Optimization. A first course on mathematics for economists
Problem set 9: Dynamic optimization

Xavier Martinez-Giralt

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9.1 Consider a company that has a license to exploit a mine for the next three years. The license will not be renewed. The mine contains 128 tons of ore remaining. The price is fixed at 1€ per ton. The cost of extraction is q_t^2/x_t where q_t is the rate of extraction and x_t is the stock of ore. For simplicity, ignore discounting.

Determine the optimal (profit maximizing) extraction plan.

9.2 Consider the consumer of problem 9.1, but now he lives for T periods. Let c_t denote the consumption in period t and w_t the wealth (measured in units of the composite good) at the beginning of period t. Solve for the optimal consumption plan.

9.3 Consider a company that has a license to exploit a mine for the next three years. The license will not be renewed. The mine contains 128 tons of ore remaining. The price is fixed at 1€ per ton. The cost of extraction is q_t^2/x_t where q_t is the rate of extraction and x_t is the stock of ore. For simplicity, ignore discounting. Determine the optimal (profit maximizing) extraction plan.

9.4 Consider the following optimal growth model à la Stokey-Lucas. There is an economy producing a composite good y with two inputs, labor l, and capital k by means of a technology described by a production function

$$y_t = f(k_t, l_t), \tag{1}$$

where k_t denotes the stock of capital and l_t the labor force available at the beginning of the period. Time horizon is finite $t = 0, \dots T$.

Output y_t is either devoted to consumption c_t or to investment i_t. That is $y_t = c_t + i_t$

Capital depreciates at a constant rate δ so that the stock of capital available at the beginning of $t + 1$ is

$$k_{t+1} = (1 - \delta)k_t + i_t. \tag{2}$$

Suppose labor supply is constant along time, so that $l_t = 1, \forall t$.

The total supply of goods at the end of a period is given by the production of the current period plus the stock capital at the beginning of the period: $F(k_t) = f(k_t, 1) + (1 - \delta)k_t$, so that

$$F(k_t) = c_t + i_t = c_t + k_{t+1} \tag{3}$$

where we have used (1) and (2). We can read (3) as

$$c_t = F(k_t) - k_{t+1} \tag{4}$$

showing that there is a trade-off between current consumption and future output.

Consumption c_t yields satisfaction captured by a concave utility function $u(c_t)$. Future utility is discounted at a rate β per period.

Find the Euler equation characterizing the optimal trade-off between consumption and investment in each period to maximize total discounted utility.

9.5 Consider an agent that lives for three periods and maximizes a utility function of the form

$$V_1 = U_1 + \alpha U_2 + \beta U_3$$

where utility in period t is a function of current and future consumption. In particular,

$$U_1(c_1, c_2, c_3) = \ln(c_1 c_2 c_3)$$
$$U_2(c_2, c_3) = \ln(c_2 c_3)$$
$$U_3(c_3) = \ln c_3$$

The budget constraint is $A_{t+1} = A_t - c_t$ where A is wealth and we assume A_1 is given and $A_4 = 0$.

(i) Compute the optimal consumption plan from the perspective of period 1, $c^1 = (c_1^1, c_2^1, c_3^1)$

(ii) Consider what happens as the agent begins to implement the consumption plan. At $t = 1$ consumes c_1^1, obtains utility U_1 and has wealth $A_2 = A_1 - c_1^1$. Then, the problem is to maximize utility over the remaining two periods:

$$\max V_2 = \alpha U_2 + \beta U_3$$

subject to $A_2 = c_2 + c_3$. Compute the new optimal consumption plan. Compare it with the one obtained in (i).