# Optimization. A first course on mathematics for economists
# Problem set 7: Differential equations

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7.1 Let the demand of a certain commodity be given by $D(p) = a - bp$ and its supply by $S(p) = \alpha + \beta p$, where $a, b, \alpha, \beta > 0$. Assume the price $p$ varies with time $t$, i.e. $p = p(t)$. Also, assume the market for the commodity is competitive so that price is determined by the excess demand function. Find the price trajectory of prices and study its stability.

7.2 Consider the following model of growth in a developing economy:

$$X(t) = \sigma K(t) \tag{1}$$
$$K'(t) = \alpha X(t) + H(t) \tag{2}$$
$$N(t) = N_0 e^{\rho t} \tag{3}$$

where $X(t)$ denotes the GDP per year, $K(t)$ is the capital stock, $H(t)$ is the flow of foreign aid, and $N(t)$ is the population.

(a) Derive a differential equation of $K(t)$

(b) Let $H(t) = H_0 e^{\mu t}$. Find the solution of the differential equation assuming $K(0) = K_0$ and $\alpha\sigma \neq \mu$

(c) Find an expression for the production per capita.

7.3 Consider the following macroeconomic model

$$Y(t) = C(t) + I(t) \tag{4}$$
$$I(t) = kC'(t) \tag{5}$$
$$C(t) = aY(t) + b \tag{6}$$

where $Y(t), I(t)$ and $C(t)$ denote GDP, investment, and consumption respectively at any time $t$. Suppose $b, k > 0$ and $a \in (0, 1)$.

(a) Derive a differential equation for the GDP

(b) Solve the differential equation for the GDP assuming $Y(0) = Y_0 > b/(1-a)$. Find the corresponding function for $I(t)$

(c) Compute $\lim_{t\to\infty} Y(t)/I(t)$.

7.4 Consider an economy described by

$$\begin{aligned}\frac{N'(t)}{N(t)} =& \alpha - \beta\frac{N(t)}{X(t)}\\ X(t) =& AN(t)^a\end{aligned}$$

where $N(t)$ and $X(t)$ denote the population and the GDP. Suppose $\alpha, \beta, a$ are positive, and $a \neq 1$. Denote by $x(t)$ the GDP per capita.

(a) Derive a differential equation for $x(t)$

(b) Solve the differential equation for the $x(t)$

(c) Find expression for $N(t)$ and $X(t)$

(d) Compute the $\lim_{t\to\infty}$ for $x(t), N(t), X(t)$ when $a \in (0, 1)$

7.5 Solve $y'(t) = a^t$ when $a \neq 1$ and when $a = 1$

7.6 Consider the following second-order differential equation

$$y''(t) - a^2 y(t) = 0,\ a \neq 0 \tag{7}$$

(a) Solve the equation.

(b) Shown that the trajectory of $y(t)$ always diverges regardless of the sign of $a$.

7.7 Solve the following second-order differential equation

$$y''(t) + y'(t) - 2y(t) = -10 \tag{8}$$

7.8 Consider a market described by the following supply and demand curves:

$$\begin{aligned}D(p) &= 9 - p(t) + p'(t) + 3p''(t)\\ S(p) &= -1 + 4p(t) - p'(t) + 5p''(t)\end{aligned}$$

Let $p(0) = 4$ and $p'(0) = 4$.

(a) Find the trajectory of the equilibrium price $p(t)$.

(b) Assess whether the $p(t)$ is convergent, divergent, or cyclical.

7.9 Solve the following system of differential equations

$$x'(t) + 2y'(t) + 2x(t) + 5y(t) = 77 \tag{9}$$

$$y'(t) + x(t) + 4y(t) = 61 \tag{10}$$

with initial conditions $x(0) = 6$ and $y(0) = 12$.