Optimization. A first course on mathematics for economists
Problem set 1: Topology

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1.1 Find the length of the line segment joining $(1, 1, 1)$ to $(3, 2, 0)$.

Solution: *This is the length of the vector* $(3, 2, 0) - (1, 1, 1) = (2, 1, -1)$. *The length is* $\|(2, 1, -1)\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

1.2 For real numbers, prove that

(a) $x \leq |x|, -|x| \leq x$

Solution: *If* $x \geq 0$*, then* $|x| = x$*. If* $x < 0$*, then* $|x| \geq x$ *since* $|x| \geq 0$*. In any case,* $x \leq |x|$*. The other assertion follows a similar argument.*

(b) $|x| \leq a \Leftrightarrow -a \leq x \leq a$, with $a \geq 0$.

Solution: *If* $x \geq 0$*, we must show that* $0 \leq x \leq a \Leftrightarrow -a \leq x \leq a$*. This is obvious. If* $x < 0$*, then we must show that* $0 \leq -x \leq a \Leftrightarrow -a \leq x \leq a$*. Again this is obvious. It is so because if* $c \leq 0$*, it follows that* $0 \leq x \leq y \Leftrightarrow 0 \geq cx \geq cy$*.*

(c) $|x + y| \leq |x| + |y|$

Solution: *By (a),* $-|x| \leq x \leq |x|$ *and* $-|y| \leq y \leq |y|$*. Adding, we obtain* $-(|x|+|y|) \leq x+y \leq |x|+|y|$*. Then, by (b)* $|x+y| \leq |x|+|y|$*.*

1.3 (a) Let $x \geq 0$ be a real number such that for any $\varepsilon > 0, x \leq \varepsilon$. Show that $x = 0$.

Solution: *Suppose* $x > 0$*. Let* $\varepsilon = x/2$*. Then,* $x < x/2$ *implies* $0 < x/2 < 0$*, a contradiction. Hence,* $x = 0$*.*

(b) Let $S = (0, 1)$. Show that for any $\varepsilon > 0$, there exists $x \in S$ such that $x < \varepsilon, x \neq 0$.

Solution: *Let* $x = \min\{\varepsilon/2, 1/2\}$*.*

1.4 Let $S = \{(x, y) \in \mathbb{R}^2 | 0 < x < 1\}$ Show that S is open.

Solution: *See figure 1 to verify that around each point* $(x, y) \in S$ *we can draw a disc of radius* $r = \min\{x, 1 - x\}$ *and it is entirely contained in* S*.Hence, by definition,* S *is open.*

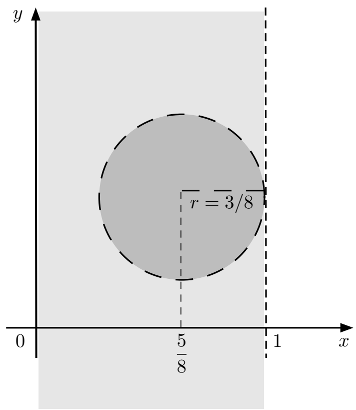

Figure 1: Problem 1.4

1.5 Let $S = \{(x, y) \in \mathbb{R}^2 | 0 < x \leq 1\}$ Is S is open?

Solution: *No, because any disc about* $(1, 0) \in S$ *contains points* $(x, 0)$ *with* $x > 1$.

1.6 Let $A \subset \mathbb{R}^n$ be open and $B \subset \mathbb{R}^n$.
Define $A + B = \{x + y \in \mathbb{R}^n | x \in A, y \in B\}$. Prove that $A + B$ is open.

Solution: *Let* $x \in A$ *and* $y \in B$ *so that* $x + y \in A + B$. *By definition,* $\exists \varepsilon > 0$ *so that* $D(x, \varepsilon) \subset A$. *We claim that* $D(x + y, \varepsilon) \subset A + B$. *Indeed, let* $z \in D(x+y, \varepsilon)$ *so that* $d(x+y, z) < \varepsilon$. *But* $d(x+y, z) = d(x, z-y)$ *so* $z - y \in A$, *and then* $z = (z - y) + y \in A + B$. *Thus,* $D(x+y, \varepsilon) \subset A+B$, *so* $A + B$ *is open.*

1.7 Let $S = \{(x, y) \in \mathbb{R}^2 | 0 < x \leq 1\}$ Find $int(S)$.

Solution: *To determine the interior points, we just need to locate points about which it is possible to draw a* ε*-disc entirely contained in* S. *By considering figure 1, we see that these are points* (x, y) *where* $0 < x < 1$. *Thus,* $int(S) = \{(x, y) | 0 < x < 1\}$.

1.8 Let $S = \{(x, y) \in \mathbb{R}^2 | 0 < x \leq 1, 0 \leq y \leq 1\}$ Is S closed?

Solution: *See figure 2. Intuitively,* S *is not closed because the portion of its boundary on the* y*-axis is not in* S. *Also, the complement is not open because any* ε*-disc about a point on the* y*-axis, say* $(0, 1/2)$ *will intersect* S, *and hence is not in* $\mathbb{R} \setminus S$.

1.9 Let $S = \{x \in \mathbb{R} | x \in [0, 1], x \text{ is rational}\}$. Find the accumulation points of S.

Solution: *The set of accumulation points consists of all points in* $[0, 1]$. *Indeed, let* $y \in [0, 1]$ *and* $D(y, \varepsilon) = (y - \varepsilon, y + \varepsilon)$ *be a neighborhood of* y. *Now we know we can find rational points in* $[0, 1]$ *arbitrarily close to* y

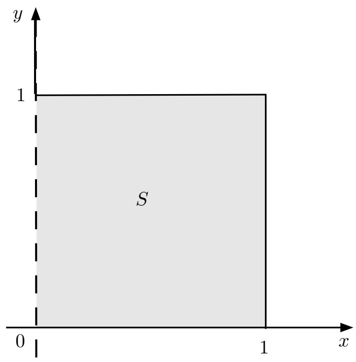

Figure 2: Problem 1.8

(other than y) and in particular in $D(y, \varepsilon)$. Hence, y is an accumulation point. Any point $y \notin [0, 1]$ is not an accumulation point because y has an ε-disc containing it which does not meet $[0, 1]$ and therefore S.

1.10 Recall the theorem that says that a set $A \subset \mathbb{R}$ is closed iff all the accumulation points of A belong to A. Verify the theorem for the set
$A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, \text{or } x = 2\}$.

Solution: *Figure 3 represents set A. Clearly, A is closed. The accumulation points of A consist exactly of A itself which lie in A. Note that on $\mathbb{R}$, $[0, 1] \cup \{2\}$ has accumulation points $[0, 1]$ without the point $\{2\}$.*

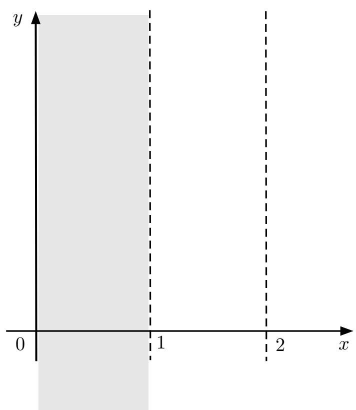

Figure 3: Problem 1.10

1.11 Determine which of the following sets are compact

(a) $\{x \in \mathbb{R} | x \geq 0\}$
Solution: *Non-compact because it is unbounded.*

(b) $[0, 1] \cup [2, 3]$
Solution: *Compact because is closed and bounded.*

(c) $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$
Solution: *Non-compact because in not closed.*