Optimization. A first course on mathematics for economists
Problem set 7: Differential equations

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7.1 Let the demand of a certain commodity be given by $D(p) = a - bp$ and its supply by $S(p) = \alpha + \beta p$, where $a, b, \alpha, \beta > 0$. Assume the price p varies with time t, i.e. $p = p(t)$. Also, assume the market for the commodity is competitive so that price is determined by the excess demand function. Find the price trajectory of prices and study its stability.

Solution: At any point in time

$$p'(t) = \lambda(D(p) - S(p)),\ \lambda > 0.$$

or rearranging,

$$p'(t) + \lambda(b + \beta)p(t) = \lambda(a - \alpha)$$

This is a first-order differential equation.

- General solution of the homogeneous equation
  The corresponding homogeneous equation is

  $$p'(t) + \lambda(b + \beta)p(t) = 0$$

  so that

  $$\frac{p'(t)}{p(t)} = -\lambda(b + \beta)$$

  that we can express as

  $$\frac{d \log p(t)}{dt} = -\lambda(b + \beta)$$

  and integrating on both sides

  $$\log p(t) = -\lambda(b + \beta)t + C$$

  Taking antilogs, we obtain the solution to the homogeneous equation:

  $$p(t) = e^{-\lambda(b+\beta)t+C} = e^C e^{-\lambda(b+\beta)t} \equiv Ae^{-\lambda(b+\beta)t} \tag{1}$$

  Note that because $-\lambda(b + \beta) < 0, \lim_{t\to\infty} p(t) = 0$.

- Particular solution for the non-homogeneous equation
  The non-homogeneous equation is

$$p'(t) + \lambda(b+\beta)p(t) = \lambda(a-\alpha)$$

  We try as solution $\overline{p}(t) = \mu$. Then,

$$\lambda(b+\beta)\mu = \lambda(a-\alpha)$$

  and

$$\mu = \frac{a-\alpha}{b+\beta}$$

- The solution of the first-order differential equation is

$$p(t) = Ae^{-\lambda(b+\beta)t} + \frac{a-\alpha}{b+\beta}$$

  To compute the value of A, evaluate $p(t)$ at $t=0$. Then,

$$p(0) = A + \frac{a-\alpha}{b+\beta}$$

  so that

$$A = p(0) - \frac{a-\alpha}{b+\beta}$$

  Therefore, the solution of the differential equation (1) is

$$p(t) = \Big(p(0) - \frac{a-\alpha}{b+\beta}\Big)e^{-\lambda(b+\beta)t} + \frac{a-\alpha}{b+\beta}$$

- Note that the equilibrium price is the solution of $D(p) = S(p)$ yielding

$$p^* = \frac{a-\alpha}{b+\beta}$$

  Hence we conclude that the trajectory $p(t)$ converges monotonically towards the equilibrium price p^*, and the equation is stable.

7.2 Consider the following model of growth in a developing economy:

$$X(t) = \sigma K(t) \tag{2}$$
$$K'(t) = \alpha X(t) + H(t) \tag{3}$$
$$N(t) = N_0 e^{\rho t} \tag{4}$$

where $X(t)$ denotes the GDP per year, $K(t)$ is the capital stock, $H(t)$ is the flow of foreign aid, and $N(t)$ is the population.

(a) Derive a differential equation of $K(t)$

(b) Let $H(t) = H_0 e^{\mu t}$. Find the solution of the differential equation assuming $K(0) = K_0$ and $\alpha\sigma \neq \mu$

(c) Find an expression for the production per capita.

Solution:

(a) Substituting (2) in (3) and rearranging, we obtain

$$K'(t) = \alpha\sigma K(t) + H(t) = \alpha\sigma K(t) + H_0 e^{\mu t}$$

or

$$K'(t) - \alpha\sigma K(t) = H_0 e^{\mu t}$$

(b)

- General solution of the homogeneous equation
  The homogeneous equation is

  $$K'(t) - \alpha\sigma K(t) == 0$$

  so that

  $$\frac{K'(t)}{K(t)} = \alpha\sigma$$

  That can be written as

  $$\frac{d \log K(t)}{dt} = \alpha\sigma$$

  Integrating both sides yields

  $$\log K(t) = \alpha\sigma t + C$$

  Taking antilogs we obtain

  $$K(t) = e^{\alpha\sigma t + C} = e^C e^{\alpha\sigma t} = A e^{\alpha\sigma t} \tag{5}$$

- Particular solution of the general equation
  The non-homogeneous equation is

  $$K'(t) - \alpha\sigma K(t) = H_0 e^{\mu t} \tag{6}$$

  the expression on the right-hand side is an exponential equation. Thus, we try as general solution

  $$\overline{K}(t) = C e^{\mu t} \tag{7}$$

  Substituting (7) in (6) we obtain

  $$\mu C e^{\mu t} - \alpha\sigma C e^{\mu t} = H_0 e^{\mu t}$$

  or

  $$e^{\mu t}\Big(C(\mu - \alpha\sigma) - H_0\Big) = 0$$

Since $e^{\mu t} \neq 0$, it follows that

$$C = \frac{H_0}{\mu - \alpha\sigma}$$

Therefore,

$$\overline{K}(t) = \frac{H_0 e^{\mu t}}{\mu - \alpha\sigma}$$

- The solution of the differential equation is

$$K(t) = Ae^{\alpha\sigma t} + \frac{H_0 e^{\mu t}}{\mu - \alpha\sigma} \tag{8}$$

For $t = 0$, we obtain

$$K(0) = K_0 = A + \frac{H_0}{\mu - \alpha\sigma}$$

so that

$$A = K_0 - \frac{H_0}{\mu - \alpha\sigma} \tag{9}$$

Finally substituting (9) into (8) we obtain the solution of the differential equation:

$$K(t) = \Big(K_0 - \frac{H_0}{\mu - \alpha\sigma}\Big)e^{\alpha\sigma t} + \frac{H_0}{\mu - \alpha\sigma}e^{\mu t} \tag{10}$$

(c) Production per capita is

$$x(t) \equiv \frac{X(t)}{N(t)} = \frac{\sigma K(t)}{N_0 e^{\rho t}} \tag{11}$$

where we have used (2) and (4).
Substituting (10) into (11) we obtain

$$\begin{aligned} x(t) = \frac{\sigma}{N_0 e^{\rho t}}\Big[\Big(K_0 - \frac{H_0}{\mu - \alpha\sigma}\Big)e^{\alpha\sigma t} + \frac{H_0}{\mu - \alpha\sigma}e^{\mu t}\Big] &= \\ \frac{\sigma}{N_0 e^{\rho t}}\Big[K_0 e^{\alpha\sigma t} + \frac{H_0}{\mu - \alpha\sigma}(e^{\mu t} - e^{\alpha\sigma t})\Big] &= \\ \frac{\sigma}{N_0}e^{-\rho t}\Big[K_0 e^{\alpha\sigma t} + \frac{H_0}{\mu - \alpha\sigma}(e^{\mu t} - e^{\alpha\sigma t})\Big] &= \\ \frac{\sigma}{N_0}\Big[K_0 e^{(\alpha\sigma - \rho)t} + \frac{H_0}{\mu - \alpha\sigma}(e^{\mu t} - e^{\alpha\sigma t})e^{-\rho t}\Big] &= \\ \frac{\sigma K_0}{N_0}e^{(\alpha\sigma - \rho)t} + \frac{\alpha}{N_0}\Big[\frac{H_0}{\mu - \alpha\sigma}(e^{\mu t} - e^{\alpha\sigma t})e^{-\rho t}\Big] &= \\ x(0)e^{-\rho t} + \frac{\sigma}{\mu - \alpha\sigma}\frac{H_0}{N_0}e^{-\rho t}(e^{\mu t} - e^{\alpha\sigma t}) & \end{aligned} \tag{12}$$

7.3 Consider the following macroeconomic model

$$Y(t) = C(t) + I(t) \tag{13}$$

$$I(t) = kC'(t) \tag{14}$$

$$C(t) = aY(t) + b \tag{15}$$

where $Y(t), I(t)$ and $C(t)$ denote GDP, investment, and consumption respectively at any time t. Suppose $b, k > 0$ and $a \in (0, 1)$.

(a) Derive a differential equation for the GDP

(b) Solve the differential equation for the GDP assuming $Y(0) = Y_0 > b/(1-a)$. Find the corresponding function for $I(t)$

(c) Compute $\lim_{t\to\infty} Y(t)/I(t)$.

Solution:

(a) From (15) we obtain

$$C'(t) = aY'(t) \tag{16}$$

Substituting (16) in (14) we obtain

$$I(t) = kaY'(t) \tag{17}$$

Finally, substituting (15) and (17) into (13) we obtain

$$Y(t) = aY(t) + b + kaY'(t)$$

or

$$Y'(t) - \frac{1-a}{ka}Y(t) = \frac{-b}{ka}$$

(b) • General solution of the homogeneous equation
The corresponding homogeneous equation is

$$Y'(t) - \frac{1-a}{ka}Y(t) = 0$$

so that

$$\frac{Y'(t)}{Y(t)} = \frac{1-a}{ka}$$

that can be written as

$$\frac{d\log Y(t)}{dt} = \frac{1-a}{ka}$$

and integration on both sides yields

$$\log Y(t) = \frac{1-a}{ka}t + C$$

Taking antilogs we obtain the solution to the homogeneous equation:

$$Y(t) = e^{\frac{1-a}{ka}t+C} = e^{C}e^{\frac{1-a}{ka}t} = Ae^{\frac{1-a}{ka}t}$$

- Particular solution of the non-homogeneous equation
  The non-homogeneous equation is

  $$Y'(t) - \frac{1-a}{ka} Y(t) = \frac{-b}{ka}$$

  The expression on the right-hand side is a constant function. Thus, we try as solution $\overline{Y}(t) = \mu$. Then,

  $$-\frac{1-a}{ka}\mu = \frac{-b}{ka}$$

  and

  $$\mu = \frac{\frac{-b}{ka}}{-\frac{1-a}{ka}} = \frac{b}{1-a}$$

- The solution of the differential equation is

  $$Y(t) = Ae^{\frac{1-a}{ka}t} + \frac{b}{1-a} \tag{18}$$

  to obtain an expression for A, evaluate (18) at $t = 0$ to obtain

  $$Y(0) = Y_0 = A + \frac{b}{1-a}$$

  so that

  $$A = Y_0 - \frac{b}{1-a}$$

  Finally, the differential equation for the GDP is

  $$Y(t) = \Big(Y_0 - \frac{b}{1-a}\Big)e^{\frac{1-a}{ka}t} + \frac{b}{1-a} \tag{19}$$

  Note that $\Big(Y_0 - \frac{b}{1-a}\Big) > 0$ and $\frac{1-a}{ka} > 0$. Therefore, $Y(t)$ shows a monotonic trajectory to ∞.

- To find the corresponding function for $I(t)$ we first compute

  $$Y'(t) = \frac{1-a}{ka}\Big(Y_0 - \frac{b}{1-a}\Big)e^{\frac{1-a}{ka}t}$$

  and substitute it in (17) to obtain:

  $$I(t) = (1-a)\Big(Y_0 - \frac{b}{1-a}\Big)e^{\frac{1-a}{ka}t} = (1-a)Y(t) - b \tag{20}$$

  The trajectory of $I(t)$ is induced by the trajectory of $Y(t)$. Given that $(1-a) > 0$, $I(t)$ also shows a monotonic trajectory to ∞.

(c) Using (20) we compute

$$\frac{Y(t)}{I(t)} = \frac{Y(t)}{(1-a)Y(t)-b}$$

Then.

$$\lim_{t\to\infty} \frac{Y(t)}{I(t)} = \frac{1}{1-a}$$

7.4 Consider an economy described by

$$\frac{N'(t)}{N(t)} = \alpha - \beta\frac{N(t)}{X(t)} \tag{21}$$

$$X(t) = AN^a(t) \tag{22}$$

where $N(t)$ and $X(t)$ denote the population and the GDP. Suppose α, β, a are positive, and $a \neq 1$. Denote by $x(t)$ the GDP per capita.

(a) Derive a differential equation for $x(t)$

(b) Solve the differential equation for the $x(t)$

(c) Find expression for $N(t)$ and $X(t)$

(d) Compute the $\lim_{t\to\infty}$ for $x(t), N(t), X(t)$ when $a \in (0,1)$

Solution:

(a) By definition, $x(t) = \frac{X(t)}{N(t)}$. Taking logs,

$$\log x(t) = \log \frac{X(t)}{N(t)} = \log X(t) - \log N(t)$$

Differentiating wrt t

$$\frac{d \log x(t)}{dt} = \frac{d \log X(t)}{dt} - \frac{d \log N(t)}{dt}$$

or

$$\frac{x'(t)}{x(t)} = \frac{X'(t)}{X(t)} - \frac{N'(t)}{N(t)} \tag{23}$$

From (22) it follows that

$$\log X(t) = \log AN(t)^a$$

so that

$$\frac{d \log X(t)}{dt} = \frac{d \log N(t)^a}{dt}$$

or

$$\frac{X'(t)}{X(t)} = \frac{aN'(t)}{N(t)} \tag{24}$$

Substituting (24) in (23) yields

$$\frac{x'(t)}{x(t)} = \frac{N'(t)(a-1)}{N(t)} = (a-1)\Big(\alpha - \beta\frac{N(t)}{X(t)}\Big) = (a-1)\Big(\alpha - \beta\frac{1}{x(t)}\Big)$$

Finally simplifying we obtain

$$x'(t) = (a-1)\alpha x(t) - (a-1)\beta \tag{25}$$

(b) Solving (25)

- General solution of the homogeneous equation
  The corresponding homogeneous equation is

  $$x'(t) - (a-1)\alpha x(t) = 0$$

  so that

  $$\frac{x'(t)}{x(t)} = \alpha(a-1)$$

  that can be rewritten as

  $$\frac{d\log x(t)}{dt} = \alpha(a-1)$$

  and integrating on both sides gives

  $$\log x(t) = \alpha(a-1)t + C$$

  Taking antilogs we obtain the solution to the homogeneous equation:

  $$x(t) = e^{\alpha(a-1)t+C} = e^{C}e^{\alpha(a-1)t} = Ae^{\alpha(a-1)t}$$

  The trajectory of $x(t)$ depends on the sign of $(a-1)$. If $a < 1$ it will monotonically convergo to zero; if $a > 1$ it will monotonically diverge to ∞.
- Particular solution of the non-homogeneous equation
  The non-homogeneous equation is

  $$x'(t) - (a-1)\alpha x(t) = -(a-1)\beta \tag{26}$$

  given that the right-hand side of (26) is a constant function, we try as particular solution $\overline{x}(t) = \mu$. Then, substituting it in (26) we obtain,

  $$-(a-1)\alpha\mu = -(a-1)\beta$$

  and

  $$\mu = \frac{\beta}{\alpha}$$

  so that

  $$\overline{x}(t) = \frac{\beta}{\alpha}$$

- The solution of (26) is

$$x(t) = Ae^{\alpha(a-1)t} + \frac{\beta}{\alpha} \tag{27}$$

To find an expression for A, evaluate (27) at $t = 0$:

$$x(0) = A + \frac{\beta}{\alpha}$$

so that

$$A = x(0) - \frac{\beta}{\alpha}$$

Finally, the solution of the differential equation (25) is

$$x(t) = \left(x(0) - \frac{\beta}{\alpha}\right)e^{\alpha(a-1)t} + \frac{\beta}{\alpha} \tag{28}$$

(c) Next we have to obtain expressions for $N(t)$ and $X(t)$.
Recall that $x(t) = X(t)/N(t)$. Then, using (22), we can write

$$x(t) = \frac{AN^a(t)}{N(t)} = AN^{a-1}(t)$$

Hence,

$$N^{a-1}(t) = \frac{x(t)}{A}$$

or

$$N(t) = \left(\frac{x(t)}{A}\right)^{\frac{1}{a-1}} = A^{\frac{1}{1-a}}x(t)^{\frac{1}{a-1}} \tag{29}$$

From (22) and (29) we obtain

$$X(t) = AN^a(t) = A\left[A^{\frac{1}{1-a}}x(t)^{\frac{1}{a-1}}\right]^a = A^{\frac{1}{1-a}}x(t)^{\frac{a}{a-1}} \tag{30}$$

(d) Let $a \in (0, 1)$. Then,

- From (28), $\alpha(a-1) < 0$ and $x(t)$ converges monotonically towards β/α, i.e.

$$\lim_{t\to\infty} x(t) = \frac{\beta}{\alpha}$$

- Note that using (29), the trajectory of $N(t)$ is induced by the trajectory of $x(t)$. Therefore,

$$\lim_{t\to\infty} N(t) = \left(\frac{\beta}{A\alpha}\right)^{\frac{1}{a-1}}$$

- Similarly, from (30), the trajectory of $X(t)$ is also induced by the trajectory of $x(t)$. Therefore,

$$\lim_{t\to\infty} X(t) = A^{\frac{1}{1-a}}\left(\frac{\beta}{\alpha}\right)^{\frac{a}{a-1}}$$

7.5 Solve $y'(t) = a^t$ when $a \neq 1$ and when $a = 1$

Solution: Suppose $a \neq 1$. Then, integrating on both sides we obtain

$$y(t) = a^t \ln a + C$$

that only defined if $a > 0$.

If $a = 1$, the the equation reduces to $y'(t) = 1$ that has as solution $y(t) = t + C$.

7.6 Consider the following second-order differential equation

$$y''(t) - a^2 y(t) = 0, \ a \neq 0 \tag{31}$$

(a) Solve the equation.

(b) Shown that the trajectory of $y(t)$ always diverges regardless of the sign of a.

Solution:

(a) Let's conjecture that the solution will be an exponential function $e^{-\lambda t}$ where λ is a parameter to be determined. This means,

$$\begin{aligned} y(t) =& e^{-\lambda t} \\ y'(t) =& -\lambda e^{-\lambda t} \\ y''(t) =& \lambda^2 e^{-\lambda t} \end{aligned}$$

Substituting these expressions in (31), we obtain

$$\lambda^2 e^{-\lambda t} - a^2 e^{-\lambda t} = 0$$

or

$$e^{-\lambda t}(\lambda^2 - a^2) = 0$$

This eqaution has tow solutions $\lambda_1 = a, \lambda_2 = -a$ so that both e^{at} and e^{-at} satisfy (31). Applying theorem 2, the solution of (31) is

$$y(t) = A_1 e^{-at} + A_2 e^{at} \tag{32}$$

To determine the values of A_1 and A_2 we need two additional conditions. Suppose $y(0) = y_0$ and $y'(0) = 0$.
From (32), compute

$$y'(t) = -aA_1 e^{-at} + aA_2 e^{at}$$

Therefore,

$$y'(0) = a(A_2 - A_1) = 0$$

implying,

$$A_1 = A_2 = \overline{A} \tag{33}$$

Next we evaluate (32) at $t = 0$ to obtain

$$y(0) = A_1 + A_2 = 2\overline{A} = y_0 \tag{34}$$

where we have used (33). From (34) it follows that

$$\overline{A} = \frac{1}{2} y_0$$

so that the solution of the differential equation (31) is

$$y(t) = \frac{1}{2} y_0 (e^{-at} + e^{at}) \tag{35}$$

(b) Note that the trajectory of e^{kt} depends on the sign of k. When $k > 0, e^{kt} \to \infty$, while when $k < 0, e^{kt} \to 0$.
In the solution (35) regardless of the sign of a there is always one term that diverges to ∞. thus, $y(t)$ always diverges.

7.7 Solve the following second-order differential equation

$$y''(t) + y'(t) - 2y(t) = -10 \tag{36}$$

Solution:

- General solution of the homogeneous equation
  The homogeneous equation is

$$y''(t) + y'(t) - 2y(t) = 0 \tag{37}$$

Propose a solution of the type $e^{-\lambda t}$ where λ is a parameter to be determined. This means,

$$\begin{aligned} y(t) &= e^{-\lambda t} \\ y'(t) &= -\lambda e^{-\lambda t} \\ y''(t) &= \lambda^2 e^{-\lambda t} \end{aligned}$$

Substituting these expressions in (37), we obtain

$$\lambda^2 e^{-\lambda t} - \lambda e^{-\lambda t} - 2e^{-\lambda t} = 0$$

or

$$e^{-\lambda t}(\lambda^2 - \lambda - 2) = 0 \tag{38}$$

This equation has two solutions $\lambda_1 = 2$ and $\lambda_2 = -1$ so that both e^{2t} and e^{-t} satisfy (37). Applying theorem 2, the solution of (37) is

$$y(t) = A_1 e^{2t} + A_2 e^{-t} \tag{39}$$

Remark that the characteristic equation is (38) shows one change of sign and one continuation of sign. This means that $y(t)$ follows a monotonic divergent trajectory.

- Particular solution of the non-homogeneous equation

  The non-homegenous equation (36) shows a constant function of the right-hand side. Therefore we propose a constant function as a particular solution, $\overline{y}(t) = \mu$. Accordingly, $y''(t) = y'(t) = 0$ ans substituting it into (36) we obtain

  $$-2\mu = -10$$

  so that

  $$\mu = 5$$

- The solution of (36) is

  $$y(t) = A_1 e^{2t} + A_2 e^{-t} + 5 \tag{40}$$

  To determine the values of A_1 and A_2 let's suppose $y(0) = 12$ and $y'(0) = -2$.
  From (40) we obtain

  $$y'(t) = 2A_1 e^{2t} - A_2 e^{-t}$$

  and evaluating it at $t = 0$ we obtain

  $$y'(0) = 2A_1 - A_2 = -2 \tag{41}$$

  Next, evaluate (39) at $t = 0$ to obtain

  $$y(0) = A_1 + A_2 + 5 = 12 \tag{42}$$

  Solving the system (41)-(42) yields $A_1 = 5/3$ and $A_2 = 16/3$ so that the solution of (36) is

  $$y(t) = \frac{5}{3}e^{2t} + \frac{16}{3}e^{-t} + 5$$

7.8 Consider a market described by the following supply and demand curves:

$$D(p) = 9 - p(t) + p'(t) + 3p''(t)$$
$$S(p) = -1 + 4p(t) - p'(t) + 5p''(t)$$

Let $p(0) = 4$ and $p'(0) = 4$.

(a) Find the trajectory of the equilibrium price $p(t)$.

(b) Assess whether the $p(t)$ is convergent, divergent, or cyclical.

Solution:

(a) Assuming the market is in equilibrium at every period t, we obtain

$$2p''(t) - 2p'(t) + 5p(t) = 10 \tag{43}$$

- General solution of the homogeneous equation
  The homogeneous equation is

$$2p''(t) - 2p'(t) + 5p(t) = 0 \tag{44}$$

  Propose a solution of the type $e^{-\lambda t}$ where λ is a parameter to be determined. This means,

$$\begin{aligned} y(t) =& e^{-\lambda t} \\ y'(t) =& -\lambda e^{-\lambda t} \\ y''(t) =& \lambda^2 e^{-\lambda t} \end{aligned}$$

  Substituting these expressions in (44), we obtain

$$2\lambda^2 e^{-\lambda t} + 2\lambda e^{-\lambda t} + 5e^{-\lambda t} = 0$$

  or

$$e^{-\lambda t}(2\lambda^2 + 2\lambda + 5) = 0$$

  This equation has two imaginary roots: $\lambda_1 = \frac{-1+3i}{2}$ and $\lambda_2 = \frac{-1-3i}{2}$ so that both, $e^{\frac{-1+3i}{2}}$ and $e^{\frac{-1-3i}{2}}$ satisfy equation (44). Applying theorem 2, the solution of the homogeneous equation is

$$p(t) = A_1 e^{\frac{-1+3i}{2}} + A_2 e^{\frac{-1-3i}{2}} = e^{\frac{-t}{2}} \left(A_1 e^{\frac{3i}{2}t} + A_2 e^{\frac{-3i}{2}t} \right)$$

  Applying Euler's formula (remember, $e^{\pm i\theta t} = \cos\theta t \pm i \sin\theta t$), we obtain

$$p(t) = e^{\frac{-t}{2}} \left[(A_1 + A_2)\cos\frac{3t}{2} + (A_1 - A_2) i \sin\frac{3t}{2} \right] \tag{45}$$

- Particular solution of the non-homogeneous equation
  The non-homegenous equation (43) shows a constant function of the right-hand side. Therefore we propose a constant function as a particular solution, $\overline{p}(t) = \mu$. Accordingly, $p''(t) = p'(t) = 0$ and substituting it into (43) we obtain

$$5\mu = 10$$

  so that

$$\mu = 2$$

- The solution of (43) is

$$p(t) = e^{\frac{-t}{2}}\left[(A_1 + A_2)\cos\frac{3t}{2} + (A_1 - A_2)i\sin\frac{3t}{2}\right] + 2 \quad (46)$$

To determine the values of A_1 and A_2 we need two additional conditions. Let's suppose $p(0) = 6$ and $p'(0) = 4$.
From (46) we obtain

$$p'(t) = \frac{1}{2}e^{\frac{-t}{2}}\Big[-\cos\frac{3t}{2}\Big((A_1 + A_2) - 3i(A_1 - A_2)\Big) \\ - \sin\frac{3t}{2}\Big(3(A_1 + A_2) - i(A_1 - A_2)\Big)\Big] \quad (47)$$

Evaluating (47) at $t = 0$ yields

$$p'(0) = \frac{1}{2}\Big(3i(A_1 - A_2) - (A_1 + A_2)\Big) = 4$$

or

$$3i(A_1 - A_2) - (A_1 + A_2) = 8 \quad (48)$$

Next, evaluate (46) at $t = 0$ to obtain

$$p(0) = (A_1 + A_2) + 2 = 6$$

or

$$A_1 + A_2 = 4 \quad (49)$$

Solving the system (48)-(49) we obtain $A_1 + A_2 = 4$ and $A_1 - A_2 = 4/i$. Then, substituting these expressions into (46) we obtain the solution of (43), namely

$$p(t) = 4e^{\frac{-t}{2}}\left(\cos\frac{3t}{2} + \sin\frac{3t}{2}\right) + 2$$

(b) The trajectory of $p(t)$ is cyclical (as it depends on $\sin$ and $\cos$ functions) with period $4\pi/3$.

The amplitude of the cycles depends on $e^{\frac{-t}{2}}$. Given that $\frac{-1}{2} < 0$, the amplitude is decreasing and $p(t)$ converges to 2 as $t \to \infty$. Note that $p = 2$ is the stationary level of prices in the market.

7.9 Solve the following system of differential equations

$$\left.\begin{aligned} x'(t) + 2y'(t) + 2x(t) + 5y(t) &= 77 \\ y'(t) + x(t) + 4y(t) &= 61 \end{aligned}\right\} \quad (50)$$

with initial conditions $x(0) = 6$ and $y(0) = 12$.

Solution

(a) General solution of the homogeneous system
The homogeneous system is given by

$$\left.\begin{aligned} x'(t) + 2y'(t) + 2x(t) + 5y(t) =& 0 \\ y'(t) + x(t) + 4y(t) =& 0 \end{aligned}\right\} \tag{51}$$

Let us try as solutions,

$$\left.\begin{aligned} y(t) &= \alpha_1 e^{\lambda t} \\ x(t) &= \alpha_2 e^{\lambda t} \end{aligned}\right\} \tag{52}$$

where α_1 and α_2 are to be determined.
From (52) we can derive

$$\left.\begin{aligned} y'(t) &= \alpha_1 \lambda e^{\lambda t} \\ x'(t) &= \alpha_2 \lambda e^{\lambda t} \end{aligned}\right\} \tag{53}$$

and substituting (52) and (53) into (51) we obtain (after simplifying)

$$\left.\begin{aligned} e^{\lambda t}[\alpha_1(5 + 2\lambda) + \alpha_2(2 + \lambda)] = 0 \\ e^{\lambda t}[\alpha_1(4 + \lambda) + \alpha_2] = 0 \end{aligned}\right\} \tag{54}$$

System (54) reduces to solve the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

that has two real roots $\lambda_1 = -3$ and $\lambda_2 = -1$.

- Consider $\lambda_1 = -3$. Substituting it in (54), we obtain $\alpha_2 = -\alpha_1$. Let us normalize $\widehat{\alpha}_1 = 1$ so that $\widehat{\alpha}_2 = -1$.
- Consider $\lambda_1 = -1$. Substituting it in (54), we obtain $\alpha_2 = -3\alpha_1$. Let us normalize $\widetilde{\alpha}_1 = 1$ so that $\widetilde{\alpha}_2 = -3$.

Accordingly, we have two solutions

$$\left.\begin{aligned} y(t) &= \widehat{\alpha}_1 e^{\lambda_1 t} \\ x(t) &= \widehat{\alpha}_2 e^{\lambda_1 t} \end{aligned}\right\}$$

and

$$\left.\begin{aligned} y(t) &= \widetilde{\alpha}_1 e^{\lambda_2 t} \\ x(t) &= \widetilde{\alpha}_2 e^{\lambda_2 t} \end{aligned}\right\}$$

Applying theorem 2, we have as solution of (51)

$$\begin{aligned} y(t) &= A_1 \widehat{\alpha}_1 e^{\lambda_1 t} + A_2 \widetilde{\alpha}_1 e^{\lambda_2 t} \\ x(t) &= A_1 \widehat{\alpha}_2 e^{\lambda_1 t} + A_2 \widetilde{\alpha}_2 e^{\lambda_2 t} \end{aligned}$$

and after substituting the corresponding values of $\alpha's$ and $\lambda's$ reduces to

$$\begin{aligned} y(t) &= A_1 e^{-3t} + A_2 e^{-t} \\ x(t) &= -A_1 e^{-3t} - 3A_2 e^{-t} \end{aligned}$$

(b) Particular solution of the non-homogenous system

The non-homogeneous system is given by (50). The two equations show in their righ-hand sides a constant function. So we try with constant functions as particular solutions of the non-homogeneous system.

$$\left.\begin{aligned}\overline{y}(t) &= \mu\\ \overline{x}(t) &= \eta\end{aligned}\right\}$$

In turn this implies $y'(t) = x'(t) = 0$ and substituting in (50) gives

$$\left.\begin{aligned}2\eta + 5\mu &= 77\\ \eta + 4\mu &= 61\end{aligned}\right\}$$

Solving this system gives $\mu = 15$ and $\eta = 1$, so that the particular solution becomes

$$\left.\begin{aligned}\overline{y}(t) &= 15\\ \overline{x}(t) &= 1\end{aligned}\right\}$$

(c) The solution of the system is given by

$$\left.\begin{aligned}y(t) &= A_1 e^{-3t} + A_2 e^{-t} + 15\\ x(t) &= -A_1 e^{-3t} - 3A_2 e^{-t} + 1\end{aligned}\right\} \tag{55}$$

To obtain the values of A_1 and A_2 we use the additional conditions $x(0) = 6$ and $y(0) = 12$. Evaluating (55) at $t = 0$ yields

$$\begin{aligned}A_1 + A_2 + 15 =& 12\\ -A_1 - 3A_2 + 1 =& 6\end{aligned}$$

so that $A_1 = -2$ and $A_2 = -1$. Then the solution of the system (50) is

$$\left.\begin{aligned}y(t) &= -2e^{-3t} - e^{-t} + 15\\ x(t) &= 2e^{-3t} + 3e^{-t} + 1\end{aligned}\right\}$$

Finally, note that since both trajectories $y(t)$ and $x(t)$ share the expressions $e^{-\lambda_i t}$ both will be monotonic.

Also, since the roots $(\lambda_1, \lambda_2) = (-3, -1)$ are negative both trajectories will converge to their equilibrium values $(\overline{y}(t), \overline{x}(t)) = (15, 1)$.