Optimization. A first course of mathematics for economists

Xavier Martinez-Giralt
Universitat Autònoma de Barcelona
xavier.martinez.giralt@uab.eu

IV. Dynamic optimization

A motivating story

- Little girl gets a cake (Wälti (2002))
- Decides to eat it all alone
- When?
  - all right away
    - today better than tomorrow
    - but decreasing marginal utility+satiation
  - a bit everyday
    - finish it before spoilt
    - how much every day?
      - same quantity every day
      - diminishing amounts along time

A motivating story - Formal statement

- c_t: amount of cake eaten in day t
- $u(c_t)$: instantaneous utility, $u' > 0, u'' < 0$
- $\beta \in (0,1)$: discount rate
- V: present value in $t = 0$ of the consumption path
- $t = 0, 1, 2, \ldots, T$
- k_0: original size of the cake (given)
- Problem

$$\max_{\{c_1,\ldots,c_T\}} V(c_1, \ldots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.}$$
$$k_{t+1} - k_t = -c_t$$
$$k_{t+1} \geq 0, \quad k_0 \text{ given}$$
$$c_t \geq 0$$

A motivating story - Formal statement (2)

Solution

- Optimal consumption path $c_t,\ t = 0, \ldots, T$
- Methods: Numerical, Analitical: Optimal control, dynamic programming

Other examples

- Individuals planning savings for retirement
- fossil fuels (extraction, exploration, polution policy, ...)
- forest managers (age of tress before harvesting), etc, etc, etc

Common features

- managment of stock of an asset over time
- decisions in t affect future opportunities and payoffs
- decisions are functions (time paths of actions)

Introduction

Definition

- Dynamic economic problem:
  Allocation of scarce resources among competing ends over a period of time.

Elements of the problem

1. time
2. state variables
3. control variables
4. equations of motion
5. Objective functional

1. Time

Time may be in continuous or discrete units. Time horizon may be finite $t \in [t_0, T]$ or infinite $t \in [t_0, \infty)$.

Introduction (2)

2. State variables

- At any time t, the state of the system is described by state variables, $\mathbf{s}(t)$.
- State variables describe those elements of the system over which the decision-maker does not have capacity of choice.

3. Control variables

- At any time t, the decisions (actions) to be taken by a decision-maker are described by control variables, $\mathbf{a}(t)$.
- Control variables belong to a set A usually assumed compact, convex, and time invariant.
- The evolution of the control variables along time is a control trajectory.

Introduction (3)

4. Equations of motion

- The evolution of the state variables along time is a state trajectory, $\{\mathrm{s}(t)\}$
- The state trajectory of each state variable is characterized by equations of motion.
- *[Continuous time]*: An equation of motion is a differential equation giving the time rate of change of the corresponding state variable as a function of the state variables, the control variables, and time:
  $\dot{\mathrm{s}}(t) = \mathbf{g_t}(\mathbf{s}(t), \mathbf{a}(t), t)$
- *[Discrete time]*: An equation of motion is a difference equation involving the state and control variable, where the state is the unknown and the control variable is a parameter to be chosen: $s(t+1) = g_t(s(t), a(t))$

Introduction (4)

5. Objective functional

- The objective functional is a mapping from control trajectories to a point on the real line, the value of which is to be maximized:
  $V(\mathbf{a}(t)) = \int_{t_0}^{T-1} f_t(\mathbf{s}(t), \mathbf{a}(t), t)dt + v(s_T, T)$
- $f_t(\mathbf{s}(t), \mathbf{a}(t), t)$ is the intermediate function. It describes how $(\mathbf{s}(t), \mathbf{a}(t), t)$ determine the contemporaneous value of the period-by-period return function (profits, ...)
- $v(s_T, T)$ is the final function. It shows the value of the final state s_T (e.g. the value of the stock of production left at T, of the assets remaining at T, ...).

6. Approach

- Focus of analysis: discrete time and finite horizon problems.

Introduction (5)

The general control problem in continuous time and finite horizon

$$\max_{\{\mathbf{a}(t)\}} V = \int_{t_0}^{T-1} f_t(\mathbf{s}(t), \mathbf{a}(t), t)dt + v(\mathbf{s_T}, T) \text{ s.t.}$$

$$\dot{\mathbf{s}}(t) = \mathbf{g}_t(\mathbf{s}(t), \mathbf{a}(t), t)$$

$$t_0, T, \mathbf{s}(t_0), \mathbf{s_T} \text{ given}$$

$$\{\mathbf{a}(t)\} \in A \text{ (set of feasible trajectories)}$$

Solution

- Solution is $\mathbf{a}^*(t), t \in [t_0, T-1]$
- Then, $\mathbf{a}^*(t), t \in [t_0, T]$ is the solution satisfying $\dot{\mathbf{s}}(t) = \mathbf{g_t}(\mathbf{s}(t), \mathbf{a}^*(t), t)$
- The max value is $\int_{t_0}^{T-1} f_t(\mathbf{s}(t), \mathbf{a}^*(t), t)dt + v(\mathbf{s_T}, T)$

Introduction (6)

The general control problem in discrete time and finite horizon

$$\max_{\{\mathbf{a}(t)\}} V = \sum_{t_0}^{T-1} f_t(s_t, a_t) + v(s_T) \text{ s.t.}$$

$$s_{t+1} = g_t(s_t, a_t), \; t = 0, 1, \ldots, T-1$$

$$t_0, T, s_0, s_T \text{ given}$$

$$a_t \in A \text{ (set of feasible trajectories)}$$

Solution

- Solution is $a_t^*, t \in [t_0, T-1]$
- Then, $a_t^*, t \in [t_0, T]$ is the solution satisfying $s_{t+1} = g_t(s_t, a_t^*)$
- The max value is $\sum_{t_0}^{T-1} f_t(s_t, a_t^*) + v(s_T)$

Intuition

- Planning a two-day trekking trip. Take w units of food.
- How to divide the food between both days?
- c_0 is consumption today ($t = 0$) and c_1 is consumption tomorrow ($t = 1$)
- Optimization problem is

$$\max_{c_0, c_1} U(c_0, c_1) \text{ s.t. } c_0 + c_1 \leq w$$

- Optimality requires eat up all food ($c_0 + c_1 = w$) and marginal utility be equal across both days ($U'_{c_0} = U'_{c_1}$)
- i.e. the marginal cost of consumption in $t = 0$ is the consumption foregone in $t = 1$.
- Thus, intertemporal optimization requires allocation of resources that exhaust intertemporal trade opportunities.

Intuition (2)

- Suppose U is separable: $U(c_0, c_1) = U(c_0) + U(c_1)$ and stationary $U(c_0, c_1) = U(c_0) + \beta U(c_1)$
- With β is the discount rate of future consumption.
- Optimality condition is $U'(c_0) = \beta U'(c_1)$
- If U concave, $c_0 > c_1 \Leftrightarrow \beta < 1$
- (Interesting) extensions
  - allow for borrowing and lending at an interest rate r (see problem 9.1)
  - allow for a finite horizon of T periods (see problem 9.2)
  - allow for continuous time
  - allow for infinite horizon

Describing the optimization problem (finite horizon)

- At $t = 0$
  - there is an initial state s_0.
  - An agent chooses an action $a_0 \in A_0$.
  - This action generates a contemporary return $f_0(a_0, s_0)$ ...
  - ... and leads to a new state s_1.
  - Transition from s_0 to s_1 through f_0 is determined by a transition equation $s_1 = g_0(a_0, s_0)$.
- This decision making takes place every period
  - sequence of actions $(a_0, a_1, \dots, a_{T-1})$ generates a
  - sequence of states $(s_0, s_1, \dots, s_{T-1})$
  - so that $s_{t+1} = g_t(a_t, s_t)$
- and reaches a final period T with a final state s_T with value $v(s_T)$.

Describing the optimization problem (2)

- Assume separability of the objective function
- Aim of the agent: choice of $(a_0, a_1, \ldots, a_{T-1})$ to
- max discounted value of sum of contemporaneous returns + value of final state:

$$\max_{(a_0, a_1, \ldots, a_{T-1})} \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v(s_T) \text{ s.t.}$$

$$s_{t+1} = g_t(a_t, s_t), \; t = 0, 1, \ldots, T-1$$

$$T, s_0, s_T \text{ given}$$

- Remark: a_t, s_t may be vectors.

On initial and terminal conditions

Initial condition

- Together with terminal condition allow for closing the system
- Initial condition typically fixed and exogenous
- Represents the level of stock the planner starts with

Terminal condition

- Requires to specify a terminal date T and a terminal state s_T
- Each may be fixed or variable $\rightarrow$ 4 alternative scenarios
  - both fixed
  - both variable
  - one fixed, the other variable

On terminal conditions

(T, s_T) fixed

- example: a fishery regulator seeking to manage harvest over a T years ensuring a stock of fish left s_T
- Illustration: Section (a) of figure

T fixed, s_T variable

- example: mine manager planning extraction over a fixed time horizon.
- Illustration: Section (b) of figure

T variable, s_T fixed

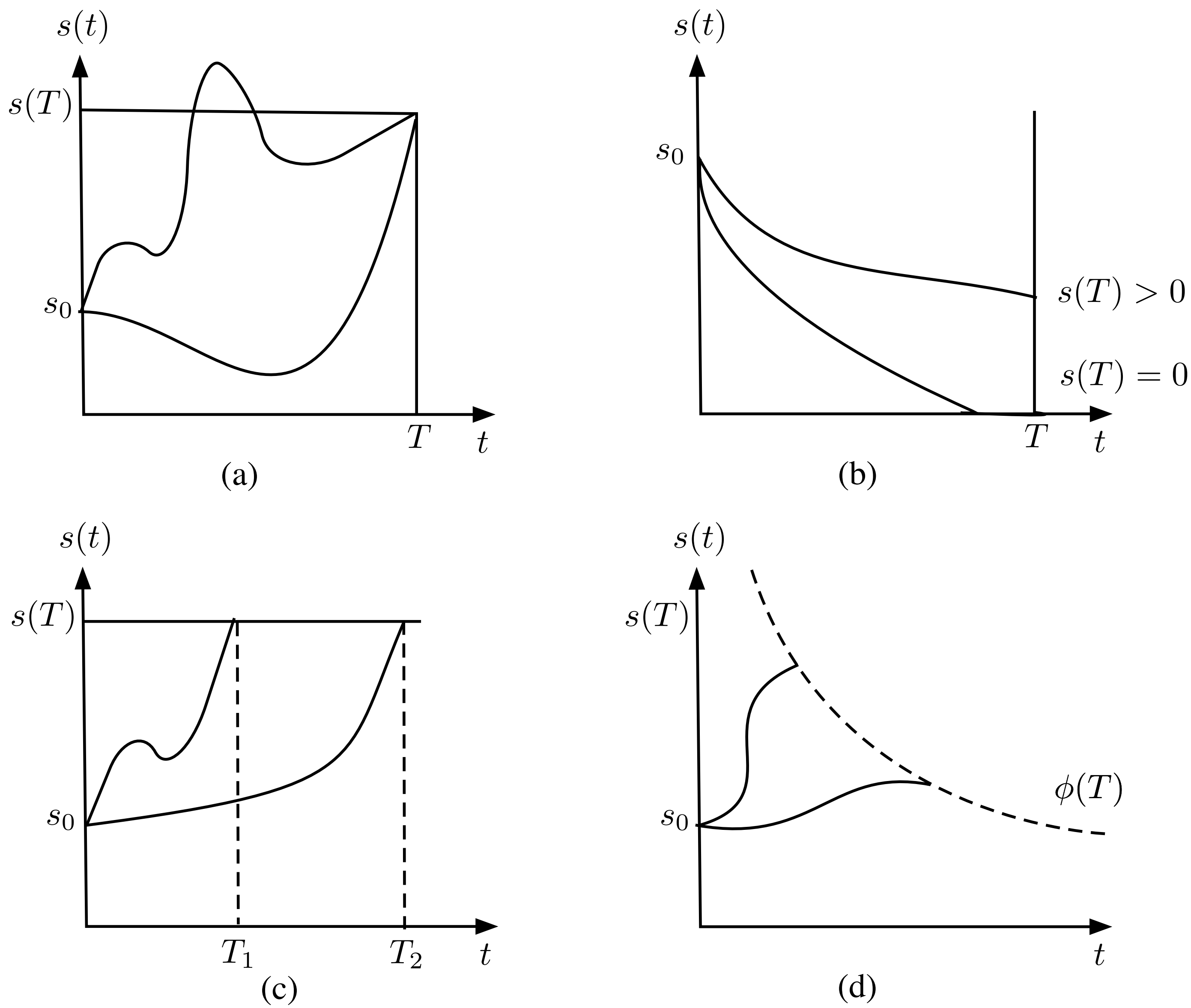
- example: environmental regulator to reach a pollution level w/o time horizon
- Illustration: Section (c) of figure

On terminal conditions (2)

(T, s_T) variable

- example: manager building capital stock to sell firm at a later date. Trade-off: the longer to accumulate capital stock the later will sell: $s_T = \phi(T)$
- Illustration: Section (d) of figure

On terminal conditions (3)



Optimal control theory

- We have examined the Kuhn-Tucker theorem to solve static optimization problems.
- The optimal control theory, applies the same theorem to a dynamic setting.
- Types of set-up:
  - Discrete time
    - Finite horizon
    - Infinite horizon
  - Continuous time
    - Finite horizon
    - Infinite horizon

OCT - discrete time - Illustration

Utility maximization; State equation

- Consumer has wealth in the bank at fixed interest rate $r > 0$.
- Consumer uses wealth $k(t)$ to consume $c(t)$ in period $t, t \in [0, T] \cap \mathbb{N}$
- Suppose borrowing not allowed, $k(t) > 0$.
- Let $k(0) = k_0$ denote the initial wealth
- Let $k(T+1)$ denote wealth left at T
- Equation of motion: $k(t+1) = (1+r)k(t) - c(t)$
- Let $k(t, k_0, c)$ denote the solution of the (difference) equation of motion

OCT - discrete time - Illustration (2)

Utility maximization; Objective function

- Consumer derives satisfaction (utility) from consumption
- Instantaneous utility: $u(c(t)), t \in [0, T] \cap \mathbb{N}$
- Control variables: choose consumption path $\{c_0, c_1, \ldots, c_T\}$ to maximize intertemporal utility function U defined as

$$U(k, c) = \sum_{t=0}^{T} \beta^t u(c(t)) + \beta^T v(k(T+1)), \text{ if finite horizon}$$

$$U(c) = \sum_{t=0}^{\infty} \beta^t u(c(t)), \text{ if infinite horizon}$$

where $\beta \in (0, 1)$, u, v strictly increasing, concave and C^2.

OCT - discrete time - Illustration (3)

Utility maximization; The problem

- Suppose u belongs to the CES family and given by

$$u_\sigma(c) = \frac{c^{1-\sigma}-1}{1-\sigma}, \text{ if } \sigma > 0, \sigma \neq 1$$
$$u_1(c) = \ln c, \text{ if } \sigma = 1$$

- Infinite horizon ($t \in \mathbb{N}$)

$$\max_c U_\sigma(c) = \sum_{t=0}^{\infty} \beta^t u_\sigma(c(t)), \text{ s.t.}$$
$$c(t) \geq 0$$
$$k(t, k_0, c) \geq 0$$
$$k(0) = k_0$$

OCT - discrete time - Illustration (4)

Utility maximization; The problem

- Finite horizon ($t \in [0, T] \cap \mathbb{N}$)

$$\max_c U_\sigma(c) = \sum_{t=0}^{T} \beta^t u_\sigma(c(t)) + \beta^T v(k(t+1)), \text{ s.t.}$$

$$c(t) \geq 0, \ k(k_0, c) \geq 0, \ k(0) = k_0$$

Macroeconomic version

- Consumer representative agent of a community
- equation of motion (re)defined as

$$k(t+1) = F(k(t)) + (1-\delta)k(t) - c(t)$$

where $F(\cdot)$ is the production function, and δ the depretiation factor.

OCT - discrete time - finite horizon

- The problem

$$\max_{(a_0,a_1,\dots,a_{T-1})} \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v(s_T) \text{ s.t.} \tag{1}$$

$$\sum_{t=0}^{T-1} \beta^{t+1}(s_{t+1} - g_t(a_t, s_t)) = 0,\ t = 0, 1, \dots, T-1$$

T, s_0, s_T given

- Lagrangean function:

$$L(a, s, \lambda) = \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v(s_T) - \sum_{t=0}^{T-1} \lambda_{t+1}\beta^{t+1}(s_{t+1} - g_t(a_t, s_t))$$

OCT - discrete time - finite horizon (2)

- FOCs

$$\frac{\partial L}{\partial a_t} = \beta^t \left[\frac{\partial f_t}{\partial a_t} + \beta \lambda_{t+1} \frac{\partial g_t}{\partial a_t}\right] = 0, \ (t = 0, \ldots, T-1)$$

$$\frac{\partial L}{\partial s_t} = \beta^t \left[\frac{\partial f_t}{\partial s_t} + \beta \lambda_{t+1} \frac{\partial g_t}{\partial s_t} - \lambda_t\right] = 0, \ (t = 1, \ldots, T-1)$$

$$\frac{\partial L}{\partial s_T} = \beta^T \left(\frac{\partial v}{\partial s_T} - \lambda_T\right) = 0, \ (t = T)$$

$$\frac{\partial L}{\partial \lambda_t} = \beta^t (s_t - g_{t-1}(a_{t-1}, s_{t-1})) = 0, \ (t = 0, \ldots, T-1)$$

OCT - discrete time - finite horizon (3)

- FOCs

$$\frac{\partial f_t}{\partial a_t} = -\beta\lambda_{t+1}\frac{\partial g_t}{\partial a_t}, \ (t = 0, \dots, T-1) \tag{2}$$

$$\lambda_t = \frac{\partial f_t}{\partial s_t} + \beta\lambda_{t+1}\frac{\partial g_t}{\partial s_t}, \ (t = 1, \dots, T-1) \tag{3}$$

$$\lambda_T = \frac{\partial v}{\partial s_T} \tag{4}$$

$$s_{t+1} = g_t(a_t, s_t) \tag{5}$$

- Remark: Condition (2) is known as the Euler equation
- Under suitable assumptions on f_t, g_t and L the problem has a solution $(a_0^*, a_1^*, \dots, a_{T-1}^*)$

OCT - discrete time - finite horizon - Interpretation

Condition (2)

- A marginal change in a_t two effects
  - on instantaneous return $\frac{\partial f_t}{\partial a_t}$
  - on next period state variable s_{t+1} through $\frac{\partial g_t}{\partial a_t}$ measured by λ_{t+1}
- Thus equation (2) measures the present value of the total impact of a marginal change in a_t
- Hence, agent in choosing a_t optimally foresees present and future consequences.

OCT - discrete time - finite horizon - Interpretation (2)

Condition (3)

- A marginal change in s_t two effects
  - on instantaneous return $\frac{\partial f_t}{\partial s_t}$
  - on next period state variable s_{t+1} through $\frac{\partial g_t}{\partial s_t}$ measured by λ_{t+1}
- Thus the present value of the total impact of a marginal change in s_t is given by the rhs of (3)
- The measure of this impact is given by the lhs of (3), namely λ_t. In other words, λ_t is the shadow price of s_t
- Equivalently, λ_t is the measure of present and future consequences of a marginal change of s_t

OCT - discrete time - finite horizon - Interpretation (3)

Condition (5)

- Condition (5) is the transition equation

Condition (4)

- Condition (4) is the transversality condition
- It says that the shadow price of s_T equals the marginal value of $v(s_T)$

OCT - discrete time - finite horizon - An example

- Profit maximizing firm over a finite horizon T
- Profits in t dependent on labor in t (control variable) and capital stock in t (state variable), $\pi_t(l_t, k_t)$
- Goal: max profits over the time horizon
- restrictions
  - capital accumulation: $k_{t+1} - k_t = g_t(l_t, k_t)$
  - initial capital stock: k_0
  - Terminal capital stock: $k(T) = \overline{K}$
- no discounting ($\beta = 1$)
- Problem:

$$\max_{l_t} \sum_{t=0}^{T} \pi_t(l_t, k_t) \text{ s.t. } k_{t+1} - k_t = g_t(l_t, k_t)$$

An example (2)

- Lagrangean function:

$$L(l,k,\lambda) = \sum_{t=0}^{T} \pi_t(l_t,k_t) - \lambda_t\Big(k_{t+1} - k_t - g_t(l_t,k_t)\Big)$$

- FOCs

$$\frac{\partial L}{\partial l_t} = \frac{\partial \pi_t}{\partial l_t} + \lambda_t \frac{\partial g_t}{\partial l_t} = 0,\ t = 0,1,\ldots,T \tag{6}$$

$$\frac{\partial L}{\partial k_t} = \frac{\partial \pi_t}{\partial k_t} + \lambda_t \frac{\partial g_t}{\partial k_t} + \lambda_t - \lambda_{t-1} = 0,\ t = 0,1,\ldots,T \tag{7}$$

$$\frac{\partial L}{\partial \lambda_t} = k_{t+1} - k_t - g_t(l_t,k_t) = 0,\ t = 0,1,\ldots,T \tag{8}$$

An example (3)

- FOC (6) requires that l_t in each period maximizes the Lagrangian given the stock of capital available in that period
- FOC (8) represents the difference equation of motion governing the accumulation of capital
- To interpret FOC (7)
  - λ_t represents the (marginal) impact on the maximum attainable value of the sum of profits of an additional unit of capital.
  - Thus, $\lambda_t - \lambda_{t-1}$ represents the rate at which capital depreciates in value.
  - Therefore, (7) requires that the depreciation in value of capital = sum of its contributions to profits + its contribution to enhancing the value of the capital stock.

OCT - discrete time - finite horizon - the Hamiltonian

- Alternative way to solve the control problem
- Define the Hamiltonian as

$$H_t(a_t, s_t, \lambda_{t+1}) = f_t(a_t, s_t) + \beta\lambda_{t+1} g_t(a_t, s_t)$$

  measuring the total return in period t.
- The choice of a_t has two effects
  - the contemporary effect $f_t(a_t, s_t)$
  - the impact on the transition of state variable $\beta\lambda_{t+1} g_t(a_t, s_t)$, i.e. the future capacity of generating returns.
- The use of the Hamiltonian allows for transforming a dynamic optimization problem into a sequence of static optimization problems related by the transition equation and by the equation determining the shadow price λ_t.

OCT - discrete time - finite horizon - the Hamiltonian (2)

- The Lagrangean function now becomes

$$L(a,s,\lambda) = H_0(a_0, s_0, \lambda_1) + \sum_{t=1}^{T-1} \beta^t \Big[H_t(a_t, s_t, \lambda_{t+1}) - \lambda_t s_t \Big]$$
$$- \beta^T \lambda_T s(T) + \beta^T v(s(T))$$

- FOCs are:

$$\frac{\partial L}{\partial a_t} = \beta^t \frac{\partial H_t(a_t, s_t, \lambda_{t+1})}{\partial a_t} = 0, \ t = 0, 1, \ldots, T-1 \tag{9}$$
$$\frac{\partial L}{\partial s_t} = \beta^t \Big[\frac{\partial H_t(a_t, s_t, \lambda_{t+1})}{\partial s_t} - \lambda_t \Big] = 0, \ t = 1, \ldots, T-1 \tag{10}$$
$$\frac{\partial L}{\partial s(T)} = \beta^T \Big(-\lambda_T + v'(s(T)) \Big) = 0 \tag{11}$$

and ...

OCT - discrete time - finite horizon - the Hamiltonian (3)

- Naturally, the optimal plan must also satisfy the transition equation
$$s_{t+1} = g_t(a_t, s_t),\ t = 0, 1, \ldots, T-1$$
- FOC (9) characterizes an interior maximum of the Hamiltonian along the optimal path.
- The optimal solution stemming from the Hamiltonian is known as the Maximum principle.
- The Maximum principle prescribes that along the optimal path, a_t should be chosen to maximize the total benefits in each period.
- Of course the solution $\{a_t\}_{t=0}^{T-1}$ is the same as previously, but (in general) simpler to obtain, and with a clear economic interpretation.

Economic interpretation of the Hamiltonian

- Consider the previous example of the profit maximizing firm.
- Define the Hamiltonian as

$$H_t(l_t, k_t) = \pi_t(l_t, k_t) + \lambda_{t+1} g_t(l_t, k_t)$$

  This is the value of profits in t + the amount of capital accumulated in t, $(g_t)\times$ the marginal value of capital at time $t+1$, (λ_{t+1}).
- In other words, $\lambda_{t+1} g_t$ captures the future profit effect of l_t through the change in the capital stock. Thus, the Hamiltonian accounts for the effects on current and future profits of l_t.

Economic interpretation of the Hamiltonian (2)

- Now the Lagrangean function becomes

$$L(l,k,\lambda) = \sum_{t=0}^{T} \Big[H_t(l_t,k_t) - \lambda_t(k_{t+1}-k_t) \Big]$$

- FOCs

$$\frac{\partial L}{\partial l_t} = \frac{\partial H_t}{\partial l_t} = 0,\ t = 0,1,\ldots,T \tag{12}$$

$$\frac{\partial L}{\partial k_t} = \frac{\partial H_t}{\partial k_t} + \lambda_t - \lambda_{t-1} = 0,\ t = 1,\ldots,T \tag{13}$$

$$\frac{\partial L}{\partial \lambda_t} = \frac{\partial H_t}{\partial \lambda_t} - (k_{t+1}-k_t) = 0,\ t = 0,1,\ldots,T \tag{14}$$

Economic interpretation of the Hamiltonian (3)

- FOC (12) can be rewritten as

$$\frac{\partial H_t}{\partial l_t} = \frac{\partial \pi_t}{\partial l_t} + \lambda_t \frac{\partial g_t}{\partial l_t} = 0$$

$$\text{or } \frac{\partial \pi_t}{\partial l_t} = -\lambda_t \frac{\partial g_t}{\partial l_t} \qquad (15)$$

  - This is the same condition as (6).
  - It means that at each point in time the firm chooses l_t to balance the marginal increase in current profits against the marginal decrease in future profits through the change in the capital stock.

Economic interpretation of the Hamiltonian (4)

- FOC (13) can be rewritten as

$$-(\lambda_t - \lambda_{t-1}) = \frac{\partial H_t}{\partial k_t} = \frac{\partial \pi_t}{\partial k_t} + \lambda_t \frac{\partial g_t}{\partial k_t} \tag{16}$$

  Same as (7). It means that the increase in capital decreases the value of the capital stock (it is marginally less scarce), while $\frac{\partial \pi_t}{\partial k_t} + \lambda_t \frac{\partial g_t}{\partial k_t}$ represents the increase in current and future profits.

- Equations (15), (16) and the equation of motion $k_{t+1} - k_t = g_t(l_t, k_t)$ (exogenously given), constitute the maximum principle

- The maximum principle prescribes that along the optimal path, l_t is chosen to maximize total profits in each period.

Exercise

- Profit maximizing monopolist extracting mineral, with a license expiring on date T
- Notation
  - x_0: initial stock of mineral
  - x_t: size of the deposit at beginning of period t
  - q_t: volume extracted during period t
  - β: discount rate
  - $p_t(q_t)$: inverse demand function
  - c: constant marginal cost of extraction
- (a) Define the transition equation.
- (b) Define the instantaneous profit function.
- (c) Define the monopolist's profit maximization problem.
- (d) Find the conditions characterizing the (interior) optimal extraction path.

Exercise (2)

- (a) Transition equation: $x_{t+1} = x_t - q_t$
- (b) Instantaneous profit function: $\pi_t(q_t) = [p_t(q_t) - c]q_t$
- (c) Monopolist's problem:

$$\max_{\{q_t \geq 0\}} \sum_{t=0}^{T-1} \beta^t \pi_t(q_t) \quad \text{s.t.}$$

$$x_{t+1} = x_t - q_t \geq 0$$

$$x(0) = x_0 > 0$$

$$x(T) = x_T \geq 0$$

Exercise (3)

- (d) Lagrangian function

$$L(x,q,\lambda) = \sum_{t=0}^{T-1} \beta^t \pi_t(q_t) + \beta^T x_T - \sum_{t=0}^{T-1} \lambda_{t+1}\beta^{t+1}(x_{t+1} - x_t + q_t) =$$

$$= \sum_{t=0}^{T-1} \beta^t \pi_t(q_t) + \sum_{t=0}^{T-1} \lambda_{t+1}\beta^{t+1}(x_t - q_t) - \sum_{t=0}^{T-1} \lambda_{t+1}\beta^{t+1} x_{t+1} + \beta^T x_T =$$

$$= \pi_0(q_0) + \lambda_1 \beta (x_0 - q_0) + \sum_{t=1}^{T-1} \beta^t \pi_t(q_t) + \sum_{t=1}^{T-1} \lambda_{t+1}\beta^{t+1}(x_t - q_t) -$$

$$\sum_{t=1}^{T} \lambda_t \beta^t x_t + \beta^T x_T =$$

Exercise (4)

- (d) Lagrangian function (cont'd)

$$= \pi_0(q_0) + \lambda_1 \beta(x_0 - q_0) + \sum_{t=1}^{T-1} \beta^t \Big[\pi_t(q_t) + \lambda_{t+1}\beta(x_t - q_t) - \lambda_t x_t \Big] -$$

$$\lambda_T \beta^T x_T + \beta^T x_T =$$

$$= \pi_0(q_0) + \lambda_1 \beta(x_0 - q_0) + \sum_{t=1}^{T-1} \beta^t \Big[\pi_t(q_t) + \lambda_{t+1}\beta(x_t - q_t) - \lambda_t x_t \Big] -$$

$$\beta^T x_T (1 - \lambda_T)$$

Exercise (5)

- FOCs

$$\frac{\partial L}{\partial q_t} = \beta^t \left[\frac{\partial \pi_t(q_t)}{\partial q_t} - \beta \lambda_{t+1} \right] = 0$$

$$\frac{\partial L}{\partial x_t} = \beta^t \left[\beta \lambda_{t+1} - \lambda_t \right] = 0$$

$$\frac{\partial L}{\partial x_T} = \beta^T (1 - \lambda_t) = 0$$

$$x_{t+1} = x_t - q_t$$

Exercise (6)

- Characterizing an (interior) optimal extraction path

$$\frac{\partial \pi_t(q_t)}{\partial q_t} = \beta\lambda_{t+1} \tag{17}$$

$$\beta\lambda_{t+1} = \lambda_t \tag{18}$$

$$1 = \lambda_t \tag{19}$$

$$x_{t+1} = x_t - q_t \tag{20}$$

- Note that $\frac{\partial \pi_t(q_t)}{\partial q_t} = p_t'(q_t)q_t + p_t(q_t) - c$
- Denote marginal revenue in period t as $m_t(q_t) \equiv p_t'(q_t)q_t + p_t(q_t)$
- Rewrite (17) as

$$m_t(q_t) = c + \beta\lambda_{t+1} \tag{21}$$

Exercise (7)

- Interpretation of (21)
  - At every t marginal revenue = marginal cost
  - marginal cost = marginal cost of extraction + opportunity cost of the reduction in stock available for next period
  - opportunity cost is measured by the shadow price of the remaining resource (λ_{t+1}) discounted to the current period (t)
- Optimal extraction plan
  - Rewrite (21) as

$$m_t(q_t) - c = \beta\lambda_{t+1} \tag{22}$$

  - Evaluate (22) at $t+1$:

$$m_{t+1}(q_{t+1}) - c = \beta\lambda_{t+2}, \text{ or}$$

$$\beta(m_{t+1}(q_{t+1}) - c) = \beta^2\lambda_{t+2} \tag{23}$$

Exercise (8)

- Optimal extraction plan (cont'd)
  - Evaluate (18) at $t+1$ and multiply by β to obtain

$$\beta\lambda_{t+1} = \beta^2\lambda_{t+2} \tag{24}$$

  - Substituting (24) into (23) we obtain

$$\beta(m_{t+1}(q_{t+1}) - c) = \beta\lambda_{t+1} \tag{25}$$

  - Substituting (22) into (25) gives

$$\beta(m_{t+1}(q_{t+1}) - c) = m_t(q_t) - c$$

  The opportunity cost of selling an additional unit in $t+1$ discounted to period t = net profit of selling an additional unit in period t

- Hence, the optimal extraction path is organized such that there are no profitable opportunities left to reallocate extraction between any two adjacent periods.

Exercise (9)

Characterizing the solution applying the Maximum Principle

- Define the Hamiltonian as $H_t(q, x, \lambda) = \pi_t(q_t) + \beta\lambda_{t+1}(x_t - q_t)$
- Lagrangian function becomes

$$L = \pi_0(q_0)+\lambda_1\beta(x_0-q_0)+\sum_{t=1}^{T-1}\beta^t\Big[H_t(q,x,\lambda)-\lambda_t x_t\Big]+\beta^T x_T(1-\lambda_T)$$

- FOCs

$$\frac{\partial L}{\partial q_t} = \frac{\partial H_t}{\partial q_t} = 0 \Longrightarrow \frac{\partial \pi_t(q_t)}{\partial q_t} = \beta\lambda_{t+1}$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial H_t}{\partial x_t} - \lambda_t = 0 \Longrightarrow \beta\lambda_{t+1} = \lambda_t$$

that are precisely conditions (17) and (18). Optimality also requires satisfying the transition equation (20) and the terminal condition (19).

OCT - discrete time - infinite horizon

- If a problem does not have a terminal date $\rightarrow$ model as an ∞-horizon problem.
- Problem is

$$\max_{a_t \in A_t} \sum_{t=0}^{\infty} \beta^t f_t(a_t, s_t) \text{ s.t.}$$

$$s_{t+1} = g_t(a_t, s_t), \; t = 0, 1, 2, \ldots$$

$$s_0 \text{ given}$$

- To ensure that the total discounted value of the objective function is finite, suppose
  - f_t is bounded $\forall t$
  - $\beta < 1$

OCT - discrete time - infinite horizon - Illustration

- Consider an individual's life-time consumption problem
  - c_t: consumption in period t;
  - $u(c_t)$: concave utility function;
  - $\beta \in (0,1)$: discount rate;
  - $U(\{c_t\}_0^\infty)$: life-time utility function;
  - A_0: initial wealth (no other income source)
  - r: interest rate
- Life-time utility function:

$$U(\{c_t\}_0^\infty) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Budget constraint: $A_0 = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$

Illustration (2)

- Problem to solve:

$$\max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } A_0 = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$$

- Lagrangean function:

$$L(c,\lambda) = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda\Big(A_0 - \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}\Big)$$

- BUT $L(c,\lambda)$ has infinite FOCs.
- How to cope with it?

Illustration (3)

- Compute FOCs for two consecutive arbitrary periods t and $t+1$:

$$\frac{\partial L}{\partial c_t} = \beta^t \frac{\partial u}{\partial c_t} - \lambda \frac{1}{(1+r)^t} = 0, \tag{26}$$

$$\frac{\partial L}{\partial c_{t+1}} = \beta^{t+1} \frac{\partial u}{\partial c_{t+1}} - \lambda \frac{1}{(1+r)^{t+1}} = 0 \tag{27}$$

- Divide (26) by (27) to obtain:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r) \tag{28}$$

This is the equilibrium condition describing the optimal consumption path.

Illustration (4)

- Equilibrium condition (28) says that the MRS of two consecutive periods = marginal value in t of income in $t+1$, i.e.
- The individual is willing to delay (a unit of) consumption for one period until the point where the value in t of the increased income (not spent in t) compensates.
- To give a feeling of the content of (28), suppose

$$u(c_t) = \ln(c_t)$$

- Then, (28) becomes $\frac{c_{t+1}}{c_t} = \beta(1+r)$, or

$$c_{t+1} = \beta(1+r)c_t \tag{29}$$

this is a linear difference equation.

Illustration (5)

- the solution of (29) is

$$c_t = c_0[(1+r)\beta]^t \tag{30}$$

- to pin down the value of c_0, substitute (30) in the budget constraint:

$$A_0 = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_0[(1+r)\beta]^t}{(1+r)^t} = \sum_{t=0}^{\infty} c_0\beta^t = \frac{c_0}{1-\beta}$$

- That is $c_0 = A_0(1-\beta)$
- and the optimal consumption path is

$$c_t = A_0(1-\beta)[(1+r)\beta]^t$$

On the meaning of the discount rate

Discrete time

- Discount rate β: present value of 1€ invested at (annual) interest rate r.
- In other words, to obtain 1€ when the interest rate is r the amount to be invested is

$$\beta(1+r) = 1, \quad \text{or} \quad \beta = \frac{1}{1+r}$$

- If the interest is accumulated n times during the year (daily, monthly, quarterly), the investment to obtain 1€ is

$$\beta\left(1+\frac{r}{n}\right)^n = 1, \quad \text{or} \quad \beta = \frac{1}{\left(1+\frac{r}{n}\right)^n}$$

On the meaning of the discount rate (2)

Continuous time

- To compute the discount rate β in continuous time, note that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

- Then, the present value of 1€ with continuous compounding over one period is

$$\beta = e^{-r}$$

- and the present value of 1€ with continuous compounding over t periods is

$$\beta = e^{-rt}$$

OCT - continuous time - finite horizon

- The general dynamic optimization problem in continuous time is

$$\max_{a(t)} \int_0^T e^{-rt} f(a(t), s(t), t)dt + e^{-rT} v(s(T)) \text{ s.t. } \dot{s} = g(a(t), s(t), t)$$

- with $s(0) = s_0$ given.
- Comparing with discrete time
  - replace $\sum$ by $\int$
  - replace difference equation describing the equation of motion by a differential equation describing the transition equation.
  - replace multipliers $(\lambda_1, \ldots, \lambda_T)$ by a functional $\lambda(t)$ on $[0, T]$.
  - To define the Lagrangean, first multiply constraint by e^{-rt}.

OCT - continuous time - finite horizon (2)

$$L = \int_0^T e^{-rt} f(a(t), s(t), t)dt + e^{-rT} v(s(T))$$

$$- \int_0^T e^{-rt} \lambda(t)(\dot{s} - g(a(t), s(t), t))dt =$$

$$= \int_0^T e^{-rt} \Big(f(a(t), s(t), t) + \lambda(t) g(a(t), s(t), t) \Big) dt$$

$$- \int_0^T e^{-rt} \lambda(t) \dot{s} dt + e^{-rT} v(s(T)) =$$

$$= \int_0^T e^{-rt} H(a(t), s(t), \lambda(t), t)dt - \int_0^T e^{-rt} \lambda(t) \dot{s} dt + e^{-rT} v(s(T))$$

where $H(a(t), s(t), \lambda(t), t) = f(a(t), s(t), t) + \lambda(t) g(a(t), s(t), t)$ is the Hamiltonian.

OCT - continuous time - finite horizon (3)

- Suppose $\lambda(t)$ is differentiable.
- Integrate $\int_0^T e^{-rt}\lambda(t)\dot{s}dt$ by parts to obtain

$$\int_0^T e^{-rt}\lambda(t)\dot{s}dt = e^{-rT}\lambda(T)s(T) - \lambda(0)s(0) -$$

$$-\int_0^T e^{-rt}s(t)\dot{\lambda}dt + r\int_0 Te^{-rt}s(t)\lambda(t)dt$$

- Substituting in the Lagrangean

$$L = \int_0^T e^{-rt}\Big(H(a(t), s(t), \lambda(t), t) + s(t)\dot{\lambda} - rs(t)\lambda(t)\Big)dt +$$

$$+ e^{-rT}v(s(T)) - e^{rT}\lambda(T)s(T) + \lambda(0)s(0)$$

OCT - continuous time - finite horizon (4)

- FOCs - Maximum Principle

$$\frac{\partial L}{\partial a(t)} = e^{-rt}\frac{\partial H(a(t), s(t), \lambda(t), t)}{\partial a(t)} = 0$$

$$\frac{\partial L}{\partial s(t)} = e^{-rt}\Big(\frac{\partial H(a(t), s(t), \lambda(t), t)}{\partial s(t)} + \dot{\lambda} - r\lambda(t)\Big) = 0$$

$$\frac{\partial L}{\partial s(T)} = e^{-rt}\Big(v'(s(T)) - \lambda(T)\Big) = 0$$

- together with $\dot{s} = \frac{\partial H(a(t),s(t),\lambda(t),t)}{\partial \lambda(t)} = g(a(t), s(t), t)$
- and where $e^{-rt} > 0$
- Maximum principle requires the Hamiltonian being maximized along the optimal path.

OCT - continuous time - finite horizon - Illustration

- Back with the girl and the cake.
- Initial size of the cake: $k(0)$
- Cake to be consumed over a continuous interval $[0, T]$
- Comsumption of cake generate utility $u[c(t)]$
- Size of cake evolves according to $\dot{k} = -c(t)$
- Terminal condition is $k(T) = \overline{k} > 0$
- value of consumtion:

$$\int_0^T u[c(t)]dt$$

- Assume away discounting.

Illustration (2)

- Solving the problem
- Construct the Hamiltonian

$$H = u[c(t)] - \lambda(t)c(t)$$

- FOCs

$$\begin{aligned}
\frac{\partial H}{\partial c(t)} =& u'[c(t)] - \lambda(t) = 0 \\
\frac{\partial H}{\partial k(t)} =& -\dot{\lambda}(t) = 0 \\
\frac{\partial H}{\partial \lambda(t)} =& -c(t) = \dot{k}(t)
\end{aligned}$$

and the transversality condition $\lambda(T)[k(t) - \overline{k}] = 0$.

OCT - continuous time - infinite horizon

- The problem is now

$$\max_{a(t)} \int_0^\infty e^{-rt} f(a(t), s(t), t) dt \text{ s.t. } \dot{s} = g(a(t), s(t), t)$$

- with $s(0) = s_0$ given.
- The solution is characterized by the FOCs

$$\frac{\partial H(a(t), s(t), \lambda(t), t)}{\partial a(t)} = 0$$

$$\dot{\lambda} = r\lambda(t) - \frac{\partial H(a(t), s(t), \lambda(t), t)}{\partial s(t)}$$

$$\dot{s} = \frac{\partial H(a(t), s(t), \lambda(t), t)}{\partial \lambda(t)} = g(a(t), s(t), t)$$

OCT - continuous time - infinite horizon - Illustration

- Let's recover the illustration used in the discrete time case.
- Now with continuous time, the life-time utility functions is

$$U(\{c_t\}_0^\infty) = \int_{t=0}^{\infty} e^{-\beta t} u(c(t))$$

- Budget constraint:

$$\dot{A}(t) = rA(t) - c(t)$$

and $A(0) = A_0$

Illustration (2)

- The problem:

$$\max_{\{c_t\}_0^\infty} U(\{c_t\}_0^\infty) \text{ s.t. } \dot{A}(t) = rA(t) - c(t),\ \ A(0) = A_0$$

- to solve the problem, use the Hamiltonian. Two alternatives
  - Current value Hamiltonian: $H^c = u(c_t) + \lambda(t)\dot{A}$
  - Present value Hamiltonian: $H^p = e^{-\beta t} H^c = e^{-\beta t} u(c_t) + \mu(t) A$
- FOCs

H^c	H^p
$\frac{\partial H^c}{\partial c} = 0$	$\frac{\partial H^p}{\partial c} = 0$
$\frac{\partial H^c}{\partial \lambda} = \dot{A}$	$\frac{\partial H^p}{\partial \mu} = \dot{A}$
$\frac{\partial H^c}{\partial A} = \beta\lambda - \dot{\lambda}$	$\frac{\partial H^p}{\partial A} = -\dot{\mu}$

Illustration (3)

- Let $u(c_t) = \ln(c_t)$
- Then

$$H^c = \ln(c_t) + \lambda_t(rA - c)$$

- FOCs

$$\frac{\partial H}{\partial c} = \frac{1}{c_t} - \lambda_t = 0 \tag{31}$$

$$\frac{\partial H}{\partial \lambda} = rA - c_t = \dot{A} \tag{32}$$

$$\frac{\partial H}{\partial A} = \lambda_t r = \beta\lambda_t - \dot{\lambda} \tag{33}$$

Illustration (4)

- From (48) we obtain $\frac{1}{c_t} = \lambda_t$
- taking logs $ln(\frac{1}{c_t}) = \ln(\lambda_t)$ or $-\ln(c_t) = \ln(\lambda_t)$
- differentiating wrt t

$$\frac{\partial \ln(c_t)}{\partial t} = -\frac{\partial \ln(\lambda_t)}{\partial t}$$
$$\frac{\partial \ln(c_t)}{\partial c_t}\frac{\partial c_t}{\partial t} = -\frac{\partial \ln(\lambda_t)}{\partial \lambda_t}\frac{\partial \lambda_t}{\partial t}$$
$$\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda} \tag{34}$$

- From (50) we obtain

$$\beta - r = \frac{\dot{\lambda}}{\lambda} \tag{35}$$

Illustration (5)

- From (52) and (53) we obtain the differential equation

$$\frac{\dot{c}}{c_t} = r - \beta$$

- that has as solution

$$c_t = c_0 e^{(r-\beta)t}$$

- To find the initial condition for c_0 we use the budget constraint. In particular, the present discounted value of consumption must equal the initial assets, i.e.

$$A_0 = \int_0^\infty e^{-rt} c(t) dt = \int_0^\infty e^{-rt} c_0 e^{(r-\beta)t} dt = c_0 \int_0^\infty e^{(-\beta)t} dt = c_0 \frac{1}{\beta}$$

  so that $c_0 = \beta A_0$

- Therefore, the general solution is

$$c_t = \beta A_0 e^{(r-\beta)t}$$

OCT - optimal economic growth model

- Let $U(c(t))$ denote the instantaneous utility function dependent on aggregate consumption per capita $c(t)$.
- $U(c(t))$ is assumed differentiable, $U' > 0, U'' < 0$
- $U(c(t))$ is thus a measure of the social utility at a point in time
- Thus a social welfare index obtains from integrating $U(\cdot)$ over the time horizon.
- $Y = F(k(t))$ represent total output obtained through a production function F expressed in terms of capital per capita.
- Total output is allocated between aggregate consumption C and investment $I = \dot{K}$.
- Thus $F(k(t)) = c(t) + \dot{k}$
- The value of the capital per capita k is given to the economy at the initial date, namely $k(0) = k_0$.

Optimal growth (2)

- The problem: find a per capita consumption function $c(t)$ that solves the infinite horizon optimal control problem

$$\max_{c(t)} \int_0^\infty e^{-rt} U(c(t))dt \quad \text{s.t.} \quad \dot{k} = F(k(t)) - c(t)$$

with k_0 given.

- Langrangean function

$$L = \int_0^\infty e^{-rt} U(c(t))dt - \int_0^\infty e^{-rt} \lambda(t) \Big(\dot{k} - [F(k(t)) - c(t)] \Big) dt =$$

$$= \int_0^\infty e^{-rt} \Big[U(c(t)) + \lambda(t)[F(k(t)) - c(t)] \Big) \Big] dt - \int_0^\infty e^{-rt} \lambda(t) \dot{k} dt =$$

$$= \int_0^\infty e^{-rt} H\Big(c(t), k(t), \lambda(t) \Big) - \int_0^\infty e^{-rt} \lambda(t) \dot{k} dt$$

Optimal growth (3)

- where $H(\cdot)$ denotes the Hamiltonian.
- integrating by parts

$$\int_0^\infty e^{-rt}\lambda(t)\dot{k} = -\int_0^\infty e^{-rt}k(t)\dot{\lambda}dt + r\int_0^\infty e^{-rt}k(t)\lambda(t)dt$$

- FOCs

$$\begin{aligned}
\frac{\partial L}{\partial c} &= e^{-rt}\Big(\frac{\partial H}{\partial c}\Big) = 0 \\
\frac{\partial L}{\partial k} &= e^{-rt}\Big(\frac{\partial H}{\partial k} + \dot{\lambda} - r\lambda(t)\Big) = 0 \\
\dot{k} &= F(k(t)) - c(t)
\end{aligned}$$

Optimal growth (4)

- FOCs (cont'd)

$$\frac{\partial H}{\partial c} = 0 \tag{36}$$

$$\frac{\partial H}{\partial k} + \dot{\lambda} - r\lambda(t) = 0 \tag{37}$$

$$\dot{k} = F(k(t)) - c(t) \tag{38}$$

- where

$$\frac{\partial H}{\partial c} = U'(c(t)) - \lambda(t) \tag{39}$$

$$\frac{\partial H}{\partial k} = \lambda(t)F'(k(t)) \tag{40}$$

Optimal growth (5)

- Substituting (39) and (40) into (36)-(38) yields

$$U'(c(t)) - \lambda(t) = 0 \tag{41}$$

$$\lambda(t)F'(k(t)) + \dot{\lambda} - r\lambda(t) = 0 \tag{42}$$

$$\dot{k} = F(k(t)) - c(t) \tag{43}$$

- From (41)

$$U'(c(t)) = \lambda(t) \tag{44}$$

- Differentiating (44) with respect to t we obtain

$$\dot{\lambda} = U''(c(t))\dot{c} \tag{45}$$

Optimal growth (6)

- Substituting (44) and (45) into (42) we obtain

$$U''(c(t))\dot{c} = -U'(c(t))\Big(F'(k(t)) - r\Big)$$

or

$$\dot{c} = -\frac{U'(c(t))}{U''(c(t))}\Big(F'(k(t)) - r\Big) \tag{46}$$

This is the Euler equation.

- Observe that sign $\dot{c} =$ sign $(F' - r)$.
- The dynamics of the model are described by (43) and (46):

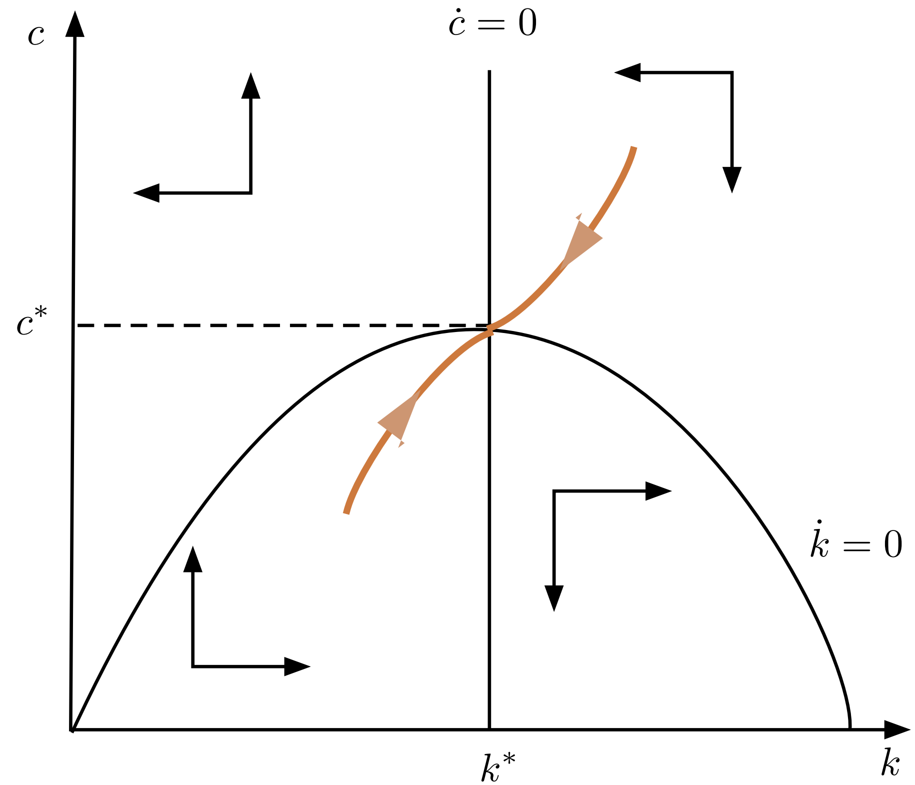
$$\dot{k} = F(k(t)) - c(t)$$

$$\dot{c} = -\frac{U'(c(t))}{U''(c(t))}\Big(F'(k(t)) - r\Big)$$

Optimal growth - phase diagrams

- Solution of system (43) and (46) is a pair of functions $c(t), k(t)$
- Represent them in the space $(k, c) \rightarrow$ phase diagram
- A steady state, (c^*, k^*), requires both $\dot{c} = 0$ and $\dot{k} = 0$.
- Each condition partitions the space (k, c) in regions where $\dot{c} > 0$ and $\dot{c} < 0$; $\dot{k} > 0$ and $\dot{k} < 0$ respectively
- Suppose $\dot{c}$ and $\dot{k}$ together partition the space $(c.k)$ into four regions as shown in the figure
- Suppose under (above) $\dot{k} = 0$ the flow of k is increasing (decreasing)
- Suppose to the left (right) of $\dot{c} = 0$ the flow of k is increasing (decreasing)
- Then a unique path passes through every point in (c, k)

Phase diagrams (2)



Phase diagrams (3)

- From the direction of the flows in each region, only if the initial condition k_0 lies either in the bottom-left or top-right region, the path will converge towards the stationary equilibrium.
- We conclude that for each initial condition k_0 in either bottom-left or top-right region ,there is a unique optimal trajectory towards the steady state.
- Any other alternative initial condition will give rise to a path leading away from the steady state.

Dynamic programming - Introduction

- Alternative approach to dynamic optimization
- Suitable to incorporate uncertainty
- Main instrument:
  Bellman's principle of optimality
- Fundamental idea:
  the optimal path for the control variable will be the same whether we solve the problem over the entire time horizon or for future periods as a function of the initial conditions given by past optimal solutions

Dynamic programming

- The problem

$$\max_{a_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v_T(s_T) \text{ s.t.} \tag{47}$$

$$\beta^{t+1}(s_{t+1} - g_t(a_t, s_t)) = 0, \; t = 0, 1, \ldots, T-1$$

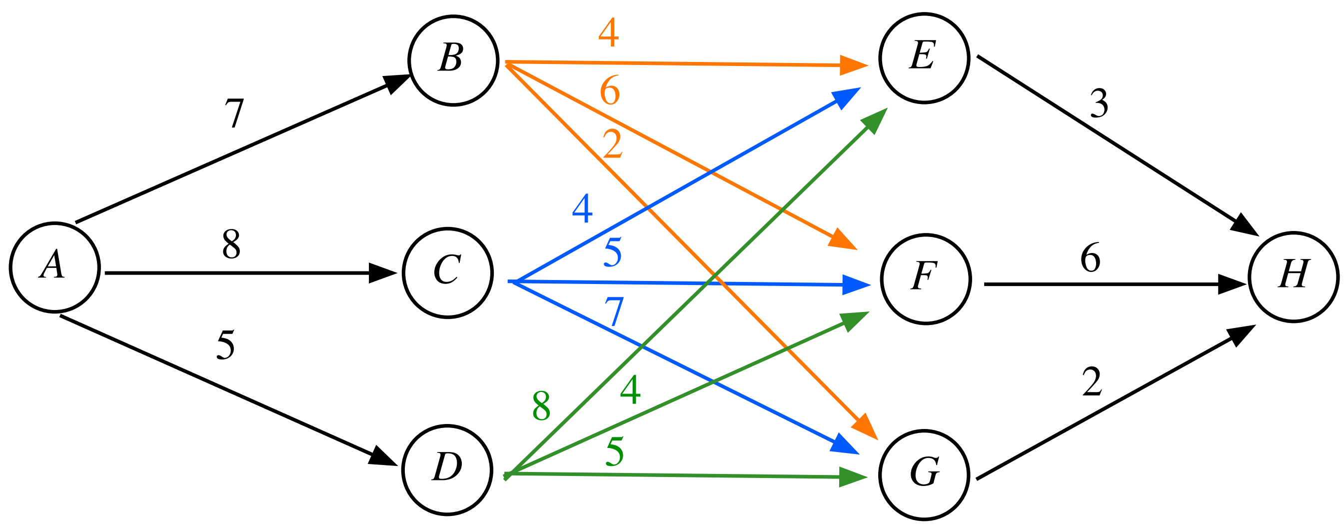
$$T, s_0, s_T \text{ given}$$

- Main feature of this approach: allows for solving the problem by backward induction.
- Particularly convenient in computational terms.
- Main elements of dynamic programming approach: value function and Bellman's equation.

An intuitive approach to Bellman's equation

The stagecoach problem

- Planning a trip from city A to city H minimizing distance.
- Figure shows the road network and distances (in km x 100)



- Three stage planning trip
  - Stage 1: travel from A to either B, C or D
  - Stage 2: travel from B, C or D to E, F or G
  - Stage 3: travel from E, F or G to H

An intuitive approach to Bellman's equation (2)

- Let $M(X)$ denote min distance from city X to city H
- Stage 3: $M(E) = 3, M(F) = 6, M(G) = 2$ (no decision).
- Stage 2: traveler may be in either B, C or D
  - If in $B, M(B) = \min\{4 + M(E), 6 + M(F), 7 + M(G)\} = \min\{7, 12, 4\} = 4$
  - If in $C, M(C) = \min\{4 + M(E), 5 + M(F), 7 + M(G)\} = \min\{7, 11, 9\} = 7$
  - If in $D, M(D) = \min\{8 + M(E), 4 + M(F), 5 + M(G)\} = \min\{11, 10, 7\} = 7$
- Stage 1: traveler may go to either B, C or D. Then,
  - $M(A) = \min\{7 + M(B), 8 + M(C), 5 + M(D)\} = \min\{11, 15, 12\} = 11$
- Hence, distance from A to H is minimized going through B and G. Distance is 11.

An intuitive approach to Bellman's equation (3)

- The backward induction reasoning is captured by Bellman's Principle of Optimality.
- It asserts that "from any point on an optimal path, the remaining trajectory is optimal for the corresponding problem initiated at that point".
- Remark: A myopic individual optimizing stage-by-stage, would have chosen to go from A to D. This is not in the optimal trajectory from A to H.

Dynamic programming (2)

The value function

- At time $t = 0$ the (maximum) value function for (47) is

$$v_0(s_0) = \max_{a_t} \left\{ \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v_T(s_T) \middle| \right.$$
$$\left. s_{t+1} = g_t(a_t, s_t), t = 0, 1, \ldots, T-1 \right\}$$

- Similarly, the value function at time t is

$$v_t(s_t) = \max_{a_\tau} \left\{ \sum_{\tau=t}^{T-1} \beta^{\tau-t} f_\tau(a_\tau, s_\tau) + \beta^T v_T(s_T) \middle| \right.$$
$$\left. s_{\tau+1} = g_\tau(a_\tau, s_\tau), \tau = t, \ldots, T-1 \right\} \tag{48}$$

Dynamic programming (3)

Bellman's equation

- the value function measures the best that can be obtained given the current state and the remaining time.
- Clearly, we can relate v_t and v_{t+1} as

$$
\begin{aligned}
v_t(s_t) &= \max_{a_t}\{f_t(a_t, s_t) + \beta v_{t+1}(s_{t+1}) | s_{t+1} = g_t(a_t, s_t)\} \\
&= \max_{a_t}\{f_t(a_t, s_t) + \beta v_{t+1}(g_t(a_t, s_t))\} \qquad (49)
\end{aligned}
$$

- This is Bellman's equation. It shows a recursive relation between today's value f_t and all future values $\beta v_{t+1}(\cdot)$
- The solution of Bellman's equation determines the optimal policy: *"An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."* (Bellman, 1957)

Dynamic programming (4)

The principle of optimality

- This property is known as the Principle of Optimality and guarantees the intertemporal consistency of the optimal policy.
- Formally, we are looking at the FOC of Bellman's equation.
  - The FOC maximizing Bellman's equation is

$$\frac{\partial f_t}{\partial a_t} + \beta v'_{t+1}(s_{t+1})\frac{\partial g_t}{\partial a_t} = 0, \; t = 0, \ldots, T-1$$

  - Let $\lambda_{t+1} = v'_{t+1}(s_{t+1})$. Then, we can rewrite the FOC as

$$\frac{\partial f_t}{\partial a_t} + \beta \lambda_{t+1}\frac{\partial g_t}{\partial a_t} = 0, \; t = 0, \ldots, T-1 \tag{50}$$

- Note that (50) is precisely the Euler equation (2) in the Lagrangean approach

Dynamic programming (5)

Equivalence with the Lagrangean approach

- To see why $\lambda_t = v'(s_t)$ suppose $a_t = h_t(s_t)$ defines the policy function. Then, we can rewrite (49) as

$$v_t(s_t) = f_t(h_t(s_t), s_t) + \beta v_{t+1}(g_t(h_t(s_t), s_t)))$$

Next, compute v_t':

$$\begin{aligned} v_t'(s_t) &= \frac{\partial f_t}{\partial s_t} + \frac{\partial f_t}{\partial a_t}\frac{\partial h_t}{\partial s_t} + \beta v_{t+1}'\left(\frac{\partial g_t}{\partial s_t} + \frac{\partial g_t}{\partial a_t}\frac{\partial h_t}{\partial s_t}\right) \\ &= \frac{\partial f_t}{\partial s_t} + \frac{\partial f_t}{\partial a_t}\frac{\partial h_t}{\partial s_t} + \beta v_{t+1}'\frac{\partial g_t}{\partial s_t} + \beta v_{t+1}'\frac{\partial g_t}{\partial a_t}\frac{\partial h_t}{\partial s_t} \\ &= \frac{\partial f_t}{\partial s_t} + \beta v_{t+1}'\frac{\partial g_t}{\partial s_t} + \frac{\partial h_t}{\partial s_t}\left(\frac{\partial f_t}{\partial a_t} + \beta v_{t+1}'\frac{\partial g_t}{\partial x s_t}\right) \end{aligned} \tag{51}$$

Dynamic programming (6)

Equivalence with the Lagrangean approach (cont'd)

- Using $\lambda_{t+1} = v'_{t+1}(s_{t+1})$, (50) can be written as

$$\lambda_t = \frac{\partial f_t}{\partial s_t} + \beta\lambda_{t+1}\frac{\partial g_t}{\partial s_t} + \frac{\partial h_t}{\partial s_t}\left(\frac{\partial f_t}{\partial a_t} + \beta\lambda_{t+1}\frac{\partial g_t}{\partial x s_t}\right) \tag{52}$$

- Substituting (50) in (52), it simplifies to

$$\lambda_t = \frac{\partial f_t}{\partial s_t} + \beta\lambda_{t+1}\frac{\partial g_t}{\partial s_t}, \; t = 1, 2, \ldots, T-1 \tag{53}$$

  which is precisely FOC (3).

- Finally, (50), (53) plus the transition equation and terminal condition constitute the equivalent system of FOCs as in the Lagrangean approach characterizing the optimal policy.

The Bellman equation - Illustration

An example to “construct” the Bellman equation

- Consider the discrete-time finite horizon optimization problem:

$$\max_{\{a_t\}} \sum_{t=0}^{T} f_t(a_t, s_t) \text{ s.t.}$$

$$s_{t+1} = g_t(a_t, s_t)$$

$$s_0 \text{ given}$$

The Bellman equation - Illustration (2)

The backwards induction argument

- Assume we are at $t = T$ and compute the optimal path in the last period:

$$v_T(s_T) = \max_{a_T} f_T(a_T, s_T)$$

This a static optimization problem. Assume it has a solution. Denote it by $a_T^*(s_T)$. Then,

$$v_T(s_T) = f_T(a_T^*(s_T), s_T). \tag{54}$$

The Bellman equation - Illustration (3)

The backwards induction argument (cont'd)

- Assume we are at $t = T - 1$ and compute the optimal path in the last two periods:
- s_{T-1} affects the (instantaneous) payoff through f_{T-1} and the future payoff though the equation of motion $s_T = g_{T-1}(a_{T-1}, s_{T-1})$
- Then,

$$\begin{aligned} v_{T-1}(s_{T-1}) &= \max_{a_{T-1}} \Big[f_{T-1}(a_{T-1}, s_{T-1}) + \beta v_T(s_T) \Big] \\ &= \max_{a_{T-1}} \Big[f_{T-1}(a_{T-1}, s_{T-1}) + \beta v_T(g_{T-1}(a_{T-1}, s_{T-1})) \Big] \end{aligned}$$

- Suppose the solution is $a^*_{T-1}(s_{T-1})$. Then,

$$v_{T-1}(s_{T-1}) = f_{T-1}(a^*_{T-1}(s_{T-1}), s_{T-1}) + \beta v_T(g_{T-1}(a^*_{T-1}(s_{T-1}), s_{T-1}))$$

The Bellman equation - Illustration (4)

The backwards induction argument (cont'd)

- Assume we are at $t = T-2$ and compute the optimal path in the last three periods:
- s_{T-2} affects the (instantaneous) payoff through f_{T-2} and the future payoff though the equation of motion $s_{T-1} = g_{T-2}(a_{T-2}, s_{T-2})$
- Then,

$$\begin{aligned} v_{T-2}(s_{T-2}) &= \max_{a_{T-2}} \Big[f_{T-2}(a_{T-2}, s_{T-2}) + \beta^2 v_{T-1}(s_{T-1}) \Big] \\ &= \max_{a_{T-2}} \Big[f_{T-2}(a_{T-2}, s_{T-2}) + \beta^2 v_{T-1}(g_{T-2}(a_{T-2}, s_{T-2})) \Big] \end{aligned}$$

- Suppose the solution is $a^*_{T-2}(s_{T-2})$. Then, $v_{T-2}(s_{T-2}) = f_{T-2}(a^*_{T-2}(s_{T-2}), s_{T-2}) + \beta^2 v_{T-1}(g_{T-2}(a^*_{T-2}(s_{T-2}), s_{T-2}))$

The Bellman equation - Illustration (5)

The backwards induction argument (cont'd)

- We repeat the argument until reaching $t = 0$
- To determine a^* we solve the FOC of the Bellman's equation in each period:
  - At $t = T, a^*_T(s_T)$ is the solution of
    $$\frac{\partial v_T(s_T)}{\partial a_T} = \frac{\partial f_T}{\partial a_T} = 0$$
  - At $t = T-1, a^*_{T-1}(s_{T-1})$ is the solution of
    $$\frac{\partial v_{T-1}(s_{T-1})}{\partial a_{T-1}} = \frac{\partial f_{T-1}}{\partial a_{T-1}} + \beta \frac{\partial v_T}{\partial s_T} \frac{\partial s_T}{\partial g_{T-1}} \frac{\partial g_{T-1}}{\partial a_{T-1}} = 0$$
  - At $t = T-2, a^*_{T-2}(s_{T-2})$ is the solution of
    $$\frac{\partial v_{T-2}(s_{T-2})}{\partial a_{T-2}} = \frac{\partial f_{T-2}}{\partial a_{T-2}} + \beta^2 \frac{\partial v_{T-1}}{\partial s_{T-1}} \frac{\partial s_{T-1}}{\partial g_{T-2}} \frac{\partial g_{T-2}}{\partial a_{T-2}} = 0$$
  - ... and so on and so forth

An example

Setting of the problem

- Consider an investment a in a capital stock s that adds to the stock generating a return of 10€ per unit.
- The stock of capital in period $t+1$ is given by $s_{t+1} = s_t + a_t$
- Investment is costly according to $0.1a^2$,
- Time interval is $t = 0, 1, 2, 3$. Assume away discounting.
- The initial stock of capital is zero.
- The problem is to find the optimal path of investment and capital stock.

Solution

- Formally, we want to solve $\max_{\{a_t\}} \sum_{t=0}^{3} (10 s_t - (0.1) a_t^2)$ subject to $s_{t+1} = s_t + a_t,\ s_0 = 0,\ a_t \geq 0$.

An example (2)

Solution (cont'd)

- We start computing the optimal path at $t = 3$:
  - $v_3(s_3) = \max_{a_3}(10s_3 - (0.1)a_3^2)$
  - This is a static optimization problem.
  - Note that $10s_3 - (0.1)a_3^2$ is decreasing in a_3.
  - Therefore, $a_3^* = 0$ and $v_3^* \equiv v_3(s_3)|_{a_3^*} = 10s_3$

An example (3)

Solution (cont'd)

- Next, compute the optimal path at $t = 2$:
  - $v_2(s_2) = \max_{a_2}(10s_2 - (0.1)a_2^2 + v_3(a_3^*))$ s.t.
    $v_3(a_3^*) = 10s_3, \quad s_3 = s_2 + a_2$
  - It can be rewritten as
    $v_2(s_2) = \max_{a_2}(10s_2 - (0.1)a_2^2 + 10s_3) =$
    $\max_{a_2}(10s_2 - (0.1)a_2^2 + 10(s_2 + a_2))$
  - Note that $(10s_2 - (0.1)a_2^2 + 10(s_2 + a_2))$ is strictly concave in a_2.
  - Solving for the FOC, it follows $a_2^* = 50$
  - ... yielding $v_2^* \equiv v_2(s_2)|_{a_2^*} = 20s_2 + 250$

An example (4)

Solution (cont'd)

- Next, compute the optimal path at $t = 1$:
  - $v_1(s_1) = \max_{a_1}(10s_1 - (0.1)a_1^2 + v_2(s_2^*))$ s.t.
    $v_2(s_2^*) = 20s_2 + 250, \quad s_2 = s_1 + a_1$
  - It can be rewritten as
    $v_1(s_1) = \max_{a_1}(10s_1 - (0.1)a_1^2 + 20s_2 + 250) =$
    $\max_{a_1}(10s_1 - (0.1)a_1^2 + 20(s_1 + a_1) + 250)$
  - Note that $(10s_1 - (0.1)a_1^2 + 20(s_1 + a_1) + 250)$ is strictly concave in a_2.
  - Solving for the FOC, it follows $a_1^* = 100$
  - ... yielding $v_1^* \equiv v_1(s_1)|_{a_1^*} = 30s_1 + 1250$

An example (5)

Solution (cont'd)

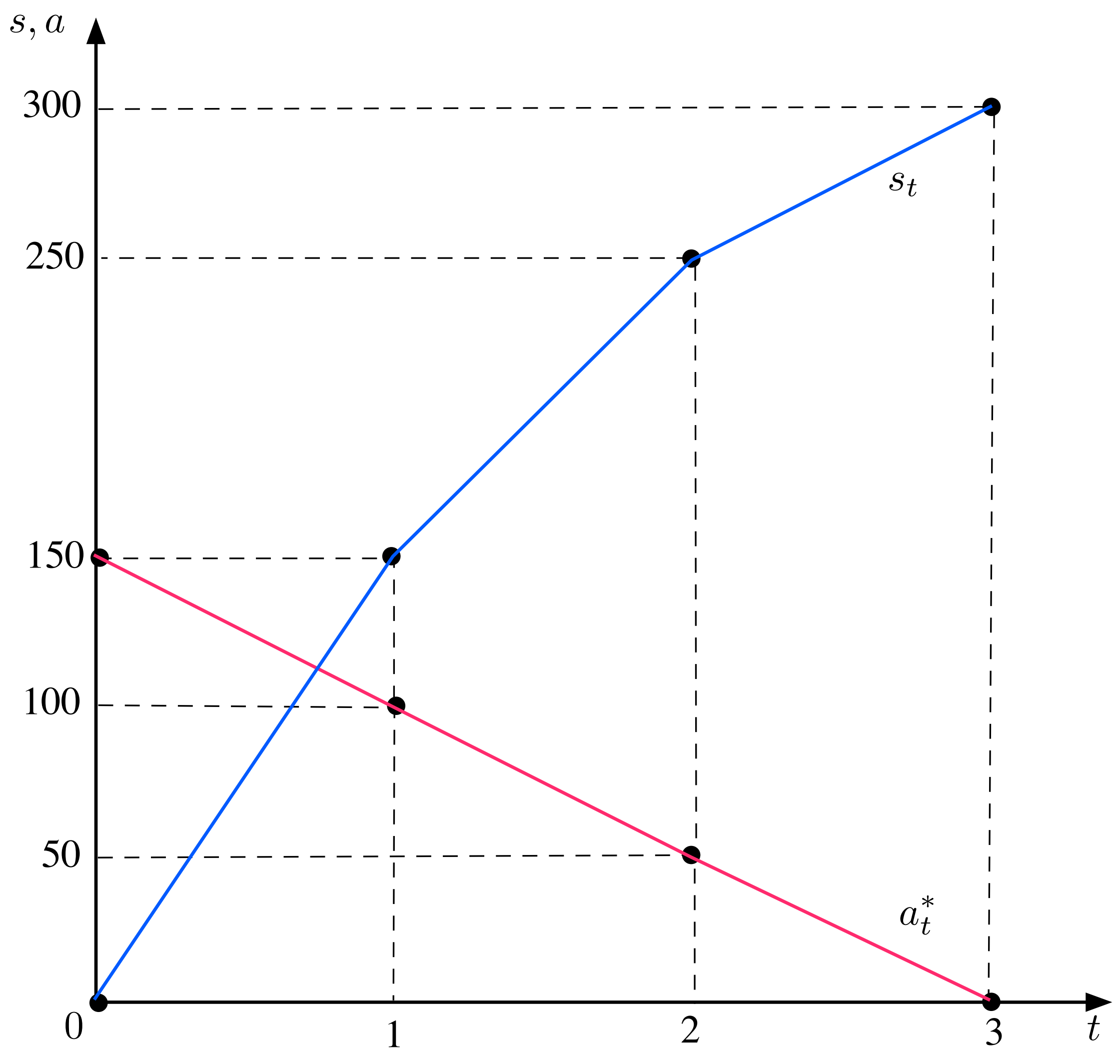
- Finally, compute the optimal path at $t = 0$:
  - $v_0(s_0) = \max_{a_0}(10s_0 - (0.1)a_0^2 + v_1(s_1^*))$ s.t.
    $v_1(s_1^*) = 30s_1 + 1250, \quad s_1 = s_0 + a_0, \quad s_0 = 0$
  - Combining the constraints, we obtain $s_1 = a_0$ and
    $v_1(s_1) = 30a_0 + 1250$
  - Substituting them in $v_0(s_0)$ we obtain
    $v_0(s_0) = \max_{a_0}(-(0.1)a_0^2 + 30a_0 + 1250)$
  - Solving for the FOC, it follows $a_0^* = 150$
  - ... yielding $v_0^* \equiv v_0(s_0)|_{a_0^*} = 3500$

An example (6)

a_t optimal path

t	a_t^*	$s_{t+1} = s_t + a_t^*;\ s_0 = 0$
0	150	0
1	100	150 $(s_1 = (s_0 + a_0^*) = a_0^*)$
2	50	250 $(s_2 = s_1 + a_1^*)$
3	0	300 $(s_3 = s_2 + a_2^*)$

An example (7)



The Principle of Optimality

- The additive separability of the objective function, the simple structure of the law of motion, and the fact that the total return is the sum of the period return functions, imply that the total payoff associated over the whole planning horizon is simply the sum of the payoffs associated with different portions of the sequence over the corresponding subperiods.
- More formally, any portion of an optimal trajectory is an optimal trajectory for an suitable subproblem in which the endpoint values of the state vector are constrained to be equal to the corresponding terms of the optimal sequence for the whole problem.
- Recall:*"An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state rwsulting from the first decision."* (Bellman, 1957)

The Principle of Optimality (2)

- This property is known as The Principle of Optimality (proof see e.g. De la Fuente (2002, ch.12))
- It guarantees the time-consistency of the optimal policy.
- This means that if at some point in time we recalculate the optimal solution from the current time and state, the solution to this new problem will be the remainder of the original optimal plan.
- The following figure illustrates (with the liberty of representing continuous time):
  $s(t)$ represents the trajectory over the time interval $[t_0, T]$ induced by the optimal path a_t^*. At $t = \tau$ we can envisage a new problem of finding the optimal path in the time interval $[\tau, T]$ with inital state variable $s(\tau)$. The Principal of Optimality says that such path (in red) is the same as the original one.

The Principle of Optimality (3)