n$ is convex if for all $(x,y) \in C \times C$, $[x,y]$ is included in C.

Examples : For all pairs of elements (x,y) of $\mathbb{R}^n$, $[x,y]$ is a convex subset of $\mathbb{R}^n$. All affine or linear subspaces of $\mathbb{R}^n$ are convex. All closed or open balls are convex whatever is the norm. All sets of solutions of a system of linear equalities and inequalities is convex. In $\mathbb{R}$, the convex subsets are the intervals.

Proposition 73 *(i) Let $(C_i)_{i\in I}$, be a family of convex subsets of $\mathbb{R}^n$. Then $\cap_{i\in I} C_i$ is convex.*

(ii) Let $(C_i)_{i\in I}$, be a family of convex subsets of $\mathbb{R}^n$ such that for all $(i,j) \in I \times I$, there exists $k \in I$ such that $C_i \cup C_j \subset C_k$. then $\cup_{i\in I} C_i$ is convex.

(iii) Let $(C_i)_{i\in I}$, be a finite family of convex subsets of $\mathbb{R}^n$. Then $\sum_{i\in I} C_i = \{\sum_{i\in I} c_i \mid (c_i) \in \prod_{i\in I} C_i\}$ is convex.

(iv) Let $(C_i)_{i\in I}$, be a finite family of convex subsets of $\mathbb{R}^{n_i}$. Then $\prod_{i\in I} C_i$ is a convex subset of $\prod_{i\in I} \mathbb{R}^{n_i}$.

(v) Let C be convex subset of $\mathbb{R}^n$ and let $\lambda \in R$. Then $\lambda C = \{\lambda c \mid c \in C\}$ is convex.

(vi) Let f be an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$, and let C be a convex subset of $\mathbb{R}^n$. Then $f(C)$ is a convex subset of $\mathbb{R}^p$.

(vii) Let f be an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$, and let C be a convex subset of $\mathbb{R}^p$. Then $f^{-1}(C)$ is a convex subset of $\mathbb{R}^n$.

The proof is left to the reader. In the following, we often use the simplex of $\mathbb{R}^n$ defined by:

$$S^n = \{\lambda \in R^n_+ \mid \sum_{i=1}^{n} \lambda_i = 1\}$$

as a reference convex set. We remark that S^n is convex, closed and bounded, so it is compact.

Definition 44 Let $(x_i)_{i=1}^k$, be k points of $\mathbb{R}^n$. A convex combination of $(x_i)_{i=1}^k$ is an element x of $\mathbb{R}^n$ such that there exists $\lambda \in S^k$ and $x = \sum_{j=1}^k \lambda_j x_j$.

If (x, y) is a pair of elements in $\mathbb{R}^n$, the set of convex combination of x and y is the segment $[x, y]$.

Proposition 74 *Let C be a subset of $\mathbb{R}^n$. C is convex if and only if C contains all convex combinations of the finite families of elements of C.*

Proof of Proposition 74. It is obvious that if C contains all convex combinations of finite families of elements of C then C is a convex subset.

Conversely, we are using an induction argument on the number of elements of the family. If the family contains one or two elements, the very definition of a convex subset shows that all convex combinations of this family belongs to C. Let us assume that is true for all families with at most k elements. Let $(x_1, \ldots, x_k, x_{k+1})$, be a family of elements of C. Let $\lambda \in S_{k+1}$ and let $x = \sum_{j=1}^{k+1} \lambda_j x_j$. Since $\sum_{j=1}^{k+1} \lambda_j = 1$, it exists a least one λ_j different from 0. Assume without any loss of generality that $\lambda_1 \neq 0$. Then

$$x = \big(\sum_{j=1}^{k} \lambda_j\big)\big(\sum_{j=1}^{k} \frac{\lambda_j}{\sum_{j=1}^{k} \lambda_j} x_j\big) + \lambda_{k+1} x_{k+1}$$

From our induction hypothesis

$$x' = \sum_{j=1}^{k} \frac{\lambda_j}{\sum_{j=1}^{k} \lambda_j} x_j$$

is an element of C since it is a convex combination of a family of k elements of C. Furthermore, since $\left(\sum_{j=1}^k \lambda_j\right) + \lambda_{k+1} = 1$, x is a convex combination of x' and x_{k+1}. So x belongs to C, which ends our proof. $\square$

Definition 45 Let A, be a subset of $\mathbb{R}^n$. The convex hull of A, denoted $\mathrm{co}(A)$, is the intersection of all convex subsets of $\mathbb{R}^n$ containing A.

Since $\mathbb{R}^n$ is convex, if A is non-empty, $\mathrm{co}(A)$ is also nonempty. Since convex subsets are stable under intersection, $\mathrm{co}(A)$ is the smallest (for the intersection) convex set containing A..

Proposition 75 *Let A, be a subset of $\mathbb{R}^n$. $\mathrm{co}(A)$ is the set of all convex combination of finite families of elements of A.*

Proof of Proposition 75. We denote by B the set of all convex combinations of finite families of elements of A. $B \subset \mathrm{co}(A)$ since $\mathrm{co}(A)$ is a convex subset of $\mathbb{R}^n$ containing A.

Let us show that $\mathrm{co}(A) \subset B$. Clearly, $A \subset B$. So, to prove the inclusion, it suffices to show that B is convex from the very definition of $\mathrm{co}(A)$. Let x and y be two elements of B and $t \in [0,1]$. It exists two finite families $(x_1 \ldots, x_k)$ and $(y_1, \ldots, y_p)$ of elements of A, $\lambda \in S^k$ and $\mu \in S^p$ such that $x = \sum_{j=1}^k \lambda_j x_j$ and $y = \sum_{j=1}^p \mu_j y_j$. So

$$tx + (1-t)y = \sum_{j=1}^k t\lambda_j x_j + \sum_{j=1}^p (1-t)\mu_j y_j$$

So $tx + (1-t)y$ is a convex combination of $(x_1, \ldots, x_k, y_1 \ldots, y_p)$ since

$$\sum_{j=1}^k t\lambda_j + \sum_{j=1}^p (1-t)\mu_j = t + (1-t) = 1$$

and thus, $(t\lambda_1, \ldots, t\lambda_k, (1-t)\mu_1 \ldots, (1-t)\mu_p)$ belongs to S^{k+p}. $\square$

A polytope is a convex hull of a finite subset of $\mathbb{R}^n$. For example, the simplex of $\mathbb{R}^n$ is the polytope generated by the elements of the canonical basis of $\mathbb{R}^n$.

Theorem 33 *(Carathéodory) Let A, be a nonempty subset of $\mathbb{R}^n$. Then $\mathrm{co}(A)$ is the set of the convex combinations of the families of elements of A containing at most $n+1$ elements.*

Proof of Theorem 33. To prove the result, it suffices to show that a convex combination of a family containing $p > n+1$ elements is also a convex combination of another family containing at most $p-1$ elements. Let $p > n+1$, Let $(x_1, \ldots, x_p) \in A^p$, let $\lambda \in S^p$ and let $x = \sum_{i=1}^p \lambda_i x_i$. Since $p > n+1$, the vectors $(x_2 - x_1, \ldots, x_p - x_1)$ are linearly dependent in $\mathbb{R}^n$. So, it exists a non zero vector

$(\mu_2, \ldots, \mu_p)$ such that $\sum_{i=2}^{p} \mu_i(x_i - x_1) = 0$. Let us define $\mu_1 = -\sum_{i=2}^{p} \mu_i$. Then $\mu \in \mathbb{R}^p$ is a non zero vector and the sum of the components is equal to 0. Furthermore $\sum_{i=1}^{p} \mu_i x_i = 0$. So the set $I_+ = \{i \in \{1, \ldots, p\} \mid \mu_i > 0\}$ is nonempty. Let

$$t = \min\{\frac{\lambda_i}{\mu_i} \mid i \in I_+\}$$

and let i_0 such that $t = \dfrac{\lambda_{i_0}}{\mu_{i_0}}$. Let us define now $\beta_i = \lambda_i - t\mu_i$ for all i. Clearly, the definition of t implies that the vector β has positive components and $\beta_{i_0} = 0$. Since, $\sum_{i=1}^{p} \mu_i = 0$, $\sum_{i \neq i_0} \beta_i = 1$. Furthermore $\sum_{i=1}^{p} \mu_i x_i = 0$,

$$x = \sum_{i=1}^{p} \beta_i x_i = \sum_{i \neq i_0} \beta_i x_i$$

So x is a convex combination of a family containing $p - 1$ elements. This ends the proof. □

Exercise 74 Let C be a closed convex subset of $\mathbb{R}^n$ satisfying $\cup_{t>0} tC = \mathbb{R}^n$.
1) Show that $0 \in C$ and for all $(t, t') \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $tC \subset t'C$ if $t < t'$.
2) Let $(e^1, e^2, \ldots, e^n)$ be the canonical basis of $\mathbb{R}^n$. Show that for all $i = 1, \ldots, n$, it exists $r^i > 0$ such that re^i and $-re^i$ belongs to C.
3) Show that 0 belongs to the interior of C. Hint: we can use a similar argument as the one of the first step of the proof of Theorem 30.

Definition 46 A subset K of $\mathbb{R}^n$ is a cone of vertex 0 if for all $x \in K$ and for all $t > 0$, tx belongs to K.

Proposition 76 *A cone K is convex if and only if it is stable under addition.*

Proof of Proposition 76. Let K be a convex cone. Let x and y two elements of K. Then $\frac{1}{2}(x + y)$ belongs to K since it is convex. Now $x + y = 2\frac{1}{2}(x + y)$ belongs to K since it is a cone. So K is stable under addition.

Let K be a cone stable under addition and let x and y be two elements of K. For all $t \in]0, 1[$, tx and $(1 - t)y$ are elements of K since it is a cone. As K is stable under addition, $tx + (1 - t)y$ belongs to K. For $t = 0$ or $t = 1$, it is obvious that $tx + (1 - t)y$ belongs to K and so K is convex. □

Examples : all linear subspaces of $\mathbb{R}^n$ are convex cones. All sets of solutions of a system of linear homogeneous equalities and inequalities is a convex cone. The image or the inverse image of a convex cone by a linear mapping is a convex cone. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ are convex cone.

Definition 47 Let A be a subset of $\mathbb{R}^n$. The conic hull of A is the smallest convex cone containing A. It is denoted cone(A).

One easily shows that all intersections of convex cones is also a convex cone. So, cone(A) is the intersection of all convex cones containing A. It is also easy to prove that cone(A) is the set of all non negative linear combination of finite families of elements of A.

We state now a result which is the adaptation of Caratheodory's Theorem to conic hull. The proof is left to the reader since it is very similar to the one for convex hull and even simpler.

Theorem 34 *Let A, be a nonempty subset of $\mathbb{R}^n$. Then, for all $y \in cone(A) \backslash \{0\}$, it exists a linearly independent family $(a_1, \dots, a_p)$ of elements of A and a vector $\lambda \in R^p_+$ such that $y = \sum_{i=1}^p \lambda_i a_i$.*

Definition 48 A finitely generated cone is the conic hull of a finite family of elements in $\mathbb{R}^n$. It is the set of positive linear combinations of a finite family of elements in $\mathbb{R}^n$.

We conclude this basic properties on convex subsets by two topological properties on polytopes and finitely generated cones, which are useful in the following, specially for the Farkas Lemma.

Proposition 77 *A polytope is compact. A finitely generated cone is closed.*

Proof of Proposition 77. Let $(a_1, \dots, a_p)$ be a finite family of elements of $\mathbb{R}^n$ and C its convex hull. Let f be the linear mapping from $\mathbb{R}^p$ to $\mathbb{R}^n$ defined by $f(\lambda) = \sum_{j=1}^p \lambda_j a_j$. f is continuous and, furthermore, as C is the set of all convex combination of $(a_1, \dots, a_p)$, C is the image of the simplex of $\mathbb{R}^p$ by f. As the simplex is compact and the image of a compact subset by a continuous function is compact, we conclude that C is compact.

Let $(a_1, \dots, a_p)$ be a finite family of $\mathbb{R}^n$ and

$$C = \left\{ \sum_{i=1}^p \lambda_i a_i \mid \lambda \in R^p_+ \right\}$$

Let P, the set of nonempty subsets I of $\{1, \dots, p\}$ such that the family $(a_i)_{i \in I}$ is linearly independent. From Theorem 34,

$$C = \cup_{I \in P} \left\{ \sum_{i \in I} \lambda_i a_i \mid \lambda \in R^I_+ \right\}$$

As P is finite, it suffices to show that for all $I \in P$, $C_I = \{\sum_{i \in I} \lambda_i a_i \mid \lambda \in R^I_+\}$ is closed. Let $\mathcal{L}((a_i)_{i \in I})$ be the linear space spanned by the family $(a_i)_{i \in I}$ and let φ be the linear mapping from $\mathbb{R}^I$ to $\mathcal{L}((a_i)_{i \in I})$ defined by $\varphi(\lambda) = \sum_{i \in I} \lambda_i a_i$. As the family $(a_i)_{i \in I}$ is linearly independent, φ is one to one and onto from $\mathbb{R}^I$ to $\mathcal{L}((a_i)_{i \in I})$ and φ and φ^{-1} are continuous. C_I is the image of the closed set $\mathbb{R}^I_+$ by φ that is the inverse image of $\mathbb{R}^I_+$ by φ^{-1}. So C_I is closed in $\mathcal{L}((a_i)_{i \in I})$. As $\mathcal{L}((a_i)_{i \in I})$ is a linear subspace of $\mathbb{R}^n$, it is closed in $\mathbb{R}^n$ and so C_I is closed in $\mathbb{R}^n$. □

5.5 Projection on a closed convex set and separation theorems

We now prove the fundamental property of the nonempty convex closed subsets of $\mathbb{R}^n$, namely the fact that there exists a unique projection on C of all points of $\mathbb{R}^n$ for the Euclidean norm. We deduce from this result the separation theorem between two disjoint nonempty convex sets, one being closed and the other one being compact.

If C is a nonempty closed subset of $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$, we can consider the following minimisation problem for the Euclidean norm:

$$\begin{cases} \text{Minimiser } \|x - c\| \\ c \in C \end{cases}$$

A solution of this problem is called a projection of x on C denoted $\text{proj}_C(x)$ and the value of this problem is the distance from x to C denoted $d_C(x)$. We remark that this problem has always a solution since the objective function is coercive and the set C, on which one minimises, is closed

The existence of a solution implies that the distance function is finite on $\mathbb{R}^n$. Furthermore, this function is Lipschitz continuous of constant 1.

Proposition 78 *Let C be a nonempty closed subset of $\mathbb{R}^n$. Then for all $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$, $|d_C(x) - d_C(x')| \leq \|x - x'\|$.*

Proof. Let us assume without any loss of generality that $d_C(x') \leq d_C(x)$. Let y (resp. y') a projection of x (resp. x') on C. Then:

$$d_C(x) - d_C(x') \leq \|x - y\| - \|x' - y'\| \leq \|x - y'\| - \|x' - y'\| \leq \|x - x'\|$$

The first inequality comes from the triangular inequality applied to $x - y' = (x - x') + (x' - y')$. This leads to the desired inequality. □

An essential characteristic of the convex sets is the fact that the projection is unique.

Théorème 1 *Let C be a nonempty closed convex subset of $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$, there exists a unique element c of C called the projection of x on C denoted $\text{proj}_C(x)$ such that $\|x - \text{proj}_C(x)\| = d_C(x)$.*

Proof. Let c_1 and c_2 be two elements of C such that $\|x - c_1\| = \|x - c_2\| \leq \|x - c\|$ for all $c \in C$. Then, let us consider the element $\bar{c} = \frac{1}{2}(c_1 + c_2)$. $\bar{c} \in C$ since C is convex. Since $\|x - c_1\| = \|x - c_2\|$, $x - c_1 - x + c_2 = c_2 - c_1$ is orthogonal to $x - c_1 + x - c_2 = 2x - (c_1 + c_2) = 2(x - \bar{c})$. So the triangle $(x, \bar{c}, c_2)$ has a right angle at $\bar{c}$ and from Pythagoras' Theorem, $\|x - c_2\|^2 = \|x - \bar{c}\|^2 + \|\bar{c} - c_2\|^2$. As $\bar{c} \in C$, we have $\|x - c_2\| \leq \|x - \bar{c}\|$ and, thus, the equality gives $\|\bar{c} - c_2\| = \|\frac{1}{2}(c_1 - c_2)\| = 0$. This implies that $c_1 = c_2$, which proves the uniqueness of the projection. □

Exercise 75 Let $\bar{B}(0,1)$ be the closed unit ball of $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$, compute the projection of x on $\bar{B}(0,1)$.

Let $C = [-1,1] \times [-1,1] \subset \mathbb{R}^2$. For all $x \in \mathbb{R}^2$, compute the projection of x on C.

Let $u \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{R}$ and $H = \{x \in \mathbb{R}^n \mid u \cdot x \leq c\}$. For all $x \in \mathbb{R}^n$, compute the projection of x on H.

We now provide some properties of the projection.

Proposition 79 *Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.*
1) $c \in C$ is the projection of x on C if and only if for all $c' \in C$, $(x-c)\cdot(c'-c) \leq 0$;
2) $(\mathrm{proj}_C(x) - \mathrm{proj}_C(y)) \cdot (x-y) \geq 0$;
3) $\|\mathrm{proj}_C(x) - \mathrm{proj}_C(y)\| \leq \|x-y\|$.

Proof. 1) If $c = \mathrm{proj}_C(x)$, for all $c' \in C$, for all $t \in [0,1]$, as $(1-t)c + tc' \in C$, $\|x-c\|^2 \leq \|x - ((1-t)c + tc')\|^2 = \|(x-c) + t(c'-c)\|^2$. By developing the square of the norm, we obtain $0 \leq 2t(x-c)\cdot(c'-c) + t^2\|c'-c\|^2$. Dividing by t and taking the limit when t converges to 0^+, we obtain $0 \leq (x-c)\cdot(c'-c)$. Conversely, $\|x-c'\|^2 = \|(x-c)+(c-c')\|^2 = \|x-c\|^2 + \|c-c'\|^2 + 2(x-c)\cdot(c-c')$. So, if $(x-c)\cdot(c'-c) \leq 0$, we obtain $\|x-c'\|^2 \geq \|x-c\|^2$ and since this inequality holds true for all $c' \in C$, c is the projection of x on C.
2) We write the previous inequality at x for $\mathrm{proj}_C(y)$ and at y for $\mathrm{proj}_C(x)$, which leads to:

$(x - \mathrm{proj}_C(x)) \cdot (\mathrm{proj}_C(y) - \mathrm{proj}_C(x)) \leq 0$

$(y - \mathrm{proj}_C(y)) \cdot (\mathrm{proj}_C(x) - \mathrm{proj}_C(y)) = (\mathrm{proj}_C(y) - y) \cdot (\mathrm{proj}_C(y) - \mathrm{proj}_C(x)) \leq 0$

By summing them, we find

$(x - y + \mathrm{proj}_C(y) - \mathrm{proj}_C(x)) \cdot (\mathrm{proj}_C(y) - \mathrm{proj}_C(x)) = (x-y)\cdot(\mathrm{proj}_C(y) - \mathrm{proj}_C(x)) + \|proj_C(y) - \mathrm{proj}_C(x)\|^2 \leq 0$

from which we derive the desired inequality since $\|\mathrm{proj}_C(y) - \mathrm{proj}_C(x)\|^2 \geq 0$.
3) With the same inequalities as above, $(x-y)\cdot(\mathrm{proj}_C(x) - \mathrm{proj}_C(y)) \geq \|\mathrm{proj}_C(y) - \mathrm{proj}_C(x)\|^2$ and with the Cauchy-Schwartz inequality, one deduces that $\|x-y\|\|\mathrm{proj}_C(y) - \mathrm{proj}_C(x)\| \geq \|\mathrm{proj}_C(y) - \mathrm{proj}_C(x)\|^2$, thus, $\|\mathrm{proj}_C(y) - \mathrm{proj}_C(x)\| \leq \|x-y\|$. $\square$

Exercise 76 Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Show that for all $t \geq 0$, $\mathrm{proj}_C(x + t(x - \mathrm{proj}_C(x))) = \mathrm{proj}_C(x)$.

We deduce from the existence of a projection and its continuity a separation Theorem between two disjoint convex sets, which can be interpreted either from a geometric point of view or from an analytic point of view. We start by the analytic form.

Theorem 35 *Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let D be a nonempty compact convex subset of $\mathbb{R}^n$. $C \cap D = \emptyset$ if and only if there exists*

$u \in \mathbb{R}^n$ *such that:*

$$\sup\{u \cdot d \mid d \in D\} = \max\{u \cdot d \mid d \in D\} < \inf\{u \cdot c \mid c \in C\}$$

A particular case of this theorem is when D is a singleton $\{x_0\}$. Then, we get the following inequality $\sup\{u \cdot d \mid d \in D\} = u \cdot x_0 < \inf\{u \cdot c \mid c \in C\}$.

Proof. Let us consider the following optimisation problem:

$$\begin{cases} \text{Minimise } d_C(x) \\ x \in D \end{cases}$$

This problem has a solution since D is nonempty and compact and d_C is a continuous function. Let $\bar{d}$ be a solution and $\bar{c} = \text{proj}_C(\bar{d})$. As $C \cap D = \emptyset$ $\bar{c} \neq \bar{d}$. Let $u = \bar{c} - \bar{d}$.

From Proposition 79 (1), for all $c \in C$, $(\bar{d} - \bar{c}) \cdot (c - \bar{c}) \leq 0$ and, thus, $u \cdot c \geq u \cdot \bar{c} = \inf\{u \cdot c \mid c \in C\}$.

For all $d \in D$, $\|d - \bar{c}\| \geq d_C(d) \geq d_C(\bar{d}) = \|\bar{d} - \bar{c}\|$, which shows that $\bar{d}$ is the projection of $\bar{c}$ on D. Hence, with the same reasoning, we deduce that $\sup\{u \cdot d \mid d \in D\} = u \cdot \bar{d}$. But $u \cdot \bar{c} - u \cdot \bar{d} = \|u\|^2 > 0$, which leads to the result.
$\square$

The geometric interpretation of this theorem is the following: it exists an hyperplan H of $\mathbb{R}^n$ such that D is included in the open half-space defined by H and C is included in the other open half-space. For this, we choose a real number $\alpha \in]\sup\{u \cdot d \mid d \in D\}, \inf\{u \cdot c \mid c \in C\}[$ and we define $H = \{y \in \mathbb{R}^n \mid u \cdot y = \alpha\}$. Then D is inclued in the open half-space $\{y \in \mathbb{R}^n \mid u \cdot y < \alpha\}$ and C is included in the open half-space $\{y \in \mathbb{R}^n \mid u \cdot y > \alpha\}$. So, the hyperplan H separates D and C.

Note that the theorem may not be true for two closed convex set. Indeed, let $D = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$ and $C = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, xy \geq 1\}$. We remark that $D \cap C = \emptyset$ but:

Exercise 77 Show that it does not exist a vector $u \in \mathbb{R}^2$, such that $\sup\{u \cdot d \mid d \in D\} < \inf\{u \cdot c \mid c \in C\}$. A graphical representation of C and D may help.

Exercise 78 Let A be an affine subspace of $\mathbb{R}^n$ and $x \notin A$. Show that if $u \in \mathbb{R}^n$ separates x from A in the sense that $u \cdot x < \inf\{u \cdot a \mid a \in A\}$, then u belongs to the orthogonal of the direction of A or that the mapping from A to $\mathbb{R}$ defined by $a \to u \cdot a$ is constant on A.

Let $\bar{B}(0, 1)$ be the closed unit ball of $\mathbb{R}^n$. For all $x \notin \bar{B}(0, 1)$, find a vector u of $\mathbb{R}^n$ such that $u \cdot x < \inf\{u \cdot y \mid y \in \bar{B}(0, 1)\}$.

Let $C = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. For all $x \notin C$, find a vector u of $\mathbb{R}^n$ such that $u \cdot x < \inf\{u \cdot y \mid y \in C\}$.

Let $u \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{R}$ and $H = \{x \in \mathbb{R}^n \mid u \cdot x \leq c\}$. For all $x \notin H$, find a vector v of $\mathbb{R}^n$ such that $v \cdot x < \inf\{v \cdot y \mid y \in H\}$.

Let $u \in \mathbb{R}^n \setminus \{0\}$, $a \in \mathbb{R}$ and $H = \{x \in \mathbb{R}^n \mid u \cdot x = a\}$. We assume that $H \cap \bar{B}(0,1) = \emptyset$. Find an hyperplan which separates H from $\bar{B}(0,1)$.

Exercise 79 Let E be a linear subspace of $\mathbb{R}^n$ such that $E \cap \mathbb{R}^n_+ = \{0\}$. The goal of the exercise is to show that there exists a vector $u \in E^\perp \cap \mathbb{R}^n_{++}$.
1) Let $S = \{x \in \mathbb{R}^n_+ \mid \sum_{i=}^n x_i = 1\}$. Show that $E \cap S = \emptyset$.
2) By applying the separation Theorem, show that there exists a vector $u \in \mathbb{R}^n \setminus \{0\}$ such that $\sup\{u \cdot x \mid x \in E\} < \inf\{u \cdot s \mid s \in S\}$.
3) Show that $u \in E^\perp$ and that $\sup\{u \cdot x \mid x \in E\} = 0$.
4) Using the fact that the vectors of the canonical basis belong to S, deduce that all components of u are non negative.

Exercise 80 Let C and D be two nonempty closed convex subsets of $\mathbb{R}^n$. We assume that $C \subset D$. Show that for all $x \in \mathbb{R}^n$,

$$\|\mathrm{proj}_C(x) - \mathrm{proj}_D(x)\|^2 \leq (d_C(x))^2 - (d_D(x))^2$$

We can remark that $\mathrm{proj}_C(x) \in D$ and use a characterisation of the projection on D.

Exercise 81 The goal of this exercise is to show that a compact convex set can be approximated by a polyhedron, that is a set defined by a finite number of affine inequalities.

Let C be a nonempty convex compact subset of $\mathbb{R}^n$. Let $r > 0$ and $U = C + B(0,r)$.
1) Show that U is an open convex subset and $\bar{U}$, the closure of U is compact, $\mathrm{Fr}(U)$, the frontier of U, is compact and $\mathrm{Fr}(U) \cap C = \emptyset$.
2) Applying a separation theorem, show that, for all $x \in \mathrm{Fr}(U)$, there exists a vector $u_x \in \mathbb{R}^n \setminus \{0\}$ and a real number a_x such that $u_x \cdot x > a_x > \sup\{u_x \cdot c \mid c \in C\}$.

Let H_x^+ be the open half space defined by $H_x^+ = \{y \in \mathbb{R}^n \mid u_x \cdot y > a_x\}$.
3) Show that $\mathrm{Fr}(U) \subset \cup_{x \in \mathrm{Fr}(U)} H_x^+$.
4) Using the compactness of $\mathrm{Fr}(U)$, show that there exists a finite number of points $(x^1, x^2, \ldots, x^p)$ in $\mathrm{Fr}(U)$ such that $\mathrm{Fr}(U) \subset \cup_{j=1}^p H_{x^j}^+$.
5) Show that C is included in P the polyhedron defined by $P = \{z \in \mathbb{R}^n \mid \forall j = 1, \ldots, p, u_{x^j} \cdot z \leq a_{x^j}\}$.
6) We now prove that P is a subset of U by contraposition. Let $\bar{x} \in P$ such that $\bar{x} \notin U$. Let $\bar{c}$ be an element of C. Show that there exists $t \in [0,1[$ such that $x^t = (1-t)\bar{x} + t\bar{c} \in \mathrm{Fr}(U)$. Show that $x^t \in P$ and deduce a contradiction.
7) In $\mathbb{R}^2$, provide explicitly the inequalities defining a polyhedron P containing $\bar{B}(0,1)$ and included in $B(0,2)$. Show that it does not exists a polyhedron containing the set $C = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2\}$ and included $C + B(0,1)$. Hint: we can show that $C + B(0,1)$ is included in the epigraph of a parabola and look at the behaviour at infinity.

5.6 Farkas' Lemma

5.6.1 Polarity

We study in this section the notion of polar cones, which generalises the notion of orthogonality for linear subspaces.

Definition 49 Let A, be a subset of $\mathbb{R}^n$. The negative polar cone of A, denoted A° is defined as follows:

$$A^\circ = \{u \in \mathbb{R}^n \mid \text{ for all } a \in A, u \cdot a \leq 0\}$$

Proposition 80 *Let A, be a subset of $\mathbb{R}^n$.*
(i) A° is a closed convex cone of vertex 0;
(ii) If $A \subset B$, then $B^\circ \subset A^\circ$;
(iii) If A is a linear subpace, then $A^\circ = A^\perp$;
(iv) If A is a cone, then u belongs to A° if and only if the function $v \to u \cdot v$ is upper bounded on A.

The proof is left to the reader.

Exercice 1 Compute the polar cone of the following sets:
2) $C = \bar{B}(0,1)$;
2) $C = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x_i \geq 0\}$;
3) $u \in \mathbb{R}^n \setminus \{0\}$; $C = \{x \in \mathbb{R}^n \mid u \cdot x \geq 0\}$;
4) u and v two non zero non colinear vectors of $\mathbb{R}^2$; $C = \{x \in \mathbb{R}^2 \mid \exists(\lambda_1, \lambda_2) \in \mathbb{R}^2_+, x = \lambda_1 u + \lambda_2 v\}$;
4) $C = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 \leq z^2\}$.

Proposition 81 *Let A be a closed convex cone of $\mathbb{R}^n$ and let $x \notin A$.*
1) Let $u \in \mathbb{R}^n$ such that $u \cdot x \leq \inf\{u \cdot a \mid a \in A\}$. Then $-u \in A^\circ$.
2) Let a be the projection of x on A. Then $x - a$ is orthogonal to a and $x - a$ is the projection of x on A°.
3) Let $(a,b) \in A \times A^\circ$ such that $x = a + b$ and $a \cdot b = 0$. Then a is the projection of x on A and b is the projection of x on A°.

The second point of this proposition provides a decomposition of a vector x on a closed convex cone and its negative polar cone similar to the one we have with a linear subspace and its orthogonal. We can notice that the vector x may have several decomposition as the sum of an element of A and an element of A° but there exists only one such that the two vectors of the decomposition are orthogonal.

Proof.

1) If $-u \notin A^\circ$, the exists $a \in A$ such that $u \cdot a < 0$. For all $t > 0$, $ta \in A$ and $\lim_{t \to +\infty} u \cdot (ta) = -\infty$, so $\inf\{u \cdot a \mid a \in A\} = -\infty$ which is in contradiction with $u \cdot x \leq \inf\{u \cdot a \mid a \in A\}$.
2) For all $t > 0$, $ta \in A$ and since a is the projection of x on A, $\|x - ta\|^2 \geq \|x - a\|^2$. Thus $\|x - ta\|^2 = \|x - a + (1-t)a\|^2 = \|x - a\|^2 + 2(1-t)(x-a) \cdot a + (1-t)^2\|a\|^2$. So, $2(1-t)(x-a) \cdot a + (1-t)^2\|a\|^2 \geq 0$. For all $t < 1$, we obtain by dividing by $1-t$, $2(x-a) \cdot a + (1-t)\|a\|^2 \geq 0$ and taking the limit when t tends to 1^-, $2(x-a) \cdot a \geq 0$. For all $t > 1$, we obtain by dividing by $1-t$, $2(x-a) \cdot a + (1-t)\|a\|^2 \leq 0$ and taking the limit when t tends to 1^+, $2(x-a) \cdot a \leq 0$. Then we get the result $(x-a) \cdot a = 0$.

Let us show that $x - a \in A^\circ$. For all $a' \in A$ and for all $t \in [0,1]$, as A is convex, $a^t = (1-t)a + ta' \in A$. Hence $\|x - a\|^2 \leq \|x - a^t\|^2 = \|(x-a) + (a - a^t)\|^2$, from which one deduces that $0 \leq 2(x-a) \cdot (a - a^t) + \|a - a^t\|^2 = 2t(x-a) \cdot (a - a') + t^2\|a - a'\|^2$. Dividing by t and taking the limit when t tends to 0^+, we obtain $0 \leq 2(x-a) \cdot (a - a')$ and since $(x-a) \cdot a = 0$, $(x-a) \cdot a' \leq 0$. Finally, $x - a \in A^\circ$.

Let us show now that $x - a$ is the projection of x on A°. Let $b \in A^\circ$. Let us compute $\|x - b\|^2$. $\|x - b\|^2 = \|x - (x-a) + (x - a - b)\|^2 = \|a + (x - a - b)\|^2 = \|a\|^2 + 2a \cdot (x-a) - 2a \cdot b + \|x - a - b\|^2$. $a \cdot (x-a) = 0$ from above and, since $a \cdot b \leq 0$ knowing that $a \in A$ and $b \in A^\circ$. So $\|x - b\|^2 \geq \|a\|^2 = \|x - (x-a)\|^2$. This shows that $x - a$ is the closest point from x in A°.
3) The proof is similar to the one above to show that $x - a$ is the projection of x on A° using the fact that $a \cdot b = 0$, $a \cdot b' \leq 0$ for all $b' \in A^\circ$ and $a' \cdot b \leq 0$ for all $a' \in A$. □

Exercise 82 Let K be a closed convex cone of $\mathbb{R}^n$.
1) Show that $x \in K^\circ$ if and only if $\text{proj}_K(x) = 0$.
2) Show that for all $x \in \mathbb{R}^n$ and for all $t > 0$, $\text{proj}_K(tx) = t\text{proj}_K(x)$.

We know state the bipolar's Theorem which is a consequence of the separation's Theorem.

Theorem 36 *(Bipolar's Theorem) Let A be a nonempty closed convex cone of $\mathbb{R}^n$. Then $(A^\circ)^\circ = A$.*

Proof of Theorem 36. $(A^\circ)^\circ$ is a closed convex cone, which obviously contains A. Les us show the converse inclusion by contradiction. Let $x_0 \in (A^\circ)^\circ$ such that $x_0 \notin A$. Then, we use the Separation Theorem between x_0 and the closed convex set A. So, there exists $u \in \mathbb{R}^n$ such that $\sup\{u \cdot a \mid a \in A\} < u \cdot x_0$. As $v \to u \cdot v$ is upper bounded on the cone A, one deduces that $u \in A^\circ$ and that $0 = \sup\{u \cdot a \mid a \in A\}$. Then, $u \cdot x_0 > 0$ contradicts $x_0 \in (A^\circ)^\circ$ which ends the proof. □

Exercise 83 Let K be a closed convex cone of $\mathbb{R}^n$.
1) Show $K \cap -K$ is a linear subspace of $\mathbb{R}^n$.

2) Show that $K^\circ \subset (K \cap -K)^\perp$.
3) Show that $K = (K \cap -K) + (K \cap (K \cap -K)^\perp)$.
4) Show that K is a linear subspace of $\mathbb{R}^n$ if and only if $K = -K$.

Exercise 84 Let C be a closed convex subset of $\mathbb{R}^n$ containing 0. Let

$$K = \{u \in \mathbb{R}^n \mid \forall t > 0, tu \in C\}$$

1) Show that K is a closed convex cone.
2) Show that $K = \{0\}$ if C is bounded.
3) Show that $C + K = C$.
4) We show in this question that $K \neq \{0\}$ if C is not bounded.
a) Show that for all $\nu \in \mathbb{N}^*$, there exists $c_\nu \in C$ such that $\|c_\nu\| \geq \nu$.
b) Show that the sequence $\left(d_\nu = \frac{1}{\|c_\nu\|} c_\nu\right)$ has a converging sub-sequence $(d_{\varphi(\nu)})$ and that $d_\nu \in C$ for all ν.
We denote by d the limit of the sub-sequence. Let $t > 0$.
c) Show that $td = \lim_{\nu \to \infty} td_{\varphi(\nu)} = \lim_{\nu \to \infty} \frac{t}{\|c_\nu\|} c_\nu$ and conclude that $td \in C$.
d) Show that $d \in K$ and conclude.

5.6.2 Farkas' Lemma

Theorem 37 *(Farkas' Lemma) Let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$, two finite families of elements of $\mathbb{R}^n$. Let*

$$A = \left\{ \sum_{i \in I} \lambda_i a_i + \sum_{j \in J} \mu_j b_j \mid \lambda \in R_+^I,\ \mu \in R^J \right\}$$

and

$$B = \{v \in \mathbb{R}^n \mid a_i \cdot v \leq 0,\ \forall i \in I,\ b_j \cdot v = 0,\ \forall j \in J\}$$

Then, $A^\circ = B$ and $B^\circ = A$.

Proof of Theorem 37. Clearly, $A^\circ = B$, $A \subset B^\circ$ and B is a closed convex cone. A is finitely generated by the family $\big((a_i)_{i \in I}, (b_j)_{j \in J}, (-b_j)_{j \in J}\big)$. Thus, from Proposition 77, A is closed. The Bipolar Theorem implies that $A = B^\circ$. $\square$

Corollary 7 *Let $(a_i)_{i \in I}$ be a finite family of elements of $\mathbb{R}^n$. Let*

$$A = \left\{ \sum_{i \in I} \lambda_i a_i \mid \lambda \in R_+^I \right\}$$

and

$$B = \{v \in \mathbb{R}^n \mid a_i \cdot v \leq 0,\ \forall i \in I\}$$

Then, $A^\circ = B$ and $B^\circ = A$.

Corollary 8 *Let $(a_i)_{i \in I}$, be a finite family of elements in $\mathbb{R}^n$. Let $b \in \mathbb{R}^n$ satisfying the following property:*

for all $v \in \mathbb{R}^n$ satisfying $a_i \cdot v \leq 0$, for all $i \in I$, then $b \cdot v \leq 0$.
Then, it exists $\lambda \in R_+^I$ such that $b = \sum_{i \in I} \lambda_i a_i$.

Proof of Corollary 8. Let $A = \{\sum_{i \in I} \lambda_i a_i \mid \lambda \in R_+^I\}$ and $B = \{v \in \mathbb{R}^n \mid a_i \cdot v \leq 0, \text{ forall } i \in I\}$. Then, the condition

for all $v \in \mathbb{R}^n$ satisfying $a_i \cdot v \leq 0$, for all $i \in I$, then $b \cdot v \leq 0$.
is equivalent to $b \in B^\circ$. From the previous corollary, this implies that $b \in A$, that is b is a non negative linear combination of the vectors $(a_i)_{i \in I}$. $\square$

Exercise 85 Compute the polar cone of the following cones:
1) $C = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 0, x_1 - 2x_2 + x_3 = 0\}$;
2) $C = \{\lambda(1, 2, -3) + \mu(-1, -1, 2) \mid \lambda \in \mathbb{R}, \mu \geq 0\}$
3) $C = \{x \in \mathbb{R}^2 \mid x_1 + 2x_2 \geq 0; 2x_1 - x_2 \leq 0; -x_1 + x_2 \geq 0\}$.

Exercise 86 Let K be a finitely generated convex cone of $\mathbb{R}^n$ such that $K \cap \mathbb{R}_+^n = \{0\}$. The goal of the exercise is to show that there exists $u \in K^\circ \cap \mathbb{R}_{++}^n$. Let S^n be the simplex of $\mathbb{R}^n$.

1) Show that $K \cap S^n = \emptyset$.

2) By applying the Separation Theorem, show that there exists $u \in \mathbb{R}^n$ such that for all $(x, s) \in K \times S$, $\sup_{x \in K}\{u \cdot x\} < \inf_{s \in S}\{u \cdot s\}$.

3) Deduce first that $u \in K^\circ$ and then that $u \in \mathbb{R}^{++}$.

Chapter 6

Karush-Kuhn-Tucker Theorem

6.1 Karush-Kuhn-Tucker first order Conditions

In this chapter, we combine the results presented in the previous chapters to deal with optimisation problems with equality and inequality constraints. We present the general case before and then the convex case, which exhibits the nice feature that the first order necessary conditions are also sufficient.

We also provide second order necessary conditions and second order sufficient conditions for local solutions.

We posit the following framework: U is an open subset of $\mathbb{R}^n$. f, $(g_i)_{i=1}^p$, $(h_j)_{j=1}^q$ are $\mathcal{C}^1$ functions from U to $\mathbb{R}$. The optimisation problem is:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ g_i(x) = 0, \, \forall i = 1, \dots p \\ h_j(x) \leq 0, \, \forall j = 1 \dots, q \\ x \in U \end{cases}$$

Let $\bar{x}$ be a (local) solution of this problem. With inequality constraints, we need to distinguish binding constraints, j such that $h_j(\bar{x}) = 0$ and non binding constraints, j such that $h_j(\bar{x}) < 0$. So $J(\bar{x}) = \{j \in \{1, \dots, q\} \mid h_j(\bar{x}) = 0\}$ is the set of binding constraints. Roughly speaking, the non-binding constraints have no influence on the solution and we can forget them like in the case of unconstrained optimisation where we do not care about the open constraint defining the open set U.

To obtain the necessary optimality condition, we need to posit a so-called constraint qualification condition. For this qualification constraint, we need to distinguish among the inequality constraints between the linear ones and the non linear ones. We let $J_a \subset \{1, \dots, q\}$ be the set of linear constraints, that is $h_j(x) = a_j \cdot x + b_j$ and $J_{na} \subset \{1, \dots, q\}$ be the set of non linear constraints. So, at $\bar{x}$, the set of inequality constraints $\{1, \dots, q\}$ is partitioned into three subsets: $J_a(\bar{x})$, linear binding constraints, $J_{na}(\bar{x})$, non-linear binding constraints, and the remaining constraints, which are the non-binding constraints.

We now state the Mangasarian-Fromovitz qualification condition.

Definition 50 The point $\bar{x}$ satisfies the Mangasarian-Fromovitz constraint qualification condition if the vectors $(\nabla g_i(\bar{x}))_{i=1}^p$ are linearly independent and it exists a vector $\bar{u} \in \mathbb{R}^n$ such that $\nabla g_i(\bar{x}) \cdot \bar{u} = 0$, for all i, $\nabla h_j(\bar{x}) \cdot \bar{u} < 0$, for all $j \in J_{na}(\bar{x})$ and $\nabla h_j(\bar{x}) \cdot \bar{u} \leq 0$, for all $j \in J_a(\bar{x})$.

Note that if there is no inequality constraint, we merely recover the condition on the linear independence of the gradient vectors of the equality constraints. This qualification condition may be hard to check and we now provide a stronger one, which is often easier to check.

Definition 51 The point $\bar{x}$ satisfies the Linear Independence constraint qualification condition if the vectors $\left((\nabla g_i(\bar{x}))_{i=1}^p, (\nabla h_j(\bar{x}))_{j \in J(\bar{x})}\right)$ are linearly independent.

Proposition 82 *If the point $\bar{x}$ satisfies the Linear Independence constraint qualification condition then it satisfies the Mangasarian-Fromovitz constraint qualification condition.*

Proof. Let us consider the linear mapping ϕ from $\mathbb{R}^n$ to $\mathbb{R}^{p+\sharp J(\bar{x})}$ defined by:

$$\phi(u) = \left((\nabla g_i(\bar{x}) \cdot u)_{i=1}^p, (\nabla h_j(\bar{x}) \cdot u)_{j \in J(\bar{x})}\right)$$

From the Linear independence condition, ϕ is of rank $p + \sharp J(\bar{x})$ so it is onto. Hence, there exists an inverse image u of the element $((0, \dots, 0), (-1, \dots, -1))$ and so the MF qualification condition is satisfied.□

We are now able to state the Karush-Kuhn-Tucker Theorem, which provides the first order necessary condition.

Theorem 38 *(Karush-Kuhn-Tucker differentiable) Let $\bar{x}$ be a (local) solution of the problem $(\mathcal{P})$. We assume that it satisfies the MF qualification condition. Then it exists $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^q_+$ such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{q} \mu_j \nabla h_j(\bar{x}) = 0$$

and $\mu_j h_j(\bar{x}) = 0$ for all $j = 1, \dots, q$. If the LI qualification condition is satisfied at $\bar{x}$, then the multipliers (λ, μ) are unique.

Note that $\bar{x}$ satisfies the constraints of the problem, that is $g_i(\bar{x}) = 0$ for all i, $h_j(\bar{x}) \leq 0$ for all j. The condition $\mu_j h_j(\bar{x}) = 0$ for all $j = 1, \dots, q$ is called the complementarity condition. Actually, it just means that the multiplier $\mu_j = 0$ if the constraint j is not binding. Note that it does not imply that the multiplier is not equal to 0 when the constraint is binding. So, as mentioned above, the gradients of the non-binding constraints play no role in the formula since the coefficients μ_j are equal to 0.

We can provide the following interpretation. Under the assumptions of the theorem, $\bar{x}$ is a solution of the following linearised problem:

$$(\mathcal{P}_\ell) \begin{cases} \text{Minimise } \nabla f(\bar{x}) \cdot x \\ \nabla g_i(\bar{x}) \cdot (x - \bar{x}) = 0, \ \forall i = 1, \ldots, p \\ \nabla h_j(\bar{x})(x - \bar{x}) \leq 0, \ \forall j \in J(\bar{x}) \\ x \in \mathbb{R}^n \end{cases}$$

In other words, $\bar{x}$ is a minimum of the linear mapping $\nabla f(\bar{x}) \cdot x$, which is the first order approximation of the objective function around $\bar{x}$ on the polyhedron

$$S_\ell = \{x \in \mathbb{R}^n \mid \nabla g_i(\bar{x}) \cdot (x - \bar{x}) = 0, \forall i = 1, \ldots, p, \nabla h_j(\bar{x})(x - \bar{x}) \leq 0, \forall j \in J(\bar{x})\}$$

Sketch of the Proof of Theorem 38. From the Farkas' Lemma, it suffices to show that $\nabla f(\bar{x}) \cdot u \geq 0$ for all $u \in T(\bar{x}) = \{u \in \mathbb{R}^n \mid \nabla g_i(\bar{x}) \cdot u = 0, \forall i, \nabla h_j(\bar{x}) \cdot u \leq 0, \forall j \in J(\bar{x})\}$. From the MF qualification condition $T(\bar{x})$ is in the closure of $\mathcal{T}(\bar{x}) = \{u \in \mathbb{R}^n \mid \nabla g_i(\bar{x}) \cdot u = 0, \forall i, \nabla h_j(\bar{x}) \cdot u < 0, \forall j \in J_{na}(\bar{x}), \nabla h_j(\bar{x}) \cdot u \leq 0, \forall j \in J_a(\bar{x})\}$, so it suffices to show that $\nabla f(\bar{x}) \cdot u \geq 0$ for all $u \in \mathcal{T}(\bar{x})$.

Let us take $u \in \mathcal{T}(\bar{x})$. From the independence of the gradient vectors of the equality constraints, we consider the mapping ψ is in the proof of the Lagrange's Theorem. We prove that $\psi(t)$ satisfies the constraints of the problem $(\mathcal{P})$ for t in an interval $[0, \tau[$ for τ small enough distinguishing the affine binding constraints, the non-affine binding constraints and the non-binding constraints. Then, as in the proof of the Lagrange's Theorem, considering the function $f(\psi(t))$, we prove that $\nabla f(\bar{x}) \cdot \psi'(0) = \nabla f(\bar{x}) \cdot u \geq 0$. □

We now consider the convex case, that is we assume that f is convex, the equality constraints g_i are affine and inequality constraints are convex. We remark that the set $S = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \forall i = 1, \ldots, p, h_j(x) \leq 0, \forall j = 1, \ldots q\}$ is convex.

In this framework, we state the Slater qualification condition, which is not related to an element of S, which is a global property of S.

Definition 52 We assume that the functions $(g_i)_{i=1}^p$ are affine and that the functions $(h_j)_{j=1}^q$ are convex. We denote by J_a the subset of $\{1, \ldots, q\}$ such that g_j is linear and by J_{na} its complement. Then the set $S = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \forall i = 1, \ldots, p, h_j(x) \leq 0, \forall j = 1, \ldots q\}$ satisfies the Slater constraint qualification condition if there exists $\bar{x} \in S$ such that $h_j(\bar{x}) < 0$ for all $j \in J_{na}$.

Note that the Slater's condition is satisfied when we have only linear constraints if and only if S is nonempty.

Note that the MF qualification condition may not be satisfied if the Slater's condition holds. Indeed, the gradients of the equality constraints may not be linearly independent. Nevertheless, in this case, since the equality constraints are linear, the gradient vector does not depend on the point we consider, so we can delete the useless constraints without modifying the constraint set S. So, without any loss of generality, we can assume that the gradient vectors of the equality constraints are linearly independent. In this case, the Slater's condition implies that the MF condition is satisfied for all $x \in S$.

Proposition 83 *We assume the set* $S = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \forall i = 1, \ldots, p, h_j(x) \leq 0, \forall j = 1, \ldots q\}$ *satisfies the Slater constraint qualification condition and that the gradient vectors* $(\nabla g_i(0))_{i=1}^p$ *are linearly independent, then for all* $x \in S$*, the MF qualification condition is satisfied.*

Proof. Let $\bar{x} \in S$. Let $u = \underline{x} - \bar{x}$ where $\underline{x}$ is given by the Slater condition. For all i, since $\underline{x}$ and $\bar{x}$ belongs to S, $g_i(\underline{x}) = a_i \cdot \underline{x} + b_i = a_i \cdot \bar{x} + b_i = 0$, so $a_i \cdot u = 0$. For all $j \in J_a(\bar{x})\}$, $h_j(\underline{x}) = a_j \cdot \underline{x} + b_j \leq 0 = h_j(\bar{x}) = a_i \cdot \bar{x} + b_i$, so $a_j \cdot u \leq 0$. Finally, for all $j \in J_{na}(\bar{x})\}$, $h_j(\underline{x}) < 0 = h_j(\bar{x})$. But, since h_j is convex, $h_j(\underline{x}) - h_j(\bar{x}) \geq \nabla h_j(\bar{x}) \cdot (\underline{x} - \bar{x})$, so $\nabla h_j(\bar{x}) \cdot u < 0$. $\square$

Now we state the convex version of the KKT Theorem. We maintain the initial assumption that the equality constraint functions are linear and the inequality constraints functions are split among linear ones and non-linear ones.

Theorem 39 *(Karush-Kuhn-Tucker convex) Let us consider the following problem:*

$$(\mathcal{P}) \begin{cases} \textit{Minimiser } f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \\ h_j(x) \leq 0,\ j = 1, \ldots, q \\ x \in U \end{cases}$$

We assume that U *is an open convex subset of* $\mathbb{R}^n$*, the functions* f *and* h_j *for* $j \in J_{na}$ *are* $\mathcal{C}^1$ *convex on* U *and the functions* g_i *and* h_j *for* $j \in J_a$ *are linear. We also assume that the Slater's condition is satisfied by the set* $S = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \forall i = 1, \ldots, p, h_j(x) \leq 0, \forall j = 1, \ldots q\}$*. Let* $\bar{x}$ *be a (local) solution. Then, it exists* $\lambda \in \mathbb{R}^p$*,* $\mu \in \mathbb{R}^q_+$ *such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{q} \mu_j \nabla h_j(\bar{x}) = 0$$

and $\mu_j h_j(\bar{x}) = 0$ *for all* $j \in J$*.*

Let $\bar{x}$ *an element of* S *satisfying the above first order condition. Then* $\bar{x}$ *is a solution of Problem* $(\mathcal{P})$*.*

As before, we can interpret this result by saying that $\bar{x}$ is a solution of a linearised problem where the objective function and the constraint functions are all linear. Note that the most interesting part of this theorem is the second one, which provides a sufficient condition.

Proof. For the necessary condition, we just apply the previous theorem 38 after having deleted the redundant equality constraints if necessary to get the MF qualification condition as a consequence of the Slater's condition. At the end, we can re-enlist these deleted constraints with an associated multiplier equal to 0.

For the sufficient part, let $x \in S$. Using the fact that the functions g_i are linear, we get for all i, $g_i(x) = a_i \cdot x + b_i = a_i \cdot \bar{x} + b_i = 0$, so $a_i \cdot u = \nabla g_i(\bar{x}) \cdot (x - \bar{x}) = 0$.

For all $j \in J(\bar{x})$, since h_j is convex, $h_j(x) \leq 0 = h_j(\bar{x})$, so $\nabla h_j(\bar{x})\cdot(x-\bar{x}) \leq 0$ and $\mu_j \nabla h_i(\bar{x})\cdot(x-\bar{x}) \leq 0$ since $\mu_j \geq 0$. For all $j \notin J(\bar{x})$, $\mu_j = 0$ so $\mu_j \nabla h_i(\bar{x})\cdot(x-\bar{x}) = 0$. Consequently,

$$\left[\sum_{i=1}^{p} \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^{q} \mu_j \nabla h_j(\bar{x})\right] \cdot (x - \bar{x}) \leq 0$$

Hence $\nabla f(\bar{x}) \cdot (x - \bar{x}) \geq 0$. Since f is convex, $f(x) \geq f(\bar{x}) + \nabla f(\bar{x}) \cdot (x - \bar{x})$, so $f(x) \geq f(\bar{x})$, which shows that $\bar{x}$ is a solution of Problem $(\mathcal{P})$. $\square$

Exercise 87 Let $u = (u_1, u_2)$ be a non zero vector of $\mathbb{R}^2$ and let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x) = u \cdot x = u_1x_1 + u_2x_2$. Using simple argument and the sign of u_1 and u_2, find the solution of the following optimisation problem:
1) $\min\{f(x) \mid x_1 \geq 0, x_2 \geq 0\}$;
2) $\min\{f(x) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$;
3) $\min\{f(x) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$;
4) $\min\{f(x) \mid x_1 \in [-1, 2], x_2 \in [0, 1]\}$;
5) $\min\{f(x) \mid |x_1| + |x_2| \leq 1\}$;
6) $\min\{f(x) \mid \max\{|x_1|, |x_2|\} \leq 1\}$;

Exercise 88 Let f be a $\mathcal{C}^1$ function on $\mathbb{R}^n$. We consider the following minimisation problem

$$\begin{cases} \text{Minimise } f(x) \\ x_i \geq 0,\ i = 1, \dots, n, \\ x \in \mathbb{R}^n \end{cases}$$

Let $\bar{x}$ be a (local) solution of this problem. Show that $\nabla f(\bar{x}) \geq 0$ and $\nabla f(\bar{x})\cdot\bar{x} = 0$.

Exercise 89 Solve the following optimisation problem:

$$\begin{cases} \text{Minimise } x^2 + y^2 \\ 2x + y \leq -4 \end{cases}$$

Let us consider the following optimisation problem:

$$\begin{cases} \text{Maximise } 3x_1x_2 - x_2^3 \\ x_1 \geq 0,\ x_2 \geq 0 \\ x_1 - 2x_2 = 5 \\ 2x_1 + 5x_2 \geq 20 \end{cases}$$

Draw the feasible set and show that the positivity constraints are non binding at the solution. Write the KKT conditions and find the solution.

Exercise 90 Solve the following optimisation problems:

$$\begin{cases} \text{Maximise } \ln(x_1x_2x_3) \\ x_1^2 + x_2^2 + x_3^2 \leq 4 \\ x_1 + x_2 + x_3 = 3 \\ x_1 > 0, x_2 > 0, x_3 > 0 \end{cases}$$

$$\begin{cases} \text{Minimise } x_1^2 + x_2^2 \\ x_1 + x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Exercise 91 Let $p \in \mathbb{R}^n_{++}$ and $w > 0$. We consider the following problem:

$$\begin{cases} \text{Maximise } f(x_1, x_2, x_3) = x_1x_2 \ldots x_n \\ \sum_{i=1}^n p_ix_i \leq w \\ x \in \mathbb{R}^n_+ \end{cases}$$

1) Show that there exists an element $x \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n p_ix_i \leq w$.
2) Show that there exists a least one solution.
3) Show that if $\bar{x}$ is a solution, then $\bar{x} \in \mathbb{R}^n_{++}$.
4) Show that if $\bar{x}$ is a solution, then $\sum_{i=1}^n p_i\bar{x}_i = w$.
5) Write the KKT conditions and find the unique solution of the problem.
6) If we denote by $\bar{x}(p, w)$ the optimal solution, compute $v(p, w) = f(\bar{x}(p, w))$ and compute its partial derivatives. Show the link between the partial derivative with respect to w and the KKT multipliers.

Exercise 92 Let $p \in \mathbb{R}^n_{++}$ and $w > 0$. We consider the following problem:

$$\begin{cases} \text{Maximise } f(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \ldots + \sqrt{x_n} \\ \sum_{i=1}^n p_ix_i \leq w \\ x \in \mathbb{R}^n_+ \end{cases}$$

1) Show that there exists an element $x \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n p_ix_i \leq w$.
2) Show that there exists a least one solution.
3) Show that if $\bar{x}$ is a solution, then $\sum_{i=1}^n p_i\bar{x}_i = w$.
5) Write the KKT conditions and find the unique solution of the problem.
6) If we denote by $\bar{x}(p, w)$ the optimal solution, compute $v(p, w) = f(\bar{x}(p, w))$ and compute its partial derivatives. Show the link between the partial derivative with respect to w and the KKT multipliers.

Exercise 93 Let $p \in \mathbb{R}^n \setminus \{0\}$ Find the solution of the following problems:

$$\begin{cases} \text{Maximise } \sum_{i=1}^n p_ix_i \\ \sum_{i=1}^n x_i^2 \leq 1 \end{cases}$$

$$\begin{cases} \text{Minimise } \sum_{i=1}^n p_ix_i \\ \sum_{i=1}^n x_i^2 \leq 1 \end{cases}$$

What are the solutions when $p = 0$.

Exercise 94 We are looking for the closest point for the Euclidean norm to the point $(10, 10)$ in the closed unit ball.
1) Explain that this question is equivalent to solve the following problem:

$$\begin{cases} \text{Minimise } (x-10)^2 + (y-10)^2 \\ x^2 + y^2 \leq 1 \end{cases}$$

1) Show that this problem is a convex optimisation problem.
2) Show that this problem has a unique solution.
3) Find the solution of this problem.

Exercise 95 Let f from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $f(x) = \exp(\|x\|^2) + a \cdot x$ where a is a given vector of $\mathbb{R}^n$ and $\|\cdot\|$ is the Euclidean norm.
1) Show that f is convex.
2) Find the solution of the following problem:

$$\begin{cases} \text{Minimise } f(x) \\ \|x\| \leq r \end{cases}$$

6.2 Second order necessary condition

We are considering again the same optimisation problem:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ g_i(x) = 0, \ \forall i = 1, \dots p \\ h_j(x) \leq 0, \ \forall j = 1 \dots, q \\ x \in U \end{cases}$$

and we assume that the functions f, g_i and h_j are $\mathcal{C}^2$ on U. Let us define the Lagrangian function of this problem:

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \sum_{i=1}^{p} \lambda_i g_i(x) + \sum_{j=1}^{q} \mu_j h_j(x)$$

If $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies the KKT conditions, we denote by $J^+(\bar{x}, \bar{\lambda}, \bar{\mu}) = \{j \in J(\bar{x}) \mid \bar{\mu}_j > 0\}$, and by $J^0(\bar{x}, \bar{\lambda}, \bar{\mu}) = \{j \in J(\bar{x}) \mid \bar{\mu}_j = 0\}$. In the following, we consider the following cone:

$$A(\bar{x}) = \left\{ u \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \nabla f(\bar{x}) \cdot u = 0; \\ \nabla g_i(\bar{x}) \cdot u = 0, \ \forall i = 1, \dots, p; \\ \nabla h_j(\bar{x}) \cdot u = 0, \ \forall j \in J^+(\bar{x}, \bar{\lambda}, \bar{\mu}); \\ \nabla h_j(\bar{x}) \cdot u \leq 0, \ \forall j \in J^0(\bar{x}, \bar{\lambda}, \bar{\mu}); \end{array} \right\}$$

Proposition 84 *Let $\bar{x}$ be a local solution of Problem $(\mathcal{P})$. We assume that the linear independence qualification condition is satisfied at $\bar{x}$, that is, the vectors $((\nabla g_i(\bar{x}))_{i=1}^p, (\nabla h_j(\bar{x}))_{j \in J(\bar{x})})$ are linearly independent. Let $(\bar{\lambda}, \bar{\mu})$ the unique KKT multipliers associated to $\bar{x}$. Then for all $u \in A$,*

$$u \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(u) \geq 0$$

Proof. Let $u \in A(\bar{x})$, $u \neq 0$. Let $\bar{J}(\bar{x}) = \{j \in J(\bar{x}) \mid \nabla h_j(\bar{x}) \cdot u = 0\}$. As $u \in A(\bar{x})$, $J^+(\bar{x}) \subset \bar{J}(\bar{x})$. Since the vectors $((\nabla g_i(\bar{x}))_{i \in I}, (\nabla h_j(\bar{x}))_{j \in \bar{J}(\bar{x})})$ are linearly independent, it exists a function ψ $\mathcal{C}^1$ from a neighbourhood of 0 in $\mathbb{R}$ to $\mathbb{R}^n$ such that $\psi(0) = \bar{x}$, $\psi'(0) = u$, $g_i(\psi(t)) = 0$ for all i and $h_j(\psi(t)) = 0$ for all $j \in \bar{J}(\bar{x})$ (See the proof of the Lagrange's Theorem). Since $\nabla h_j(\bar{x}) \cdot u < 0$ for all $j \in J^0(\bar{x}) \setminus \bar{J}(\bar{x})$ and $h_j(\bar{x}) < 0$ for all $j \in J \setminus J(\bar{x})$, one deduces that for all $t > 0$ small enough, $\psi(t)$ satisfies the constraints of the problem and $h_j(\psi(t)) = 0$ for all $j \in J^+(\bar{x})$. Thus $\mathcal{L}(\psi(t), \bar{\lambda}, \bar{\mu}) = f(\psi(t)) \geq f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})$. With a second order Taylor expansion of $\mathcal{L}$ in a neighbourhood of $\bar{x}$, we get:

$$\begin{aligned} 0 &\leq \nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) \cdot (\psi(t) - \bar{x}) + \tfrac{1}{2}(\psi(t) - \bar{x}) \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(\psi(t) - \bar{x}) \\ &+\|\psi(t) - \bar{x}\|^2 \eta(\psi(t) - \bar{x}) \\ &= \tfrac{1}{2}(\psi(t) - \bar{x}) \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(\psi(t) - \bar{x}) + \|\psi(t) - \bar{x}\|^2 \eta(\psi(t) - \bar{x}) \end{aligned}$$

Dividing by t^2 and taking the limit to 0^+, noticing that $\lim_{t \to 0^+} \frac{\psi(t) - \bar{x}}{t} = u$ and $\lim_{x \to 0} \eta(x) = 0$, we get

$$0 \leq u \cdot H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(u)$$

which ends the proof.□

We can remark that, in the previous proposition, we have assume the linear independence qualification condition, which is stronger than the MF condition. We now provide an example, which illustrates that the proposition does not hold under the MF condition.

$U = \mathbb{R}^2$, $f(x, y) = -y$, $h_1(x, y) = y$ and $h^2(x, y) = y - x^2$. We remark that $(0, 0)$ is a global solution of the following problem:

$$(\mathcal{P}) \begin{cases} \text{Minimise } f(x) \\ h_1(x, y) \leq 0 \\ h_2(x, y) \leq 0 \end{cases}$$

At $(0, 0)$, $\nabla h_1(0, 0) = (0, 1)$ and $\nabla h_2(0, 0) = (0, 1)$, so the MF condition is satisfied but not the LI condition. Note that we have an infinity of KKT multipliers: $\{\lambda \in \mathbb{R}^2_+ \mid \lambda_1 + \lambda_2 = 1\}$. Depending on the multiplier, $A(0, 0) = \{(x, 0) \mid x \in \mathbb{R}\}$ if $\lambda \gg 0$ or $A(0, 0) = \{(x, y) \mid x \in \mathbb{R}, y \leq 0\}$ if $\lambda_1 = 0$ or $\lambda_2 = 1$. The Hessian matrix of the Lagrangian at $(0, 0)$ is:

$$H_{xx}\mathcal{L}(0, 0, \lambda_1, \lambda_2) = \lambda_2 \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

So the Hessian matrix of the Lagrangian is not positive semi-definite on $A(0, 0)$ if $\lambda_2 > 0$. This shows that the second order necessary condition is not satisfied.

6.3 Second order sufficient condition

As in the previous section, We are considering again the same optimisation problem with equality and inequality constraints:

$$(\mathcal{P})\begin{cases}\text{Minimise } f(x)\\ g_i(x)=0,\ \forall i=1,\dots p\\ h_j(x)\le 0,\ \forall j=1\dots,q\\ x\in U\end{cases}$$

and we assume that the functions f, g_i and h_j are $\mathcal{C}^2$ on U.

We recall that the KKT necessary conditions are the following: there exists $\lambda\in\mathbb{R}^I$, $\mu\in\mathbb{R}^J_+$ such that

$$(\text{KKT})\begin{cases}\nabla f(x)+\sum_{i=1}^p\lambda_i\nabla g_i(x)+\sum_{j=1}^q\mu_j\nabla h_j(x)=0\\ g_i(x)=0 \text{ for all } i\\ h_j(x)\le 0 \text{ and } \mu_j h_j(x)=0 \text{ for all } j\end{cases}$$

If $(\bar{x},\bar{\lambda},\bar{\mu})$ satisfies KKT conditions, we denote by $J^+(\bar{x},\bar{\lambda},\bar{\mu})=\{j\in J(\bar{x})\mid \bar{\mu}_j>0\}$ and $J^0(\bar{x},\bar{\lambda},\bar{\mu})=\{j\in J(\bar{x})\mid \bar{\mu}_j=0\}$. We consider the following cone:

$$A(\bar{x})=\left\{u\in\mathbb{R}^n\ \middle|\ \begin{array}{l}\nabla f(\bar{x})\cdot u=0;\\ \nabla g_i(\bar{x})\cdot u=0,\ \forall i;\\ \nabla h_j(\bar{x})\cdot u=0,\ \forall j\in J^+(\bar{x},\bar{\lambda},\bar{\mu});\\ \nabla h_j(\bar{x})\cdot u\le 0,\ \forall j\in J^0(\bar{x},\bar{\lambda},\bar{\mu});\end{array}\right\}$$

Definition 53 Let $(\bar{x},\bar{\lambda},\bar{\mu})$ satisfying the KKT condition. The second order sufficient condition is satisfied at this point if $u\in A(\bar{x})$, $u\neq 0$,

$$H_{xx}\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu})(u)\cdot u>0$$

If we have only linear constraints, the Hessian matrix of the Lagrangian is the Hessian matrix of the objectif function f. If we have only equality constraints or a strict complementarity slackness condition, that is, $J^+(\bar{x},\bar{\lambda},\bar{\mu})=J(\bar{x})$, then the second order necessary condition means that the restriction to the subspace orthogonal to the vectors $\nabla g_i(\bar{x})$ for all i and $\nabla h_j(\bar{x})$ for $j\in J(\bar{x})$ of the objective function f is strictly convex in a neighbourhood of $\bar{x}$.

Lemma 1 *If $(\bar{x},\bar{\lambda},\bar{\mu})$ satisfies the KKT condition and the second order necessary condition, then it exists $\rho>0$ such that for all $u\in A(\bar{x})$,*

$$H_{xx}\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu})(u)\cdot u\ge\rho\|u\|^2$$

Proof. If $A(\bar{x})=\{0\}$, there is nothing to be proved. Otherwise, it exists $\bar{u}\in A$, $\|\bar{u}\|=1$, such that

$$\rho=\bar{u}\cdot H_{xx}\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu})(\bar{u})=\min\{u\cdot H_{xx}\mathcal{L}(\bar{x},\bar{\lambda},\bar{\mu})(u)\mid u\in A(\bar{x}),\|u\|=1\}$$

Since $A(\bar{x})$ is a cone, one deduces the desired inequality. $\square$

Theorem 40 *Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfying the KKT conditions and the second order sufficient condition. Let $\rho > 0$ as given by the previous lemma. Then, for all $\varepsilon \in [0, \rho[$, there exists a neighbourhood V of $\bar{x}$ such that for all $x \in V$ satisfying $x \neq \bar{x}$, $g_i(x) = 0$ for all i, $h_j(x) \leq 0$ for all j,*

$$f(x) > f(\bar{x}) + \frac{1}{2}\varepsilon\|x - \bar{x}\|^2$$

This implies that $\bar{x}$ is a local solution and even a strict local solution of the problem $(\mathcal{P})$.

Proof. Let $C = \{x \in U \mid g_i(x) = 0,\ \forall i,\ h_j(x) \leq 0,\ \forall j\}$. By contraposition, let us assume the there exists $\varepsilon \geq 0$, $\varepsilon < \rho$ and a sequence $(x_\nu) \subset C$ which converges to $\bar{x}$ and $x_\nu \neq \bar{x}$ for all ν and $f(x_\nu) \leq f(\bar{x}) + \frac{1}{2}\varepsilon\|x_\nu - \bar{x}\|^2$. Without any loss of generality, we can assume that the sequence $(\frac{x_\nu - \bar{x}}{\|x_\nu - \bar{x}\|})$ converges to $u \in \mathbb{R}^n$ of norm 1. For all i, $0 = g_i(x_\nu) - g_i(\bar{x}) = \nabla g_i(\bar{x}) \cdot (x_\nu - \bar{x}) + \|x_\nu - \bar{x}\|\varphi_i(x_\nu - \bar{x})$ with $\lim_{v\to 0} \varphi_i(v) = 0$. Thus, one deduces that $\nabla g_i(\bar{x}) \cdot u = 0$. Identically, for all $j \in J(\bar{x})$, $h_j(x_\nu) - h_j(\bar{x}) = h_j(x_\nu) \leq 0$. So, $0 \geq \nabla h_j(\bar{x}) \cdot (x_\nu - \bar{x}) + \|x_\nu - \bar{x}\|\psi_j(x_k - \bar{x})$ with $\lim_{v\to 0} \psi_j(v) = 0$. Hence, one deduces that $\nabla h_j(\bar{x}) \cdot u \leq 0$.

We now show that $\nabla f(\bar{x}) \cdot u = 0$. $\nabla f(\bar{x}) \cdot u \geq 0$ is directly deduces from the KKT condition: $\nabla f(\bar{x}) + \sum_i \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_j \bar{\mu}_j \nabla h_j(\bar{x}) = 0$ and $\bar{\mu}_j = 0$ if $j \notin J(\bar{x})$.

Let us show the converse inequality. By assumption, we have for all ν

$$f(\bar{x}) + \frac{1}{2}\varepsilon\|x_\nu - \bar{x}\|^2 \geq f(x_\nu) = f(\bar{x}) + \nabla f(\bar{x}) \cdot (x_\nu - \bar{x}) + \|x_k\nu - \bar{x}\|\varphi(x_\nu - \bar{x})$$

with $\lim_{v\to 0} \varphi(v) = 0$. Dividing by $\|x_\nu - \bar{x}\|$, one obtains $\nabla f(\bar{x}) \cdot u \leq 0$.

We end the proof by showing that for all $j \in J^+(\bar{x}, \bar{\lambda}, \bar{\mu})$, we have $\nabla h_j(\bar{x}) \cdot u = 0$ which implies that $u \in A(\bar{x})$. As

$$\begin{aligned} 0 &= \nabla f(\bar{x}) + \textstyle\sum_i \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j\in J} \bar{\mu}_j \nabla h_j(\bar{x}) \\ &= \nabla f(\bar{x}) + \textstyle\sum_{i\in I} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j\in J^+(\bar{x},\bar{\lambda},\bar{\mu})} \bar{\mu}_j \nabla h_j(\bar{x}) \end{aligned}$$

and since $0 = \nabla f(\bar{x}) \cdot u = \nabla g_i(\bar{x}) \cdot u$ for all i, we deduces that $0 = \sum_{j\in J^+(\bar{x},\bar{\lambda},\bar{\mu})} \bar{\mu}_j \nabla h_j(\bar{x}) \cdot u$. As $\bar{\mu}_j > 0$ and $\nabla h_j(\bar{x}) \cdot u \leq 0$ for all $j \in J^+(\bar{x}, \bar{\lambda}, \bar{\mu})$, we conclude that $\nabla h_j(\bar{x}) \cdot u = 0$.

From the Taylor expansion of $\mathcal{L}$,

$$\begin{aligned} \mathcal{L}(x_\nu, \bar{\lambda}, \bar{\mu}) = \;& \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) + \nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) \cdot (x_\nu - \bar{x}) \\ & + \tfrac{1}{2}(H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(x_\nu - \bar{x}) \cdot (x_\nu - \bar{x})) \\ & + \|x_\nu - \bar{x}\|^2 \bar{\varphi}(x_\nu - \bar{x}) \end{aligned}$$

with $\lim_{v\to 0} \bar{\varphi}(v) = 0$. As $\sum_{j\in J} \bar{\mu}_j h_j(\bar{x}) = 0$, we have $\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) = f(\bar{x})$ and from the KKT conditions, $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0$. So,

$$\mathcal{L}(x_\nu, \bar{\lambda}, \bar{\mu}) = f(\bar{x}) + \frac{1}{2}(H_{xx}L(\bar{x}, \bar{\lambda}, \bar{\mu})(x_\nu - \bar{x}) \cdot (x_\nu - \bar{x})) + \|x_\nu - \bar{x}\|^2 \bar{\varphi}(x_\nu - \bar{x})$$

As $f(x_\nu) \geq L(x_\nu, \bar{\lambda}, \bar{\mu})$, using the assumption, we deduces that :

$$\begin{array}{ll} f(\bar{x}) + \frac{1}{2}\varepsilon\|x_\nu - \bar{x}\|^2 \geq f(x_\nu) & \geq \mathcal{L}(x_\nu, \bar{\lambda}, \bar{\mu}) \\ & = f(\bar{x}) \\ & +\frac{1}{2}(H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(x_\nu - \bar{x}) \cdot (x_\nu - \bar{x})) \\ & +\|x_\nu - \bar{x}\|^2 \bar{\varphi}(x_\nu - \bar{x}) \end{array}$$

Dividing by $\|x_\nu - \bar{x}\|^2$, and taking the limit when ν goes to $+\infty$, we deduces that:

$$\varepsilon \geq (H_{xx}\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu})(u) \cdot (u) \geq \rho\|u\|^2 = \rho$$

which contradicts $\varepsilon < \rho$.□

We have also a stronger sufficient condition when we have a global convesity of the Lagrangian.

Theorem 41 *Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfying the KKT conditions. If the open set U is convex and the partial function $\mathcal{L}(\cdot, \bar{\lambda}, \bar{\mu})$ is convex on U, then $\bar{x}$ is a solution of the problem $(\mathcal{P})$.*

The above condition holds true when the objective functions and the inequality constraint functions are convex and the equality constraint functions are linear, which is the convex case treated above. But, it may happens that the convexity of the inequality constraint functions compensates the lack of convexity of the objective function in such a way that the Lagrangian is convex even if the objective function is not convex. The convexity of the objective constraint can also compensates the lack of convexity of the inequality or equality constraint functions.

The proof is a direct consequence of the fact that $\nabla_x\mathcal{L}(\bar{x}, \bar{\lambda}, \bar{\mu}) = 0$ from the KKT conditions and for all x satisfying the constraints of the problem $(\mathcal{P})$, $\mathcal{L}(x, \bar{\lambda}, \bar{\mu}) \leq f(x)$.