Optimization. A first course on mathematics for economists
Problem set 3: Differentiability

Xavier Martinez-Giralt

Academic Year 2015-2016

3.1 Let $f(x, y) = x^2y$

(a) Find $\nabla f(3, 2)$

(b) Find the derivative of f in the direction of $(1, 2)$ at the point $(3, 2)$.

(c) Find the derivative of f in the direction of $(2, 1)$ at the point $(3, 2)$.

(d) Identify in which direction is the directional derivative maximal at the point $(3, 2)$. What is the directional derivative in that direction?

3.2 Let $f(x, y, z) = xye^{x^2+z^2-5}$. Calculate the gradient of f at the point $(1, 3, -2)$ and calculate the directional derivative at the point $(1, 3, -2)$ in the direction of the vector $v = (3, -1, 4)$.

3.3 Consider an industry producing a consumption good supplied according to the following supply function $S = S(w, p)$ where w represents the wage rate and p the price. Also, demand for the consumption good is captured by the demand function $D = D(m, p)$ where m denotes income. Assume

$$\frac{\partial S}{\partial p} > 0, \quad \frac{\partial S}{\partial w} < 0$$
$$\frac{\partial D}{\partial p} < 0, \quad \frac{\partial D}{\partial m} > 0$$

Assess how a change in the wage rate w and in the income m affects the equilibrium price.

3.4 Verifiy the homogeneity of

$$f(x_1, x_2, x_3, x_4) = \frac{x_1 + 2x_2 + 3x_3 + 4x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

3.5 Consider a general Cobb-Douglas production function

$$F(x_1, \ldots, x_n) = Ax_1^{a_1} \cdots x_n^{a_n}$$

(a) Show that it is homogeneous.

(b) Determine when it has constant, decreasing, or increasing returns to scale.

3.6 Show that the constant elasticity of substitution (CES) function

$$f(x) = A\Big(\sum_{i=1}^{n} \delta_i x_i^{-\rho}\Big)^{-v/\rho}$$

where $A > 0, v > 0, \delta_1 > 0, \sum_i \delta_i = 1, \rho > -1, \rho \neq 0$, is homogeneous of degree v

3.7 Consider an individual consuming two goods (x, y) available at prices (p_x, p_y). The individual determines the demand of each good given those prices and the income m defining the budget constraint $m = p_x x + p_y y$. Denote the resulting demands by $x(p_x, p_y, m)$ and $y(p_x, p_y, m)$ Show that these demands are homogeneous of degree zero in prices and income.

3.8 Approximate $\sqrt{5}$ to at least accuracy $1/100$ around $x = 4$.