Optimization. A first course on mathematics for economists
Problem set 9: Dynamic optimization

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9.1 Consider a consumer living for two periods ($t = 0, 1$). He derives utility U from consuming a composite good (c). The initial endowment of the good is w. The consumer can borrow and lend intertemporally at an interest rate r. Suppose the utility function is separable and stationary, so that $U(c_0, c_1) = U(c_0) + \beta U(c_1)$ where β stands for the discount rate. Find the optimal consumption across both periods.

Solution:

- Lagrangean approach:
  The problem to solve is

$$\max_{c_0,c_1} U(c_0) + \beta U(c_1) \;\text{ s.t. }\; c_1 = (1+r)(w - c_0)$$

The Lagrangian function is

$$L(c_0, c_1) = U(c_0) + \beta U(c_1) - \lambda(c_1 - (1+r)(w - c_0))$$

and the FOCs

$$\frac{\partial L}{\partial c_0} = U'_{c_0} - \lambda(1+r) = 0 \quad (1)$$

$$\frac{\partial L}{\partial c_1} = U'_{c_1} - \lambda = 0 \quad (2)$$

Solving the system of FOC we obtain

$$U'_{c_0} = \beta(1+r)U'_{c_1} \quad (3)$$

Meaning that the optimal distribution of consumption is achieved when marginal utility of consumption today equals the marginal utility of the consumption foregone today (i.e. marginal cost of consumption today). This is given the interest rate foregone $(1+r)$ and discounted at the rate β.

Alternatively, we can write the optimality condition as

$$\frac{U'_{c_0}}{\beta U'_{c_1}} = 1 + r$$

That is, the intertemporal marginal rate of substitution measured at $t = 0$ equals the rate at which savings today can be transformed into consumption tomorrow (marginal rate of transformation)

Finally, if $U(\cdot)$ is concave, then $c_0 > c_1 \Leftrightarrow \beta(1 + r) < 1$

- Dynamic programming approach

  The problem to solve is

$$\max_{c_0,c_1} \sum_{t=0}^{1} \beta^t U(c_t), \quad \text{s.t. } c_1 = (1 + r)(w - c_0)$$

  - At $t = 1$, the value function is

$$v_1(w) = \max_{c_1} \{U(c_1)|c_1 \leq (1 + r)(w - c_0)\}$$

    This is a static optimization problem, increasing in c_1. Therefore, $c_1^* = (1 + r)(w - c_0)$ and $v_1(w) = U((1 + r)(w - c_0))$

  - At $t = 0$, the value function is

$$v_0(c_0) = \max_{c_0} U(c_0)+\beta v_1(w) = \max_{c_0} U(c_0)+\beta U((1+r)(w-c_0))$$

    The FOC is

$$\frac{\partial U}{\partial c_0} + \beta \frac{\partial U}{\partial c_1} \frac{dc_1}{dc_0} = U'_{c_0} - \beta U'_{c_1}(1 + r) = 0 \tag{4}$$

    Note that optimality condition (4) coincides with (3).

9.2 Consider the consumer of problem 9.1, but now he lives for T periods. Let c_t denote the consumption in period t and w_t the wealth (measured in units of the composite good) at the beginning of period t. Solve for the optimal consumption plan.

Solution: The level of wealth in each period t is

$$\begin{aligned}
w_1 &= (1 + r)(w_0 - c_0) \\
w_2 &= (1 + r)(w_1 - c_1) \\
&\vdots \\
w_t &= (1 + r)(w_{t-1} - c_{t-1}) \\
&\vdots \\
w_T &= (1 + r)(w_{T-1} - c_{T-1})
\end{aligned}$$

The problem to solve is

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t U(c_t) \;\text{ s.t. }\; w_t = (1+r)(w_{t-1} - c_{t-1})$$

The Lagrangean function is

$$L = \sum_{t=0}^{T-1} \beta^t U(c_t) - \sum_{t=1}^{T} \lambda_t [w_t - (1+r)(w_{t-1} - c_{t-1})]$$

To ease the derivation of the FOCs it is convenient to express the Lagrangean distinguishing the first, the intermediate, and the last period:

$$\begin{aligned}
L &= \sum_{t=0}^{T-1} \beta^t U(c_t) - \sum_{t=1}^{T} \lambda_t [w_t - (1+r)(w_{t-1} - c_{t-1})] \\
&= \sum_{t=0}^{T-1} \beta^t U(c_t) - \sum_{t=1}^{T} \lambda_t w_t + \sum_{t=1}^{T} \lambda_t (1+r)(w_{t-1} - c_{t-1})] \\
&= \sum_{t=0}^{T-1} \beta^t U(c_t) - \sum_{t=1}^{T} \lambda_t w_t + \sum_{t=0}^{T-1} \lambda_{t+1} (1+r)(w_t - c_t)] \\
&= U(c_0) + \sum_{t=1}^{T-1} \beta^t U(c_t) - \lambda_T w_T - \sum_{t=1}^{T-1} \lambda_t w_t \\
&\quad + \lambda(1+r)(w_0 - c_0) + \sum_{t=1}^{T-1} \lambda_{t+1}(1+r)(w_t - c_t) \\
&= U(c_0) + \lambda_1(1+r)(w_0 - c_0) - \lambda_T w_T \\
&\quad + \sum_{t=1}^{T-1} \Big(\beta^t U(c_t) - \lambda_t w_t + \lambda_{t+1}(1+r)(w_t - c_t) \Big) \\
&= U(c_0) + \lambda_1(1+r)(w_0 - c_0) - \lambda_T w_T \\
&\quad + \sum_{t=1}^{T-1} \Big(\beta^t U(c_t) - \lambda_{t+1}(1+r)c_t + w_t[\lambda_{t+1}(1+r) - \lambda_t] \Big)
\end{aligned}$$

The system of FOCs is

$$\frac{\partial L}{\partial c_0} = U'_{c_0} - \lambda_1(1+r) = 0 \tag{5}$$

$$\frac{\partial L}{\partial c_t} = \beta^t U'_{c_t} - \lambda_{t+1}(1+r) = 0, (t = 1, \ldots, T-1) \tag{6}$$

$$\frac{\partial L}{\partial w_t} = \lambda_{t+1}(1+r) - \lambda_t = 0, (t = 1, \ldots, T-1) \tag{7}$$

$$\frac{\partial L}{\partial w_T} = \lambda_T = 0 \tag{8}$$

Note that the last condition, $\lambda_T = 0$, means that optimally at the end of period T all the initial endowment has to be exhausted, otherwise it is wasted.

From (6) and (7) it follows that

$$\beta^t U'_{c_t} = \lambda_t, (t = 1, \ldots, T-1)$$

or,

$$\beta^{t+1} U'_{c_{t+1}} = \lambda_{t+1} \tag{9}$$

Substituting (9) into (6) we obtain

$$\beta^t U'_{c_t} = \beta^{t+1} U'_{c_t}(1+r)$$

and dividing by β^t the optimal consumtion path is the solution of

$$U'_{c_t} = \beta(1+r)U'_{c_{t+1}} \tag{10}$$

Interestingly enough, note that optimality condition (10) coincides with (3) the optimality condition for the 2-period problem. Thus, the length of the (finite) time horizon is irrelevant for the design of the optimal consumption path. Consumption has to be allocated so that marginal utility of consumption today equals the marginal utility of the consumption foregone today.

9.3 Consider a company that has a license to exploit a mine for the next three years. The license will not be renewed. The mine contains 128 tons of ore remaining. The price is fixed at 1€ per ton. The cost of extraction is q_t^2/x_t where q_t is the rate of extraction and x_t is the stock of ore. For simplicity, ignore discounting. Determine the optimal (profit maximizing) extraction plan.

Solution:

Dynamic programming approach

Profits in a period t are

$$q_t - \frac{q_t^2}{x_t} = q_t\Big(1 - \frac{q_t}{x_t}\Big)$$

The problem to solve is

$$\max_{q_t, x_t} \sum_{t=0}^{3} q_t\Big(1 - \frac{q_t}{x_t}\Big) \quad \text{s.t.}$$

$$x_{t+1} = x_t - q_t, \; t = 0, 1, 2$$

- Observe that by assumption, at $t = 3$ the license is exhausted, so that $v_3(x_3) = 0$. Also,

  $$v_3(x_3) = q_3\Big(1 - \frac{q_3}{x_3}\Big)$$

  so that $q_3^* = x_3$

- Then, at $t = 2$, the Bellman equation is

$$v_2(x_2) = \max_{q_2}\Big[q\Big(1 - \frac{q_2}{x_2}\Big) + v_3(x_3)\Big] = \max_{q_2} q_2\Big(1 - \frac{q_2}{x_2}\Big)$$

This gives as solution $q_2 = x_2/2$. Therefore,

$$v_2(x_2) = \Big(1 - \frac{x_2}{2}\frac{1}{x_2}\Big)\frac{x_2}{2} = \frac{x_2}{4}$$

- At $t = 1$, the Bellman equation is

$$\begin{aligned} v_1(x_1) = \max_{q_1}\Big[q_1\Big(1 - \frac{q_1}{x_1}\Big) + v_2(x_2)\Big] &= \\ \max_{q_1}\Big[q_1\Big(1 - \frac{q_1}{x_1}\Big) + \frac{1}{4}x_2\Big] &= \\ \max_{q_1}\Big[q_1\Big(1 - \frac{q_1}{x_1}\Big) + \frac{1}{4}(x_1 - q_1)\Big] \end{aligned}$$

that has as solution $q_1^* = \frac{3}{8}x_1$, so that

$$v_1(x_1) = \Big(\frac{3}{8}x_1\Big)\Big(1 - \frac{3}{8}x_1\frac{1}{x_1}\Big) + \frac{1}{4}\Big(x_1 - \frac{3}{8}x_1\Big) = \frac{25}{64}x_1$$

- Finally, at $t = 0$ the Bellman equation is

$$\begin{aligned} v_0(x_0) = \max_{q_0}\Big[q_0\Big(1 - \frac{q_0}{x_0}\Big) + v_1(x_1)\Big] &= \\ \max_{q_0}\Big[q_0\Big(1 - \frac{q_0}{x_0}\Big) + \frac{25}{64}x_1\Big] &= \\ \max_{q_0}\Big[q_0\Big(1 - \frac{q_0}{x_0}\Big) + \frac{25}{64}(x_0 - q_0)\Big] \end{aligned}$$

yielding as solution $q_0^* = \frac{39}{128}x_0$. Given the initial condition $x_0 = 128$ it follows that $q_0^* = 39$

Using the transition equation, we can compute

$$\begin{aligned} x_1 &= x_0 - q_0 = 128 - 39 = 89 \quad \text{and } q_1^* = (\frac{3}{8})89 = 33.375 \\ x_2 &= x_1 - q_1 = 89 - 33.375 = 55.625 \quad \text{and } q_2^* = (\frac{1}{2})55.625 = 27.8125 \\ x_3 &= x_2 - q_2 = 55.625 - 27.8125 = 27.8125 \quad \text{and } q_3^* = x_3 \end{aligned}$$

Optimal control approach

The Lagrangian function is

$$L(q,x,\lambda)=\sum_{t=0}^{2}\Big[\Big(1-\frac{q_t}{x_t}\Big)q_t-\lambda_{t+1}(x_{t+1}-x_t+q_t)\Big]+x_3=$$

$$=\sum_{t=0}^{2}\Big[\Big(1-\frac{q_t}{x_t}\Big)q_t+\lambda_{t+1}(x_t-q_t)\Big]-\sum_{t=0}^{2}\lambda_{t+1}x_{t+1}+x_3=$$

$$=\sum_{t=0}^{2}\Big[\Big(1-\frac{q_t}{x_t}\Big)q_t+\lambda_{t+1}(x_t-q_t)\Big]-\sum_{t=1}^{3}\lambda_t x_t+x_3=$$

$$=\sum_{t=0}^{2}\Big[\Big(1-\frac{q_t}{x_t}\Big)q_t+\lambda_{t+1}(x_t-q_t)-\lambda_t x_t\Big]-\lambda_3x_3+x_3=$$

$$=\sum_{t=0}^{2}\Big[\Big(1-\frac{q_t}{x_t}\Big)q_t+\lambda_{t+1}(x_t-q_t)-\lambda_t x_t\Big]+x_3(1-\lambda_3)$$

The FOCs are,

$$\begin{aligned}\frac{\partial L}{\partial q_t}&=1-2\frac{q_t}{x_t}-\lambda_{t+1}=0,\ t=0,1,2\\ \frac{\partial L}{\partial x_t}&=\Big(\frac{q_t}{x_t}\Big)^2+\lambda_{t+1}-\lambda_t=0,\ t=1,2\\ x_{t+1}&=x_t-q_t,\ t=0,1,2\\ \lambda_3&=0,\ \text{(shadow price of remaining ore)}\end{aligned}$$

Let $z_t\equiv\frac{2q_t}{x_t}$ so that the FOCs can be rewritten as

$$\begin{aligned}z_t+\lambda_{t+1}-1&=0,\ t=0,1,2\\ \frac{1}{4}z_t^2+\lambda_{t+1}-\lambda_t&=0,\ t=1,2\\ \lambda_3&=0\end{aligned}$$

- Then, $z_2=1-\lambda_3$ or $z_2=1$, so that $q_2^*=\frac{1}{2}x_2$.
- Also, $\lambda_2=\lambda_3+\frac{1}{4}$, so that $\lambda_2=\frac{1}{4}$.
- Then, $z_1=1-\lambda_2$ or $z_1=\frac{3}{4}$, so that $q_1^*=\frac{3}{8}x_1$.
- Also, $\lambda_1=\frac{1}{4}+\frac{1}{4}(\frac{3}{4})^2=\frac{25}{16}$
- Then, $z_0=1-\lambda_1=\frac{39}{64}$, so that $q_0^*=39$ (recall $x_0=128$)

Finally,

$$
\begin{aligned}
x_1 =& x_0 - q_0^* = 128 - 39 = 89 \\
\text{Thus, } q_1^* =& \frac{3}{8}(89) = 33.375 \\
x_2 =& x_1 - q_1^* = 89 - \frac{3}{8}(89) = \frac{5}{8}(89) \\
\text{Thus, } q_2^* =& \frac{1}{2}\frac{5}{8}(89) = 27.8125 \\
x_3 =& x_2 - q_2^* = \frac{5}{8}(89) - \frac{1}{2}(89)\frac{5}{8} = \frac{5}{16}(89) \\
\text{Thus, } q_3^* =& x_3 = \frac{5}{16}(89)
\end{aligned}
$$

Bringing the Hamiltonian in

Define the Hamiltonian as

$$
H_t(q, x, \lambda) = \left(1 - \frac{q_1}{x_1}\right)q_t + \lambda_{t+1}(x_t - q_t)
$$

The Lagrangian is thus,

$$
L(q, x, \lambda) = \sum_{t=0}^{2}\Big[H_t(q, x, \lambda) - \lambda_t x_t\Big] + x_3(1 - \lambda_3)
$$

FOCs:

$$
\begin{aligned}
\frac{\partial L}{\partial q_t} &= \frac{\partial H_t}{\partial q_t} = 1 - 2\frac{q_t}{x_t} - \lambda_{t+1} = 0,\ t = 0, 1, 2 \\
\frac{\partial L}{\partial x_t} &= \frac{\partial H_t}{\partial x_t} - \lambda_t = \left(\frac{q_t}{x_t}\right)^2 + \lambda_{t+1} - \lambda_t = 0,\ t = 1, 2 \\
x_{t+1} &= x_t - q_t,\ t = 0, 1, 2 \\
\lambda_3 &= 0,\ \text{(shadow price of remaining ore)}
\end{aligned}
$$

as before.

- Summarizing the optimal policy is

t	x_t	q_t
0	128	39
1	89	33.375
2	55.625	27.8125
3	27.8125	27.8125

9.4 Consider the following optimal growth model à la Stokey-Lucas. There is an economy producing a composite good y with two inputs, labor l, and capital k by means of a technology described by a production function

$$y_t = f(k_t, l_t), \tag{11}$$

where k_t denotes the stock of capital and l_t the labor force available at the beginning of the period. Time horizon is finite $t = 0, \ldots T$.

Output y_t is either devoted to consumption c_t or to investment i_t. That is $y_t = c_t + i_t$

Capital depreciates at a constant rate δ so that the stock of capital available at the beginning of $t + 1$ is

$$k_{t+1} = (1 - \delta)k_t + i_t. \tag{12}$$

Suppose labor supply is constant along time, so that $l_t = 1, \forall t$.

The total supply of goods at the end of a period is given by the production of the current period plus the stock capital at the beginning of the period: $F(k_t) = f(k_t, 1) + (1 - \delta)k_t$, so that

$$F(k_t) = c_t + i_t = c_t + k_{t+1} \tag{13}$$

where we have used (11) and (12). We can read (13) as

$$c_t = F(k_t) - k_{t+1} \tag{14}$$

showing that there is a trade-off between current consumption and future output.

Consumption c_t yields satisfaction captured by a (concave) utility function $u(c_t)$. Future utility is discounted at a rate β per period.

Find the Euler equation characterizing the optimal trade-off between consumption and investment in each period to maximize total discounted utility.

Solution: The problem to be solved is

$$\max_{c_t} \sum_{t=0}^{T-1} \beta u(c_t) + \beta^T v(k_T) \text{ s.t. } k_{t+1} = F(k_t) - c_t \tag{15}$$

where $v(k_T)$ denotes the liquidation value of the remaining capital at T.

Bellman's equation is

$$v_t(k_t) = \max_{c_t}\{u(c_t) + \beta v_{t+1}(k_{t+1})\} = \max_{c_t}\{u(c_t) + \beta v_{t+1}(F(k_t) - c_t)\}$$

The FOC is,

$$u'(c_t) - \beta v'_{t+1}(F(k_t) - c_t) = 0 \tag{16}$$

Note that mutatis mutandis it follows that,

$$v_{t+1}(k_{t+1}) = \max_{c_{t+1}}\{u(c_{t+1}) + \beta v_{t+2}(F(k_{t+1}) - c_{t+1})\} \tag{17}$$

Its FOC is

$$u'(c_{t+1}) - \beta v'_{t+2}(F(k_{t+1}) - c_{t+1}) = 0 \tag{18}$$

Applying the envelope theorem to (17), we obtain

$$v'_{t+1}(k_{t+1}) = \beta v'_{t+2}(F(k_{t+1}) - c_{t+1})F'(k_{t+1}) \tag{19}$$

Substituting (18) in (19) gives

$$v'_{t+1}(k_{t+1}) = u'(c_{t+1})F'(k_{t+1}) \tag{20}$$

and substituting (20) in (16) we obtain the Euler equation:

$$u'(c_{t+1}) - \beta u'(c_{t+1})F'(k_{t+1}) = 0 \tag{21}$$

characterizing the optimal consumption path.

We could have obtained (21) using the Lagrangean approach. The Lagrangean function is

$$L(c,k,\lambda) = \sum_{t=0}^{T-1} \beta u(c_t) + \beta^T v(k_T) - \sum_{t=0}^{T} \beta^{t+1}\lambda_{t+1}(k_{t+1} - (F(k_t) - c_t))$$

To ease the derivation of the FOCs, let's separate the first, last and intermediate periods, so that,

$$\begin{aligned}
L(c,k,\lambda) &= \sum_{t=0}^{T-1} \beta u(c_t) + \sum_{t=0}^{T-1} \beta^{t+1}\lambda_{t+1}(F(k_t) - c_t) - \sum_{t=0}^{T-1} \beta^{t+1}\lambda_{t+1}k_{t+1} + \beta^T v(k_T) \\
&= \sum_{t=0}^{T-1} \beta u(c_t) + \sum_{t=0}^{T-1} \beta^{t+1}\lambda_{t+1}(F(k_t) - c_t) - \sum_{t=1}^{T} \beta^{t}\lambda_t k_t + \beta^T v(k_T) \\
&= u(c_0) + \sum_{t=1}^{T-1} \beta^t u(c_t) + \beta\lambda(F(k_0) - c_0) + \sum_{t=1}^{T-1} \beta^{t+1}\lambda_{t+1}(F(k_t) - c_t) \\
&\qquad - \beta^T \lambda_T k_T - \sum_{t=1}^{T-1} \beta^t \lambda_t k_t + \beta^T v(k_T) \\
&= u(c_0) + \beta\lambda(F(k_0) - c_0) - \beta^T\lambda_T k_T + \beta^T v(k_T) \\
&\quad + \sum_{t=1}^{T-1} \beta^t u(c_t) + \sum_{t=1}^{T-1} \beta^{t+1}\lambda_{t+1}(F(k_t) - c_t) - \sum_{t=1}^{T-1} \beta^t \lambda_t k_t \\
&= u(c_0) + \beta\lambda(F(k_0) - c_0) - \beta^T\lambda_T k_T + \beta^T v(k_T) \\
&\qquad + \sum_{t=1}^{T-1} \beta^t \Big[u(c_t) + \beta\lambda_{t+1}(F(k_t) - c_t) - \lambda_t k_t\Big]
\end{aligned}$$

The system of FOCs is

$$
\begin{aligned}
u'(c_t) =& \beta\lambda_{t+1}, && (22)\\
\lambda_t =& \beta\lambda_{t+1}F'(k_t), && (23)\\
k_{t+1} =& F(k_t) - c_t && (24)\\
\lambda_T =& v'(k_T) && (25)
\end{aligned}
$$

In period $t+1$ FOCs (30) and (32) read

$$
\begin{aligned}
u'(c_{t+1}) =& \beta\lambda_{t+2}, && (26)\\
\lambda_{t+1} =& \beta\lambda_{t+2}F'(k_{t+1}), && (27)
\end{aligned}
$$

From (35) and (36) it follows that

$$\lambda_{t+1} = u'(c_{t+1}F'(k_{t+1}) \tag{28}$$

and substituting (37) in (30) we obtain

$$u'(c_t) = \beta u'(c_{t+1})F'(k_{t+1}) \tag{29}$$

This is the same Euler equation obtained before in (21).

9.5 Consider an agent that lives for three periods and maximizes a utility function of the form

$$V_1 = U_1 + \alpha U_2 + \beta U_3$$

where utility in period t is a function of current and future consumption. In particular,

$$
\begin{aligned}
U_1(c_1, c_2, c_3) &= \ln(c_1c_2c_3)\\
U_2(c_2, c_3) &= \ln(c_2c_3)\\
U_3(c_3) &= \ln c_3
\end{aligned}
$$

The budget constraints are $A_{t+1} = A_t - c_t$ where A is wealth and we assume A_1 is given and $A_4 = 0$.

(i) Compute the optimal consumption plan from the perspective of period 1, $c^1 = (c_1^1, c_2^1, c_3^1)$

(ii) Consider what happens as the agent begins to implement the consumption plan. At $t = 1$ consumes c_1^1, obtains utility U_1 and has wealth $A_2 = A_1 - c_1^1$. Then, the problem is to maximize utility over the remaining two periods:

$$\max V_2 = \alpha U_2 + \beta U_3$$

subject to $A_2 = c_2 + c_3$. Compute the new optimal consumption plan. Compare it with the one obtained in (i).

Solution

(i) The objective function from the perspective of period 1 can be written as

$$V_1 = \ln c_1 + (1+\alpha)\ln c_2 + (1+\alpha+\beta)\ln c_3 \tag{30}$$

The budget constraints can be summarized as $A_1 = c_1 + c_2 + c_3$, or

$$c_3 = A_1 - c_1 - c_2. \tag{31}$$

Substituting it in (30) the problem of the consumer at time $t = 1$ is

$$\max_{c_1,c_2} V_1(c_1, c_2) = \ln c_1 + (1+\alpha)\ln c_2 + (1+\alpha+\beta)\ln(A_1 - c_1 - c_2) \tag{32}$$

The FOCs are

$$\frac{\partial V_1}{\partial c_1} = \frac{1}{c_1} + (1+\alpha+\beta)\frac{-1}{A_1 - c_1 - c_2} = 0 \tag{33}$$

$$\frac{\partial V_1}{\partial c_2} = (1+\alpha)\frac{1}{c_2} + (1+\alpha+\beta)\frac{-1}{A_1 - c_1 - c_2} = 0 \tag{34}$$

From (33)and (34) we obtain,

$$\frac{1}{c_1} = (1+\alpha)\frac{1}{c_2}$$

or

$$c_2 = (1+\alpha)c_1 \tag{35}$$

Substituting (35) in (33) we obtain

$$c_1^1 = \frac{A_1}{3+2\alpha+\beta} \tag{36}$$

Substituting (36) in (35) we obtain

$$c_2^1 = \frac{(1+\alpha)A_1}{3+2\alpha+\beta} \tag{37}$$

Finally, substituting (36) and (37) in (31) we obtain

$$c_3^1 = \frac{(1+\alpha+\beta)A_1}{3+2\alpha+\beta} \tag{38}$$

(ii) Suppose that our individual has already consumed c_1^1 in $t = 1$ and faces the problem of (re)-computing the optimal consumption path from the perspective of $t = 2$. His wealth left is

$$A_2 = A_1 - c_1^1 = \frac{(2+2\alpha+\beta)A_1}{3+2\alpha+\beta} \tag{39}$$

and his problem is

$$\max_{c_2,c_3} V_2(c_2, c_3) = \alpha U_2 + \beta U_3 \;\text{ s.t. }\; A_2 = c_2 + c_3$$

Following a parallel reasoning as before, the consumer solves

$$\max_{c_2} V_2(c_2) = \alpha \ln c_2 + (\alpha + \beta) \ln(A_2 - c_2) \tag{40}$$

The FOC is

$$V_2'(c_2) = \alpha \frac{1}{c_2} - \frac{\alpha + \beta}{A_2 - c_2} = 0$$

Accosrdingly,

$$c_2^2 = \frac{\alpha A_2}{2\alpha + \beta} \tag{41}$$

Substituting (39) into (41) we obtain

$$c_2^2 = \frac{\alpha}{2\alpha + \beta} \frac{(2 + 2\alpha + \beta)A_1}{3 + 2\alpha + \beta}$$

Next, multiplying and dividing by $(1 + \alpha)$ we can rewrite c_2^2 as

$$c_2^2 = \frac{\alpha(2 + 2\alpha + \beta)}{(2\alpha + \beta)(1 + \alpha)} \frac{(1 + \alpha)A_1}{3 + 2\alpha + \beta}$$

and substituting (36) we finally obtain

$$c_2^2 = \frac{\alpha(2 + 2\alpha + \beta)}{(2\alpha + \beta)(1 + \alpha)} c_2^1 \tag{42}$$

Since $\frac{\alpha(2+2\alpha+\beta)}{(2\alpha+\beta)(1+\alpha)} < 1$, we conclude that $c_2^2 < c_2^1$. Namely, the revised consumption plan for period 2 is lower than the original consumption plan for period 2. In other words, the *principle of optimality* does not hold. In other words, the consumption plan is not time consistent.