Optimization. A first course on mathematics for economists
Problem set 5: Non-linear programming

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5.1 Let $f(x_1, x_2) = -8x_1^2 - 10x_2^2 + 12x_1x_2 - 50x_1 + 80x_2$. Solve the following problem:

$$\max_{x_1,x_2} f(x_1, x_2) \text{ s.t.}$$
$$x_1 + x_2 \leq 1$$
$$8x_1^2 + x_2^2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0$$

Solution: *The Lagrangian of the problem is:*

$$L(x_1, x_2, \lambda_1, \lambda_2) = -8x_1^2 - 10x_2^2 + 12x_1x_2 - 50x_1 + 80x_2 + \lambda_1(1 - x_1 - x_2) + \lambda_2(2 - 8x_1^2 - x_2^2)$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial x_1} = -16x_1 + 12x_2 - 50 - \lambda_1 - 16\lambda_2 x_1 \leq 0$$
$$x_1 \frac{\partial L}{\partial x_1} = x_1(-16x_1 + 12x_2 - 50 - \lambda_1 - 16\lambda_2 x_1) = 0$$
$$\frac{\partial L}{\partial x_2} = -20x_2 + 12x_1 + 80 - \lambda_1 - 2\lambda_2 x_2 \leq 0$$
$$x_2 \frac{\partial L}{\partial x_2} = x_2(-20x_2 + 12x_1 + 80 - \lambda_1 - 2\lambda_2 x_2) = 0$$
$$\frac{\partial L}{\partial \lambda_1} = 1 - x_1 - x_2 \geq 0$$
$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = \lambda_1(1 - x_1 - x_2) = 0$$
$$\frac{\partial L}{\partial \lambda_1} = 2 - 8x_1^2 - x_2^2 \geq 0$$
$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = \lambda_2(2 - 8x_1^2 - x_2^2) = 0$$
$$\lambda_1 \geq 0, \lambda_2 \geq 0, x_1 \geq 0, x_2 \geq 0$$

There are four possible types of solutions:

$$\begin{aligned} x_1 \neq 0, &\quad x_2 = 0 \\ x_1 = 0, &\quad x_2 = 0 \\ x_1 \neq 0, &\quad x_2 \neq 0 \\ x_1 = 0, &\quad x_2 \neq 0 \end{aligned}$$

that have to be examined one-by-one.

Case 1: $(x_1 \neq 0, x_2 = 0)$ *In this case,* $x_1 \frac{\partial L}{\partial x_1} = 0$ *implies* $\frac{\partial L}{\partial x_1} = 0$*, that is,*

$$\frac{\partial L}{\partial x_1}(x_1, 0) = -16x_1 - 50 - \lambda_1 - 16\lambda_2 x_1 = 0$$

In turn this imples

$$x_1 = -\frac{50 + \lambda_1}{16(1 + \lambda_2)} < 0$$

contradicting the restriction $x_1 \geq 0$*. Therefore, there is no solution in Case 1.*

Case 2: $(x_1 = 0, x_2 = 0)$ *In this case,* $\frac{\partial L}{\partial \lambda_i} > 0$ *implying* $\lambda_1 = \lambda_2 = 0$*. Now evaluate*

$$\frac{\partial L}{\partial x_2}(0, 0)|_{\lambda_1 = 0} = 80 > 0$$

which is a contradiction. Thus, there is no solution in Case 2.

Case 3: $(x_1 \neq 0, x_2 \neq 0)$ *We organize the analysis of this Case in the study of four subcases:*

(3a) $\lambda_1 = \lambda_2 = 0$
Because $x_i > 0$*, it follows that* $x_i \frac{\partial L}{\partial x_i} = 0$ *implies* $\frac{\partial L}{\partial x_i} = 0$*, that is,*

$$\begin{aligned} -16x_1 + 12x_2 - 50 = 0 \\ -20x_2 + 12x_1 + 80 = 0 \end{aligned}$$

The solution of this system yields $x_2 = \frac{85}{22}$ *and* $x_1 = \frac{-5}{22} < 0$*, thus violating the restriction* $x_1 > 0$*. Hence, there is no solution in Case 3a.*

(3b) $\lambda_1 > 0, \lambda_2 = 0$
Because $x_i > 0$*, it follows that* $x_i \frac{\partial L}{\partial x_i} = 0$ *implies* $\frac{\partial L}{\partial x_i} = 0$*. Also,* $\lambda_1 > 0$ *implies* $\frac{\partial L}{\partial \lambda_1} = 0$*, that is,*

$$\begin{aligned} -16x_1 + 12x_2 - 50 - \lambda_1 = 0 \\ -20x_2 + 12x_1 + 80 - \lambda_1 = 0 \\ 1 - x_1 - x_2 = 0 \end{aligned}$$

The solution of this system yields $x_1 = \frac{-49}{30} < 0$*, thus violating the restriction* $x_1 > 0$*. Hence, there is no solution in Case 3b.*

(3c) $\lambda_1 = 0, \lambda_2 > 0$

Because $x_i > 0$*, it follows that* $x_i \frac{\partial L}{\partial x_i} = 0$ *implies* $\frac{\partial L}{\partial x_i} = 0$*. Also,* $\lambda_2 > 0$ *implies* $\frac{\partial L}{\partial \lambda_2} = 0$*, that is,*

$$\begin{aligned} -16x_1 + 12x_2 - 50 - 16x_1\lambda_2 &= 0 \\ -20x_2 + 12x_1 + 80 - 2x_2\lambda_2 &= 0 \\ 2 - 8x_1^2 - x_2^2 &= 0 \end{aligned}$$

Instead of solving this system, a more fruitful way to verify if there may be a solution, is to pay a close look at the frontier of the feasible set.

The frontier of the restriction $x_1 + x_2 \leq 1$ *is a traight line with slope* -1 *and extremes* $(0, 1)$ *and* $(1, 0)$*.*

The frontier of the restriction $8x_1^2 + x_2^2 \leq 2$ *can be written as* $x_2 = (2 - 8x_1^2)^{1/2}$*. This frontier has the following properties:*

- *the extreme points are* $(0, \sqrt{2})$ *and* $(\frac{1}{2}, 0)$
- *its slope is* $\frac{dx_2}{dx_1} = \frac{-8x_1}{(2-8x_1^2)^{1/2}} < 0$
- *also,* $\frac{dx_2}{dx_1}|_{x_1=0} = 0$ *while* $\frac{dx_2}{dx_1}|_{x_1=1/2} = -\infty$
- *compute* $\frac{d^2x_2}{dx_1^2} = -8(2 - 8x_1^2)^{1/2}[1 + 8x_1(2 - 8x_1^2)^{-1}] < 0, \forall x_1 \in [0, 1/2)$*. Thus, the frontier is concave and has a maximum at* $x_1 = 0$*.*

Summarizing the feasible set is defined by the intersection of both restriction and is shown in figure 1.

The assumption $\lambda_2 > 0$ *means that the corresponding restriction is binding and therefore if a solution exists it will belong to the frontier of the restriction* $g(x_1, x_2) \equiv 8x_1^2 + x_2^2 \leq 2$*. That is, at or below* $\widetilde{x}$*.*

(a) *Also, we know that the gradient* ∇g *points towards north-east, while the gradient* ∇f *points in the north-west direction. This implies that there cannot be a solution in Case 3c.*

(3d) $\lambda_1 > 0, \lambda_2 > 0$.

Now both restrictions are binding. Hence, we have a system of four equations with four unknowns. Looking at figure 2 the only possibility is $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$*, that is the solution of* $x_1 + x_2 - 1 = 8x_1^2 + x_2^2 - 2$*. But* $\widetilde{x}$ *cannot be a solution because at that point the gradient of* f *points north-westwards, while the gradients of the restrictions point north-eastwards.*

Summarizing, there is no solution in Case 3.

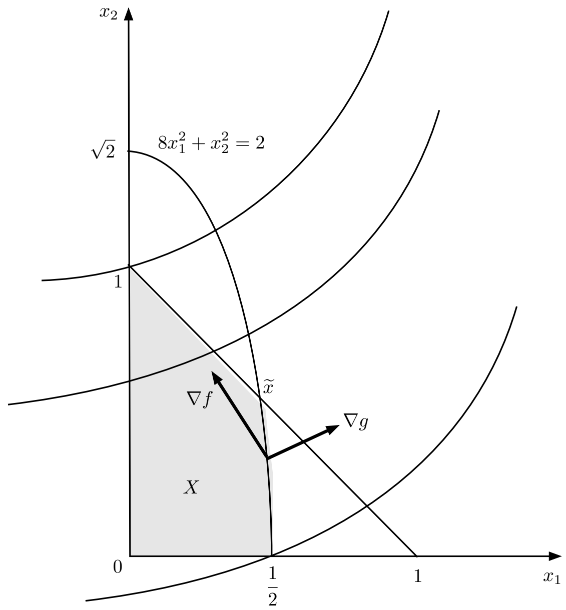

Figure 1: Problem 5.1a

Case 4: $(x_1 = 0, x_2 \neq 0)$ *In this case,* $x_2 \frac{\partial L}{\partial x_2} = 0$ *implies* $\frac{\partial L}{\partial x_2} = 0$*. Let us consider the system*

$$\begin{aligned} \frac{\partial L}{\partial x_2} &= -20x_2 + 80 - \lambda_1 - 2\lambda_2 x_2 = 0 \\ \lambda_1 \frac{\partial L}{\partial \lambda_1} &= \lambda_1(1 - x_2) = 0 \\ \lambda_2 \frac{\partial L}{\partial \lambda_2} &= \lambda_2(2 - x_2^2) = 0 \end{aligned}$$

The first equation can be rewritten as

$$x_2 = \frac{80 - \lambda_1}{2(10 + \lambda_2)}$$

Substituting the value of x_2 *into the second equation of the system, we obtain*

$$\lambda_1 \Big(1 - \frac{80 - \lambda_1}{2(10 + \lambda_2)}\Big) = \lambda_1 \Big(\frac{\lambda_1 + 2\lambda_2 - 60}{2(10 + \lambda_2)}\Big) = 0$$

That can be satisfied if $\lambda_1 = 0$ *and/or* $\lambda_1 + 2\lambda_2 - 60 = 0$*.*

(4a) *Let* $\lambda_1 = 0$*.*
Substituting in the value of x_2 *we obtain*

$$x_2 = \frac{80}{2(10 + \lambda_2)}$$

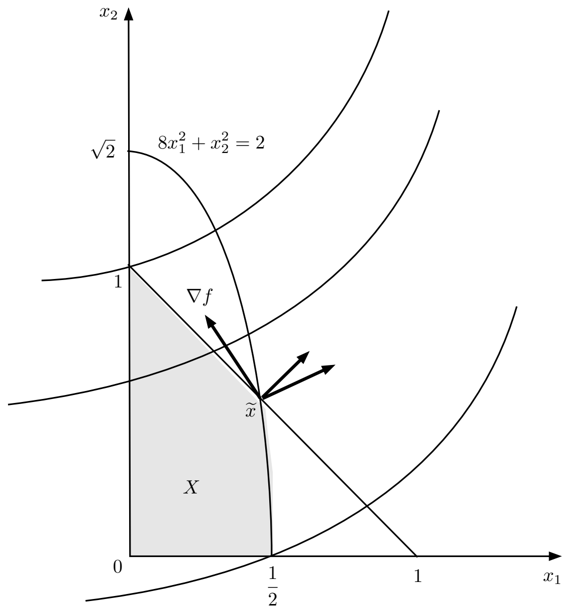

Figure 2: Problem 5.1b

Substituting this value of x_2 in the third equation of the system we obtain

$$\lambda_2(2 - x_2^2) = \lambda_2\Big(\frac{8(10+\lambda_2)^2 - 80^2}{4(10+\lambda_2)^2}\Big) = 0$$

That can be satisfied if $\lambda_2 = 0$ and/or $8(10+\lambda_2)^2 - 80^2 = 0$, yielding $\lambda_2 \approx 18.285$.

(i) Let $\lambda_2 = 0$. Then, $x_2 = 4$, but the condition $\frac{\partial L}{\partial \lambda_1} = 1 - 4 = -3 < 0$ is violated. Thus, $(x_1, x_2, \lambda_1, \lambda_2) = (0, 4, 0, 0)$ cannot be a solution.

(ii) Let $\lambda_2 \approx 18.285$. Then, $x_2 \approx \frac{80}{56.57} > 1$ and as before, the condition $\frac{\partial L}{\partial \lambda_1} \geq 0$ is violated.

Summarizing, there is no solution in Case 4a.

4(b) $\lambda_1 + 2\lambda_2 - 60 = 0$.
Substituting in the value of x_2 we obtain $x_2 = 1$, which in turn implies $\lambda_2 = 0$ because $\lambda_2(2 - x_2^2) = \lambda_2 = 0$. Given $x_2 = 1$ and $\lambda_2 = 0$, substituting these values in x_2 it follows that $\lambda_1 = 60$.

Hence we have identified a candidate solution given by

$$(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*) = (0, 1, 60, 0)$$

described in figure 3. Note that at x^* the two restrictions that are active are $x_1 \geq 0$ and $g(x_1, x_2) \equiv x_1 + x_2 \leq 1$. The gradient of f lies

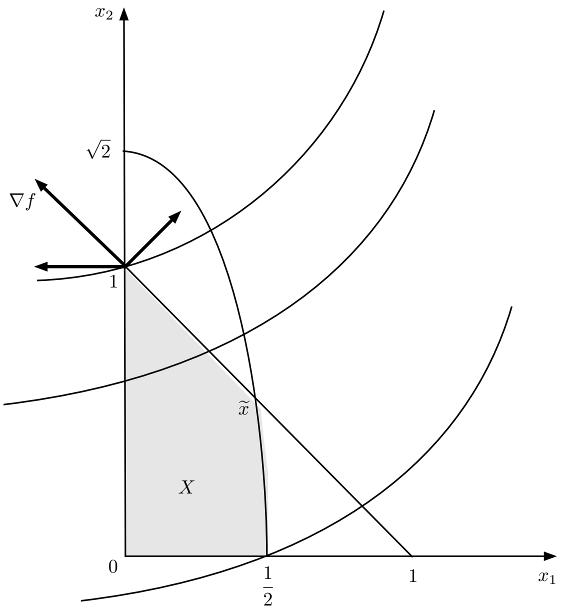

Figure 3: Problem 5.1c

in the cone formed by the gradients of those two restrictions. Note $\nabla g = (1, 1)$ *and* $\nabla f|_{(x^*,\lambda^*)} = (-38, 60)$.

5.2 Let $f(x_1, x_2) = 4x_1 + 3x_2,\ g(x_1, x_2) = 2x_1 + x_2$ and $x_1, x_2 \geq 0$. Find the candidate solutions to the problem

$$\max_{x_1,x_2} f(x_1, x_2) \text{ s.t } g(x_1, x_2) \leq 10,\ x_1 \geq 0, x_2 \geq 0$$

Solution:*The Lagrangian function is:*

$$L(x_1, x_2, \lambda) = 4x_1 + 3x_2 + \lambda(10 - 2x_1 - x_2)$$

and the necerssary Kuhn-tucker conditions to identify a maximum point are

$$\frac{\partial L}{\partial x_1} = 4 - 2\lambda \leq 0 \tag{1}$$

$$x_1 \frac{\partial L}{\partial x_1} = x_1(4 - 2\lambda) = 0 \tag{2}$$

$$\frac{\partial L}{\partial x_2} = 3 - \lambda \leq 0 \tag{3}$$

$$x_2 \frac{\partial L}{\partial x_2} = x_2(3 - \lambda) = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda} = 10 - 2x_1 - x_2 \geq 0 \tag{5}$$

$$\lambda \frac{\partial L}{\partial \lambda} = \lambda(10 - 2x_1 - x_2) = 0 \tag{6}$$

$$x_1 \geq 0, x_2 \geq 0, \lambda \geq 0 \tag{7}$$

- *Consider* (2). *It will be verified if* $x_1 = 0$ *and/or* $\lambda = 2$.
  If $\lambda = 2$ *substituting it into* (3) *leads to a contradiction. Thus, it must be the case that* $x_1 = 0$.
- *Consider* (4). *It will be verified if* $x_2 = 0$ *and/or* $\lambda = 3$.
  If $x_2 = 0$ *(together with the fact that* $x_1 = 0$*), substituting it into* (6) *implies* $\lambda = 0$. *Then, substituting* $x_1 = x_2 = \lambda = 0$ *in* (1) *leads to a contradiction. Hence it must be the case that* $\lambda = 3$.
- *Consider* (6). *Given that* $x_1 = 0$ *and* $\lambda = 3$*, it reads* $3(10 - x_2) = 0$ *so that* $x_2 = 10$.

Thus, we have identified a (unique) candidate solution $(x_1^*, x_2^*, \lambda^*) = (0, 10, 3)$.

5.3 Let $f(x_1, x_2) = 2x_1 + 3x_2,\ g(x_1, x_2) = x_1^2 + x_2^2$ and $x_1, x_2 \geq 0$. Find the solutions to the problem

$$\max_{x_1, x_2} f(x_1, x_2) \text{ s.t } g(x_1, x_2) \leq 2,\ x_1 \geq 0, x_2 \geq 0$$

Solution:*The Lagrangian function is:*

$$L(x_1, x_2, \lambda) = 2x_1 + 3x_2 + \lambda(2 - x_1^2 - x_2^2)$$

and the necerssary Kuhn-tucker conditions to identify a maximum point are

$$\frac{\partial L}{\partial x_1} = 2 - 2\lambda x_1 \leq 0 \tag{8}$$

$$x_1 \frac{\partial L}{\partial x_1} = x_1(2 - \lambda x_1) = 0 \tag{9}$$

$$\frac{\partial L}{\partial x_2} = 3 - 2\lambda x_2 \leq 0 \tag{10}$$

$$x_2 \frac{\partial L}{\partial x_2} = x_2(3 - 2\lambda x_2) = 0 \tag{11}$$

$$\frac{\partial L}{\partial \lambda} = 2 - x_1^2 - x_2^2 \geq 0 \tag{12}$$

$$\lambda \frac{\partial L}{\partial \lambda} = \lambda(2 - x_1^2 - x_2^2) = 0 \tag{13}$$

$$x_1 \geq 0, x_2 \geq 0, \lambda \geq 0 \tag{14}$$

- *Consider* (9). *It will be verified if* $x_1 = 0$ *and/or* $\lambda x_1 = 1$.
  If $x_1 = 0$ *substituting it into* (8) *leads to a contradiction. Thus, it must be the case that* $\lambda x_1 = 1$. *In turn, this implies,* $x_1 > 0, \lambda > 0$ *and* $x_1 = 1/\lambda$.
- *Consider* (11). *It will be verified if* $x_2 = 0$ *and/or* $\lambda x_1 = 3/2$.
  If $x_2 = 0$ *substituting it into* (10) *leads to a contradiction. Thus, it must be the case that* $\lambda x_2 = 3/2$. *In turn, this implies,* $x_2 > 0, \lambda > 0$ *and* $x_2 = 3/2\lambda$.
- *Consider* (13) *and substitute the values of* x_1 *and* x_2 *to obtain*

  $$\lambda\Big[2 - \Big(\frac{1}{\lambda}\Big)^2 - \Big(\frac{3}{2\lambda}\Big)^2\Big] = 0$$

  Given that we already know that $\lambda > 0$, *it follows that* $8\lambda^2 - 13 = 0$ *or* $\lambda = \sqrt{13/8}$.

Accordingly we have a unique candidate for a maximum point, namely

$$(x_1^*, x_2^*, \lambda^*) = \big(\sqrt{8/13}, \sqrt{18/13}, \sqrt{13/8}\big).$$

To assess whether this candidate is actually a maximum point, we know that if the objective function f *is differentaible and concave and the constraint* g *is differentiable and convex, then the candidate solution will maximize the value of* f. *In our problem,* f *is linear thus concave, and both* f *and* g *are differentiable. To assess the convexity of* g *we have to verify that its Hessian matrix is positive definite. The Hessian matrix of* g *is*

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Clearly $|H_1| > 0$ *and* $|H_2| > 0$ *so that* g *is convex. We thus conclude that* $(x_1^*, x_2^*) = \big(\sqrt{8/13}, \sqrt{18/13}\big)$ *is a maximum of* f.

5.4 Solve the following problem

$$\min_{x_1,x_2} x_1^2 - 4x_1 + x_2^2 - 6x_2 \text{ s.t}$$
$$x_1 + x_2 \leq 3$$
$$-2x_1 + x_2 \leq 2$$

Solution: *Following the same methodology,*

$$(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*) = (1, 2, 2, 0)$$

emerges as the only solution candidate.

5.5 Let $f(x) = (x-1)^3,\ x \leq 2$ and $x \geq 0$. Show that Kuhn-Tucker first-order conditions are necessary but not sufficient to characterize a maximum of the problem

$$\max_{x_1,x_2} f(x) \text{ s.t}$$
$$x \leq 2$$
$$x \geq 0$$

Solution: *The Lagrangian function is:*

$$L(x,\lambda) = (x-1)^3 + \lambda(2-x)$$

and the necessary Kuhn-tucker conditions to identify a maximum point are:

$$\frac{\partial L}{\partial x} = 3(x-1)^2 - \lambda \leq 0 \tag{15}$$
$$x\frac{\partial L}{\partial x} = x(3(x-1)^2 - \lambda) = 0 \tag{16}$$
$$\frac{\partial L}{\partial \lambda} = 2 - x \geq 0 \tag{17}$$
$$\lambda\frac{\partial L}{\partial \lambda} = \lambda(2-x) = 0 \tag{18}$$
$$x \geq 0, \lambda \geq 0 \tag{19}$$

- *Consider* (16). *It will be verified if* $x = 0$ *and/or* $3(x-1)^2 - \lambda = 0$. *If* $x = 0$ *substituting in* (18) *yields* $\lambda = 0$. *But these values lead to a contradiction when substituted into* (15). *Thus it must be the case that* $3(x-1)^2 - \lambda = 0$.
- *Consider* $3(x-1)^2 - \lambda = 0$ *and rewrite it as*

$$\lambda = 3(x-1)^2 \tag{20}$$

  Substituting it in (18) *we obtain*

$$3(x-1)^2(2-x) = 0$$

  that has as solutions $x = 1$ *and* $x = 2$.

- *Consider* $x = 1$. *Substituting it in* (20) *yields* $\lambda = 0$. *Therefore,* $(x^*, \lambda^*) = (1, 0)$ *is a candidate solution.*
- *Consider* $x = 2$. *Substituting it in* (20) *yields* $\lambda = 3$. *Therefore,* $(x^*, \lambda^*) = (2, 3)$ *is also a candidate solution.*

An inspection of function f *tells us that it is monotonically increasing for any value of* x. *Therefore, the maximum of the function has to be located at* $x = 2$ *where the restriction is binding.*
Hence, only one of the candidate solutions is actually solving the problem proposed. In other words, the necessary Kuhn-Tucker conditions are not sufficient to characterize the maxima of f.
Remark: at $x = 1$, f *shows an inflection point. Looking at the second derivative of* f *is easy to check that* f *is concave for* $x < 1$ *and convex for* $x > 1$.

5.6 Let $f(x, y) = \frac{1}{x^2+y^2}$, $g_1(x, y) = y - (x - 1)^3$, $g_2(x, y) = -y$, $g_3(x, y) = x - 2$, with $g_i(x, y) \leq 0$.

(a) Let S be the set defind by g_1, g_2 and g_3. Provide an argument showing that f has a maximum and a minimum over S.

(b) Show graphically that f has a maximum at $(x, y) = (1, 0)$

(c) Verify that the Kuhn-Tucker conditions do not identify that point as a critical point. Explain why.

Solution:

(a) *The first remark is that the point* $(0, 0)$ *does not belong to* S *and therefore, the function* f *is continuous over* S. *The second remark is that the set* S *is compact. Obervation of figure 4 tells us that* S *is defined by points such that* $x \in [1, 2]$ *and* $y \in [0, 1]$. *In other words* $\forall (x, y) \in S, \|(x, y)\| = \sqrt{x^2 + y^2} \leq 5$, *so that* S *is bounded. Finally, the intersection of* g_1, g_2, g_3 *defining* S *is closed. Applying Weierstrass theorem, it follows that* f *has a maximum and a minimum over* S.

(b) *The level sets of* f *are circles centered at the origin. The closer the level set to the origin the higher the value of the function. In other words, the gradient of* f *points towards the origin as illustrated in figure 5. The level set of* f *with the highest value compatible with* S *is the one passing through point* $(1, 0)$. *It cannot be a level set with smaller radius because it would violate restriction* g_2. *See figure 6. Accordingly, the point* $(1, 0)$ *maximizes* f *within* S. *Note that* $(1, 0)$ *both* g_1 *and* g_2 *are binding while* g_3 *is not.*

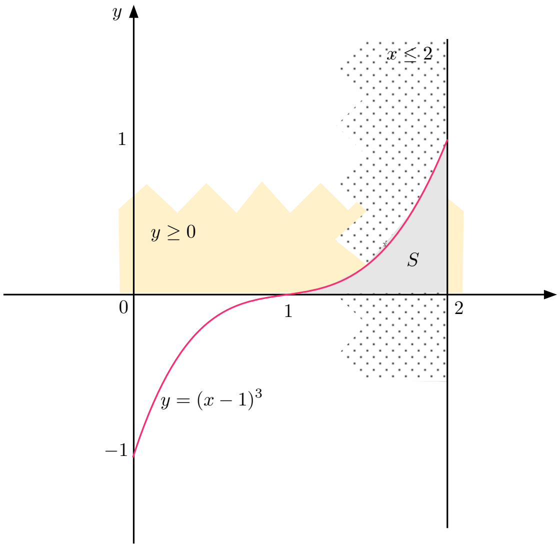

Figure 4: Problem 5.6a

(c) *The problem to solve is*

$$\max_{x,y} f(x) \text{ s.t}$$

$$y \leq (x-1)^3$$
$$y \geq 0$$
$$x \leq 2$$

The corresponding Lagrangian function is

$$L(x.y, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{x^2+y^2} - \lambda_1(y-(x-1)^3) + \lambda_2 y - \lambda_3(x-2)$$

and the necessary Kuhn-Tucker conditions to identify a maximum point

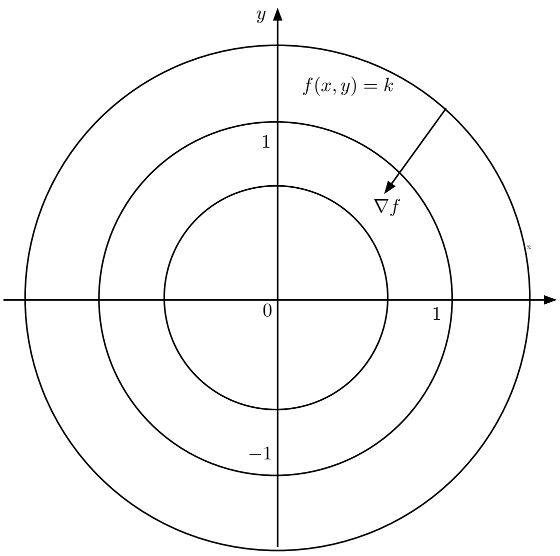

Figure 5: Problem 5.6b

are:

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= \frac{-2x}{(x^2+y^2)^2} + 3\lambda_1(x-1)^2 - \lambda_3 \leq 0 \\
x\frac{\partial L}{\partial x} & x\Big(\frac{-2x}{(x^2+y^2)^2} + 3\lambda_1(x-1)^2 - \lambda_3\Big) = 0 \\
\frac{\partial L}{\partial y} &= \frac{-2y}{(x^2+y^2)^2} - \lambda_1 + \lambda_2 \leq 0 \\
y\frac{\partial L}{\partial y} &= y\Big(\frac{-2y}{(x^2+y^2)^2} - \lambda_1 + \lambda_2\Big) = 0 \\
\frac{\partial L}{\partial \lambda_1} &= y - (x-1)^3 \leq 0 \\
\lambda_1\frac{\partial L}{\partial \lambda_1} &= \lambda_1\Big(y - (x-1)^3\Big) = 0 \\
\frac{\partial L}{\partial \lambda_2} &= y \geq 0 \\
\lambda_2\frac{\partial L}{\partial \lambda_2} &= \lambda_2 y = 0 \\
\frac{\partial L}{\partial \lambda_3} &= x - 2 \leq 0 \\
\lambda_3\frac{\partial L}{\partial \lambda_3} &= \lambda_3(x-2) = 0
\end{aligned}
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Next we evaluate the Kuhn-Tucker conditions at the point $(1,0)$ *that we have identified as maximizer. All conditions should be satisfied given that it is a maximizer. Let us list only the conditions that are trivially*

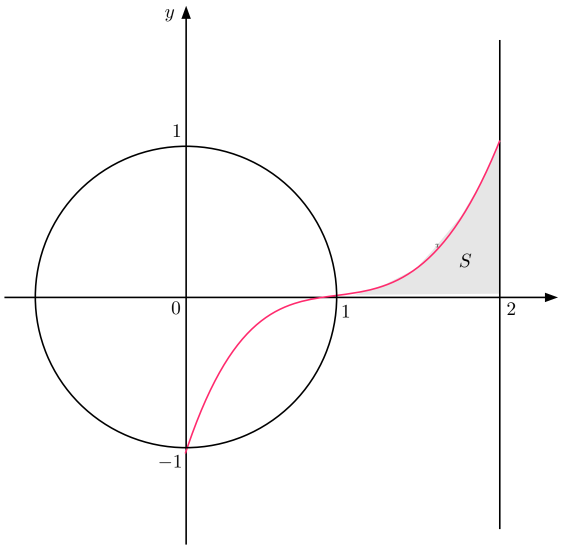

Figure 6: Problem 5.6c

satisfied:

$$\frac{\partial L}{\partial x}(1,0) = -\lambda_3 - 2 = 0 \tag{21}$$

$$\lambda_3 \frac{\partial L}{\partial \lambda_3}(1,0) = -\lambda_3 = 0 \tag{22}$$

From (22) *it follows* $\lambda_3 = 0$. *Then, substituting it in* (21) *we obtain a contradiction!*
Why the Kuhn-Tucker conditions fail to identify $(1,0)$ *as maximizer? The answer is that at* $(1,0)$ *the constraint qualification is not satisfied. Remeber that at* $(1,0)$ *restrictions* g_1 *and* g_2 *are binding. Compute the gradients of these restrictions. They are*

$$\nabla g_1(1,0) = (0,1) \quad \textit{and} \quad \nabla g_2(1,0) = (0,-1)$$

therefore they linearly dependent, and thus they do not form a cone in which ∇f *may lie.*

5.7 Let $U(x,y)$ be a utility function with indifference map represented in figure 7. Let $g(x,y) \leq k$ be the budget constraint. As the figure shows, utility is maximized (given the budget constraint) at the point (x^*, y^*). Show that at that point the indifference curve must be steeper than the budget constraint.

Solution: *To compare slopes of the indifference curve and of the restriction we apply the implicit function theorem. Assuming both* U *and* g *have all the required properties (continuity, differentiability, ...) let* $y(x)$ *implicitly denote*

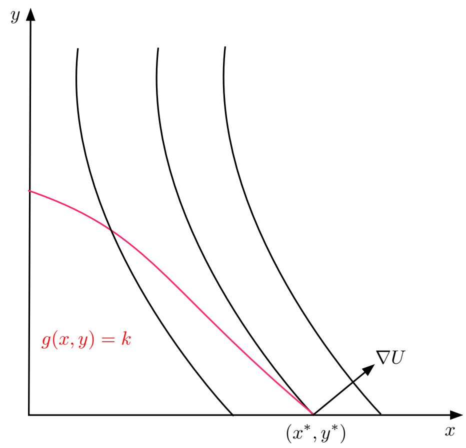

Figure 7: Problem 5.7

the constraint so that we have $g(y(x),x)=k$. Implicitly differentiating both sides with respect to y we obtain

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}dy/dx=0$$

or

$$\frac{dy}{dx}=-\frac{\frac{\partial g(x,y)}{\partial x}}{\frac{\partial g(x,y)}{\partial y}}$$

Evaluated at the point (x^*,y^*) the slope of the restriction is

$$\frac{dy}{dx}|_{(x^*,y^*)}=-\frac{\frac{\partial g(x^*,y^*)}{\partial x}}{\frac{\partial g(x^*,y^*)}{\partial y}}$$

The indifference curve is expressed as $U(x,y)=s$. In a parallel fashion, we can also let $y(x)$ implicitly denote the indifference curve so that we have $U(y(x),x)=s$. The slope of the indifference curve evaluated at (x^*,y^*) is given by

$$\frac{dy}{dx}|_{(x^*,y^*)}=-\frac{\frac{\partial U(x^*,y^*)}{\partial x}}{\frac{\partial U(x^*,y^*)}{\partial y}}$$

Formally, the idea that the indifference curve is steeper than the restriction means that in absolute value the slope of U is greater than the absolute value *of the slope of g. This gives*

$$\frac{\dfrac{\partial U(x^*,y^*)}{\partial x}}{\dfrac{\partial U(x^*,y^*)}{\partial y}} > \frac{\dfrac{\partial g(x^*,y^*)}{\partial x}}{\dfrac{\partial g(x^*,y^*)}{\partial y}}$$

Because each of these derivatives is positive, the inequality can be rearranged as

$$\frac{\dfrac{\partial U(x^*,y^*)}{\partial x}}{\dfrac{\partial g(x^*,y^*)}{\partial x}} > \frac{\dfrac{\partial U(x^*,y^*)}{\partial y}}{\dfrac{\partial g(x^*,y^*)}{\partial y}} \tag{23}$$

Remark that $(x^*,y^*) = (x^*,0)$ with $x^* > 0$.
To formally solve the maximization problem, the Lagrangean function is

$$L(x,y,\lambda) = U(x,y) + \lambda(k - g(x,y))$$

The relevant Kuhn-Tucker conditions (evaluated at (x^*,y^*)) is

$$\frac{\partial L}{\partial x} = \frac{\partial U(x^*,y^*)}{\partial x} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial x} = 0$$

or

$$\lambda^* = \frac{\dfrac{\partial U(x^*,y^*)}{\partial x}}{\dfrac{\partial g(x^*,y^*)}{\partial x}} \tag{24}$$

Combining (23) and (24) gives

$$\lambda^* > \frac{\dfrac{\partial U(x^*,y^*)}{\partial y}}{\dfrac{\partial g(x^*,y^*)}{\partial y}}$$

or

$$\frac{\partial U(x^*,y^*)}{\partial y} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial y} < 0$$

Summarizing, we have

$$\begin{aligned}
&x^* > 0,\ \frac{\partial U(x^*,y^*)}{\partial x} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial x} = 0\\
&y^* = 0,\ \frac{\partial U(x^*,y^*)}{\partial y} - \lambda^* \frac{\partial g(x^*,y^*)}{\partial y} < 0\\
&\lambda^* > 0,\ g(x^*,y^*) - k = 0
\end{aligned}$$

so that the Kuhn-Tucher conditions are satisfied. In other words, the intuition illustrated in figure 7 is captured by the Kuhn-Tucker conditions.