Optimization. A first course on mathematics for economists
Problem set 8: Difference equations

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8.1 Let the demand of a certain commodity be given by $D(p_t) = \alpha - \beta p_t$ and its supply by $S(p_t) = \gamma + \delta p_t$, where $\alpha, \beta, \delta > 0$. Assume the price p adjustes from one period to the next according to the stocks cummulated by the sellers, in the following way

$$p_{t+1} = p_t - r(S(p_t) - D(p_t)) \tag{1}$$

(a) determine the trajectory of the price along time

(b) study the properties of the trajectory when $r = 0.1, \beta = 1, \delta = 15$

(c) study the properties of the trajectory when $r = 0.3, \beta = 2, \delta = 6$

Solution:

(a) Substituting demand and supply functions in (1) and simplifying gives

$$p_{t+1} - p_t[1 - r(\beta + \delta)] = r(\alpha - \gamma) \tag{2}$$

To simplify notation, let

$$k = 1 - r(\beta + \delta)$$
$$s = r(\alpha - \gamma)$$

so that (2) reduces to

$$p_{t+1} - kp_t = s \tag{3}$$

- General solution of the homogeneous equation
  The homogeneous equation is

$$p_{t+1} - kp_t = 0 \tag{4}$$

  The candidate solution is

$$p_t = k^t A \tag{5}$$

  where A is to be determined.

- Particular solution of the non-homogeneous equation
  The non-homogeneous equation (3) shows a constant function in its right-hand side. Therefore, we propose as particular solution

$$\overline{p}_t = \mu, \forall t \tag{6}$$

  Substituting (6) in (3) gives

$$\mu - k\mu = s$$

  or

$$\mu = \frac{s}{1-k}$$

- The solution of (3) is

$$p_t = k^t A + \frac{s}{1-k} \tag{7}$$

  To determine the value of A, evaluate (7) at $t = 0$ to obtain

$$p_0 = A + \frac{s}{1-k}$$

  or

$$A = p_0 - \frac{s}{1-k}$$

  Then, the trajectory of the prices is

$$p_t = k^t\left(p_0 - \frac{s}{1-k}\right) + \frac{s}{1-k}$$

  Finally substituting back k and s,

$$p_t = [1 - r(\beta+\delta)]^t\left(p_0 - \frac{\alpha-\gamma}{\beta+\delta}\right) + \frac{\alpha-\gamma}{\beta+\delta}$$

  Note that $\frac{\alpha-\gamma}{\beta+\delta}$ is the stationary equilibrium price.

(b) The properties of the trajectory p_t depend of the value of $k = 1 - r(\beta+\delta)$. When $r = 0.1, \beta = 1, \delta = 15$ it follows that $k = -0.6$, Therefore, p_t follows a monotonic and convergent trajectory towards the stationary price.

(c) When $r = 0.3, \beta = 2, \delta = 6$ it follows that $k = -1.4$, Therefore, p_t follows a monotonic and divergent trajectory away from the stationary price.

8.2 Solve the following equation

$$x_t = \frac{1}{2}x_{t-1} + 3 \tag{8}$$

for $x_0 = 2$.

Solution:

- General solution of the homogeneous equation
  The homogeneous equation is
  $$x_t - \frac{1}{2}x_{t-1} = 0$$
  The candidate solution is
  $$x_t = \left(\frac{1}{2}\right)^t A$$
  where A is to be determined.
- Particular solution of the non-homogeneous equation
  The non-homogeneous equation shows a constant function in its right-hand side. Therefore, we propose as particular solution
  $$\overline{p}_t = \mu, \forall t$$
  Substituting it in (8) gives
  $$\mu - \frac{1}{2}\mu = 3$$
  or
  $$\mu = 6$$
- The solution of (8) is
  $$x_t = \left(\frac{1}{2}\right)^t A + 6 \tag{9}$$
  To determine the value of A, evaluate (9) at $t = 0$ to obtain
  $$x_0 = 2 = A + 6, \quad \text{or} \quad A = -4$$
  then,
  $$x_t = -4\left(\frac{1}{2}\right)^t + 6 = 6 - 2^{2-t}$$
  this trajectory is monotone convergent as shown in figure 1.

8.3 Solve the following equation
$$y_t = -3y_{t-1} + 4 \tag{10}$$
for $y_0 = 2$.

Solution:

- General solution of the homogeneous equation
  The homogeneous equation is
  $$y_t + 3y_{t-1} = 0$$
  The candidate solution is
  $$y_t = (-3)^t A$$
  where A is to be determined.

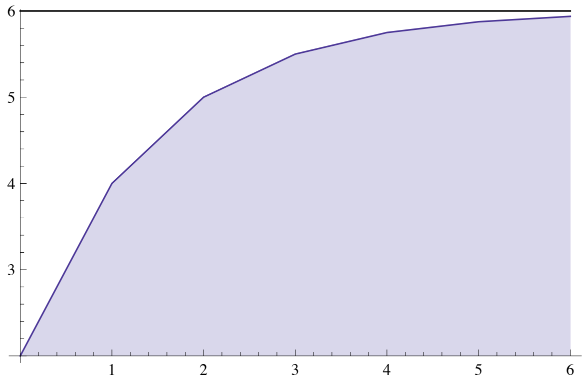

Figure 1: Problem 8.2

- Particular solution of the non-homogeneous equation
  The non-homogeneous equation shows a constant function in its right-hand side. Therefore, we propose as particular solution

  $$\overline{p}_t = \mu, \forall t$$

  Substituting it in (10) gives

  $$\mu + 3\mu = 4$$

  or

  $$\mu = 1$$

- The solution of (10) is

  $$y_t = (-3)^t A + 1 \tag{11}$$

  To determine the value of A, evaluate (11) at $t = 0$ to obtain

  $$y_0 = 2 = A + 1, \quad \text{or} \quad A = 1$$

  then,

  $$y_t = (-3)^t + 1$$

  this trajectory is cyclical divergent as shown in figure 2.

8.4 Solve the following equation

$$x_t = -\frac{1}{2} x_{t-1} + 3 \tag{12}$$

for $x_0 = 2$.

Solution:

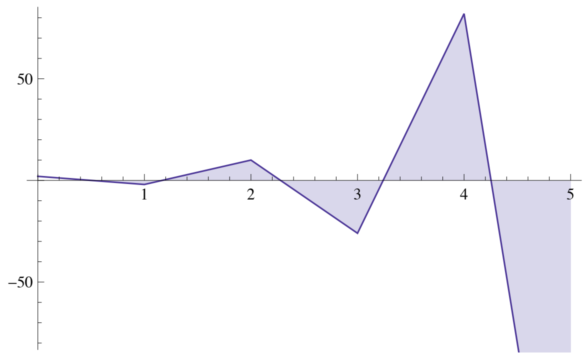

Figure 2: Problem 8.3

- General solution of the homogeneous equation
  The homogeneous equation is

$$x_t + \frac{1}{2}x_{t-1} = 0$$

  The candidate solution is

$$x_t = \left(-\frac{1}{2}\right)^t A$$

  where A is to be determined.

- Particular solution of the non-homogeneous equation
  The non-homogeneous equation shows a constant function in its right-hand side. Therefore, we propose as particular solution

$$\overline{p}_t = \mu, \forall t$$

  Substituting it in (8) gives

$$\mu - \frac{1}{2}\mu = 3$$

  or

$$\mu = 6$$

- The solution of (12) is

$$x_t = \left(-\frac{1}{2}\right)^t A + 6 \tag{13}$$

  To determine the value of A, evaluate (13) at $t = 0$ to obtain

$$x_0 = 2 = A + 6, \quad \text{or} \quad A = -4$$

  then,

$$x_t = -4\left(-\frac{1}{2}\right)^t + 6$$

  this trajectory is cyclical convergent as shown in figure 3.

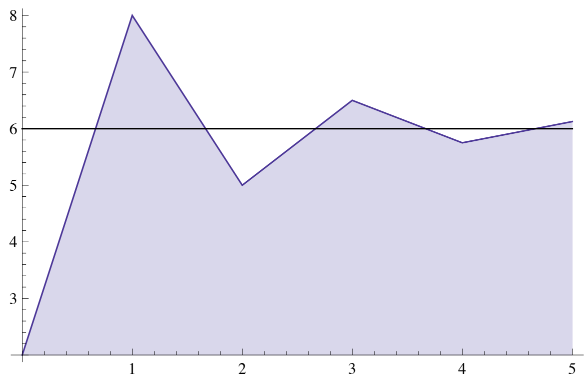

Figure 3: Problem 8.4

8.5 Solve the following equation

$$y_t = 3y_{t-1} + 4 \tag{14}$$

for $y_0 = 2$.

Solution:

- General solution of the homogeneous equation
  The homogeneous equation is

$$y_t - 3y_{t-1} = 0$$

  The candidate solution is

$$y_t = 3^t A$$

  where A is to be determined.

- Particular solution of the non-homogeneous equation
  The non-homogeneous equation shows a constant function in its right-hand side. Therefore, we propose as particular solution

$$\overline{p}_t = \mu, \forall t$$

  Substituting it in (14) gives

$$\mu - 3\mu = 4$$

  or

$$\mu = -2$$

- The solution of (14) is

$$y_t = 3^t A - 2 \tag{15}$$

To determine the value of A, evaluate (15) at $t = 0$ to obtain

$$y_0 = 2 = A - 2, \quad \text{or} \quad A = 4$$

then,

$$y_t = (3^t)4 - 2$$

this trajectory is monotone divergent as shown in figure 4.

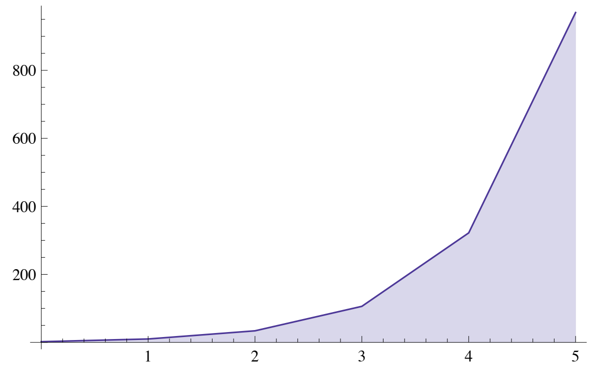

Figure 4: Problem 8.5

8.6 Consider an individual contracting a mortgage for B€ at $t = 0$. Suppose that (i) the interest rate r is constant along time, (ii) repayment per period z is also constant until mortgage is paid off after T periods. The principal b_t on the loan in period t is given by

$$b_t = (1 + r)b_{t-1} + z, \text{ with } b_0 = B, b_T = 0 \tag{16}$$

(a) Solve equation (16)

(b) Compute B and give an economic meaning to the expression obtained

(c) Compute z and give an economic meaning to the expression obtained

(d) Compute the principal repayment in period t and give an economic meaning to the expression obtained

Solution

(a) Solve equation (16)

(i) General solution of homogeneous equation
The homogeneous equation is

$$b_t - (1 + r)b_{t-1} = 0$$

The candidate solution is

$$b_t = (1 + r)^t A$$

with A to be determined.

(ii) Particular solution of the homogeneous equation
The solution will be of the type

$$\bar{b}_t = \mu, \forall t$$

Substituting it in (16) we obtain

$$\mu + (1+r)\mu = z, \text{ or } \mu = \frac{z}{r}$$

(iii) The solution of (16) is

$$b_t = (1+r)^t A + \frac{z}{r} \tag{17}$$

To determine A we need an additional condition, namely $b_0 = B$, and evaluationg (17) at $t = 0$ we obtain

$$b_0 = B = A + \frac{z}{r} \text{ or } A = b_0 - \frac{z}{r}$$

The solution of (16) is finally,

$$b_t = (1+r)^t \Big(b_0 - \frac{z}{r}\Big) + \frac{z}{r} \tag{18}$$

(b) To compute B remark that $b_T = 0$. Then, Evaluationg (18) at $t = T$ yields

$$0 = (1+r)^T \Big(b_0 - \frac{z}{r}\Big) + \frac{z}{r}$$

and solving it for B gives

$$B = z\Big[1 - (1+r)^{-T}\Big]\frac{1}{r} = z\sum_{t=1}^{T}(1+r)^{-t} \tag{19}$$

To derive the last equality, we ned to recall how to obtain the summation formula for finite geometric series. Suppose we want to compute the sum

$$s_n = 1 + k + k^2 + \cdots + k^{n-1}$$

multiplying both sides by k we obtain

$$ks_n = k + k^2 + k^3 + \cdots + k^n$$

Next compute

$$s_n - ks_n = 1 - k^n \text{ so that } s_n(1-k) = 1 - k^n \text{ or } s_n = \frac{1-k^n}{1-k}.$$

In a parallel way we can derive the corresponding formula for negative powers. Starting from

$$s_n = 1 + k^{-1} + k^{-2} + \cdots + k^{-(n-1)}$$

we obtain

$$s_n = \frac{k(1-k^{-n})}{k-1}$$

Now consider that the initial sum is $S_n = ks_n$. Then, a parallel argument will tell us that the value of the sum is

$$S_n = k\frac{1-k^n}{1-k},$$

and when powers are negative, following a similar reasoning it follows that

$$S_n = k^{-1}\frac{k(1-k^{-n})}{k-1} = \frac{1-k^{-n}}{k-1}$$

Finally just define $k = (1+r)$ to obtain (19).

Equation (19) says that the original loan B is equal to the present discounted value of T repayments z starting in period $t = 1$.

(c) To compute z we simply solve (19) for z to obtain

$$z = \frac{rB}{1-(1+r)^{-T}} = rB + \frac{(1+r)^{-T}rB}{1-(1+r)^{-T}} \tag{20}$$

This means that the instalment due each period has two destinations. One part is the interest on the original loan, rB; the other part covers the interest payments each period.

(d) To compute b_t depart from (19) and write it as

$$B - \frac{z}{r} = -\frac{z}{r}(1+r)^{-T}$$

and substitute it into (18) to obtain (recall $b_0 = B$)

$$b_t = \frac{z}{r}\Big[1-(1+r)^{(t-T)}\Big] \tag{21}$$

this is the remaing principal at time t. Therefore, at time t the interest payment on this principal at time $t-1$ is

$$rb_{t-1} = z\Big[1-(1+r)^{(t-1-T)}\Big]$$

Given that z is the total repayment in the period, it follows that

$$z - rb_{t-1} = z(1+r)^{(t-1-T)}$$

Note that this is a small amount in the early periods of the mortgage but increases exponentially (at the rate $(1+r)$). In the last period, $t = T$ the interest payment is

$$z\Big(1-\frac{1}{1+r}\Big) = \frac{rz}{1+r}$$

and the principal repayment is $z/(1+r)$.

However, at $t = 1$ the first payment z only the amount $z/(1+r)^{-T}$ repays principal. The rest is interest.

8.7 Solve the following difference equations and study the solution paths.

(a) $x_{t+2} + 3x_{t+1} - \frac{7}{4}x_t = 9$, with $x_0 = 0, x_1 = 6$

Solution:

- General solution of the homogeneous equation
  The homogeneous equation is

$$x_{t+2} + 3x_{t+1} - \frac{7}{4}x_t = 0 \tag{22}$$

  The candiate solution is $x_t = m^t, m \neq 0$. Substituting it in (22) we determine m:

$$m^{t+2} + 3m^{t+1} - \frac{7}{4}m^t = 0, \text{ or}$$
$$m^t(m^2 + 3m - \frac{7}{4}) = 0 \text{ implying}$$
$$m^2 + 3m - \frac{7}{4} = 0$$

  Applying Descarte's rule of sign we already know that one continuation and one change of sign means that there will be one positive root and one negative root. The negative root implies that the trajectory of the solution will be cyclical.
  The roots of the quadratic function are $m_1 = 1/2$ and $m_2 = -7/2$. Given that $|m_2| > 1$, we conclude that the cyclical trajectory will be divergent.
  Applying theorem 2 the solution of (22) is

$$x_t = A_1\Big(\frac{1}{2}\Big)^t + A_2\Big(\frac{-7}{2}\Big)^t \tag{23}$$

  where (A_1, A_2) are to be determined.
- Particular solution of the non-homogeneous solution
  The right-hand side of the equation is a constant function. thus, the candidate solution will also be a constant. Try $\overline{x}_t = \mu, \forall t$ Substituting it in the equation we determine the value of μ:

$$\mu + 3\mu - \frac{7}{2}\mu = 9 \Rightarrow \mu = 4$$

- The solution of the second-order difference equation is

$$x_t = A_1\Big(\frac{1}{2}\Big)^t + A_2\Big(\frac{-7}{2}\Big)^t + 4 \tag{24}$$

  To determine the values of (A_1, A_2) we need two additional conditions. Suppose $x_0 = 0, x_1 = 6$. Evaluating (24) at $t = 0$ and

$t = 1$, we obtain

$$0 = A_1 + A_2 + 4$$
$$6 = A_1 \frac{1}{2} + A_2 \frac{-7}{2} + 4$$

yielding $(A_1, A_2) = (-3, -1)$. Substituting these values in (24), the solution of the second-order difference equation is

$$x_t = -3\Big(\frac{1}{2}\Big)^t - \Big(\frac{-7}{2}\Big)^t + 4$$

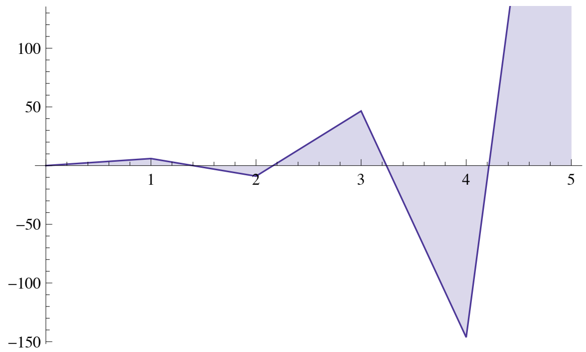

Figure 5: Problem 8.7a

(b) $x_{t+2} - x_{t+1} + \frac{1}{4}x_t = 2$, with $x_0 = 0, x_1 = 6$.
Solution:

- General solution of the homogeneous equation
  The homogeneous equation is

$$x_{t+2} - x_{t+1} + \frac{1}{4}x_t = 0 \tag{25}$$

The candiate solution is $x_t = m^t, m \neq 0$. Substituting it in (25) we determine m:

$$m^{t+2} - m^{t+1} + \frac{1}{4}m^t = 0, \text{ or}$$
$$m^t(m^2 - m + \frac{1}{4}) = 0 \text{ implying}$$
$$m^2 - m + \frac{1}{4} = 0$$

The roots of the quadratic function are $m_1 = m_2 = \widehat{m} = 1/2$. This implies a monotonic and convergent trajectory.

We consider as roots in this case $\widehat{m}^t$ and $t\widehat{m}^t$, so that applying theorem 2 gives as solution of (25)

$$x_t = A_1\left(\frac{1}{2}\right)^t + A_2 t\left(\frac{1}{2}\right)^t = (A_1 + A_2 t)\left(\frac{1}{2}\right)^t \tag{26}$$

where (A_1, A_2) are to be determined.

- Particular solution of the non-homogeneous solution
  The right-hand side of the equation is a constant function. Thus, the candidate solution will also be a constant. Try $\overline{x}_t = \mu, \forall t$ Substituting it in the equation we determine the value of μ:

$$\mu - \mu + \frac{1}{4}\mu = 2 \Rightarrow \mu = 8$$

  so $\overline{x}_t = 8$.

- The solution of the second-order difference equation is

$$x_t = (A_1 + A_2 t)\left(\frac{1}{2}\right)^t + 8 \tag{27}$$

  To determine the values of (A_1, A_2) we need two additional conditions. Suppose $x_0 = 0, x_1 = 6$. Evaluating (27) at $t = 0$ and $t = 1$, we obtain

$$\begin{aligned} 0 &= A_1 + 8 \\ 6 &= (A_1 + A_2)\frac{1}{2} + 8 \end{aligned}$$

  yielding $(A_1, A_2) = (-8, 4)$. Substituting these values in (27), the solution of the second-order difference equation is

$$x_t = -(-8 + 4t)\left(\frac{1}{2}\right)^t + 8$$

(c) $x_{t+2} + 2x_{t+1} + x_t = 9(2)^t$, with $x_0 = 0, x_1 = 6$.
Solution:

- General solution of the homogeneous equation
  The homogeneous equation is

$$x_{t+2} + 2x_{t+1} + x_t = 0 \tag{28}$$

  The candidate solution is $x_t = m^t, m \neq 0$. Substituting it in (28) we determine m:

$$\begin{aligned} &m^{t+2} + 2m^{t+1} + m^t = 0, \text{ or} \\ &m^t(m^2 + 2m + 1) = 0 \text{ implying} \\ &m^2 + 2m + 1 = 0 \end{aligned}$$

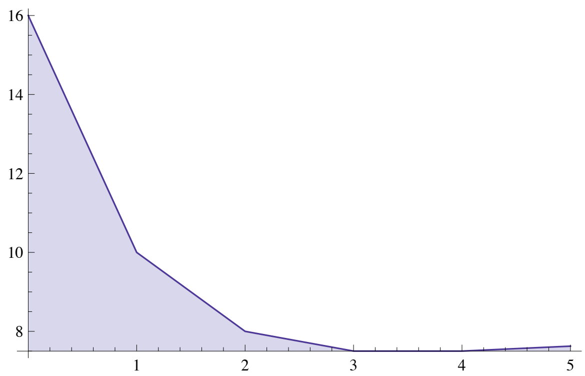

Figure 6: Problem 8.7b

The roots of the quadratic function are $m_1 = m_2 = \widehat{m} = -1$. This implies a divergent cyclical trajectory, because $|\widehat{m}| \nless 1$.
We consider as roots in this case $\widehat{m}^t$ and $t\widehat{m}^t$, so that applying theorem 2 gives as solution of (28)

$$x_t = A_1(-1)^t + A_2 t(-1)^t = (A_1 + A_2 t)(-1)^t \tag{29}$$

where (A_1, A_2) are to be determined.

- Particular solution of the non-homogeneous solution
The right-hand side of the equation is an exponential function. Thus, the candidate solution will also be a exponential. Try $\overline{x}_t = \mu 2^t, \forall t$ Substituting it in the equation we determine the value of μ:

$$\mu 2^{t+2} + 2\mu 2^{t+1} + \mu 2^t = 9(2)^t \Rightarrow \mu = 1$$

so that $\overline{x}_t = 2^t$.

- The solution of the second-order difference equation is

$$x_t = (A_1 + A_2 t)(-1)^t + 2^t \tag{30}$$

To determine the values of (A_1, A_2) we need two additional conditions. Suppose $x_0 = 0, x_1 = 6$. Evaluating (30) at $t = 0$ and $t = 1$, we obtain

$$\begin{aligned} 0 &= A_1 + 8 \\ 6 &= (A_1 + A_2)\frac{1}{2} + 8 \end{aligned}$$

yielding $(A_1, A_2) = (-2, -2)$. Substituting these values in (30), the solution of the second-order difference equation is

$$x_t = -2(1 + t)(-1)^t + 2^t$$

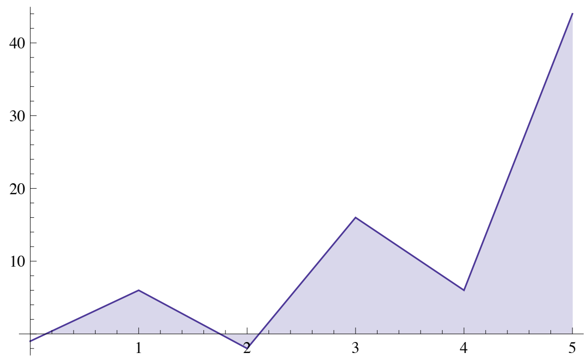

Figure 7: Problem 8.7c

8.8 Consider an economy whose GDP at time t, Y_t is defined as

$$Y_t = C_t + I_t \tag{31}$$

where I_t denotes investment, and C_t denotes consumption. Suppose that

- consumption is determined as

$$C_{t+1} = aY_t + b \tag{32}$$

where $a, b > 0$.

- Investment is defined as proportional to the change in consumption

$$I_{t+1} = c(C_{t+1} - C_t) \tag{33}$$

with $c > 0$.

Derive the expression of the GDP as a second-order difference equation and assess the stability of the solution.

Solution:

- Rewrite (31) as

$$Y_{t+2} = C_{t+2} + I_{t+2} \tag{34}$$

- Rewrite (32) and (33) as

$$C_{t+2} = aY_{t+1} + b \tag{35}$$
$$I_{t+2} = c(C_{t+2} - C_{t+1}) \tag{36}$$

- Using (32) and (35) compute

$$C_{t+2} - C_{t+1} = a(Y_{t+1} - Y_t) \tag{37}$$

- Substitute (37) into (36) to obtain

$$I_{t+2} = ca(Y_{t+1} - Y_t) \tag{38}$$

- Substitute (35) and (38) into (34) to obtain

$$Y_{t+2} = aY_{t+1} + b + ca(Y_{t+1} - Y_t)$$

  and rearranging

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b$$

  This ttrajectory will be stable (converge to b) iff

$$\begin{aligned} a(1+c) &< 1 + ac, \text{ and} \\ ac &< 1 \end{aligned}$$

  or equivalently iff $a < 1$ and $ac < 1$.