Optimization. A first course of mathematics for economists

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III.2 Economic Dynamics - Difference equations

Introduction

- Given a function $y_t = f(t)$ its first difference is defined as the difference in value of the function evaluated at $t+h$ and $t,\ \Delta y_t = f(t+h) - f(t)$.
- Usually we take $h = 1$ so that $\Delta y_t = f(t+1) - f(t)$.
- The same logic taking as reference any time period:

$$
\begin{aligned}
\Delta y_t =& f(t+1) - f(t) \\
\Delta y_{t+1} =& f(t+2) - f(t+1) \\
&\vdots \\
\Delta y_{t+\tau} =& f(t+\tau+1) - f(t+\tau)
\end{aligned}
$$

Introduction (2)

- Given Δy_t the second difference of y_t is defined as the difference in value of the first difference:
  $\Delta^2 y_t = \Delta y_{t+1} - \Delta y_t = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) =$
  $y_{t+2} - 2y_{t+1} + y_t$
- Similarly, $\Delta^2 y_{t+1} = \Delta y_{t+2} - \Delta y_{t+1} = y_{t+3} - 2y_{t+2} + y_{t+1}$ etc, etc.
- Same logic for third, fourth, ... n-th difference of y_t.
- Remark: Solution of a difference equation is independent of the time period considered:
  - solution of $ay_{t+1} + by_t = 0$ is the same as the
  - solution of $ay_{t+2} + by_{t+1} = 0$ and the same as the
  - solution of $ay_t + by_{t-1} = 0$
  - because solution is a function satisfying the equation $\forall t$.

Introduction (3)

- Focus in Linear Difference Equations with constant coefficients:
  $c_n y_{t+n} + c_{n-1} y_{t+n-1} + c_{n-2} y_{t+n-2} + \cdots + c_1 y_{t-1} + c_0 y_t = g(t)$
  where c_j are given constants, $c_n \neq 0, c_0 \neq 0$, and $g(t)$ is a known function.
- Strategy of analysis:
  - Find the general solution of the homogeneous equation, $f(t, A_1, \ldots, A_n)$
  - Fins a particular solution of the non-homogeneous equation, $\overline{y}(t)$
  - Solution of the difference equation is $y(t) = f(t, A_1, \ldots, A_n) + \overline{y}(t)$
  - Additional conditions allow for solving for $(A_1, \ldots, A_n)$

Two theorems

- Theorem 1
  If $y_1(t)$ is solution of the homogeneous equation, so is $Ay_1(t), A \in \mathbf{R}$.
- Theorem 2
  If $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, so is $A_1y_1(t) + A_2y_2(t), (A_1, A_2) \in \mathbf{R}$.
- Proofs trivial. See Gandolfo (2010, ch 1)

First-order difference equations

General form: $c_1 y_t - c_0 y_{t-1} = g(t)$ with $c_1 \neq 0, c_0 \neq 0$
Homogeneous equation:

$$c_1 y_t - c_0 y_{t-1} = 0, \quad \text{or, with } b \equiv c_0/c_1$$

$$y_t - b y_{t-1} = 0 \qquad [heq1]$$

- General solution of homogeneous equation
  - Suppose $y(0) = y_0$. Then,
  - $y_1 - b y_0 = 0 \to y_1 = b y_0$
  - $y_2 - b y_1 = y_2 - b(b y_0) = 0 \to y_2 = -b^2 y_0$
  - $\vdots$
  - Therefore, $y_t^h = b^t y_0$. In general,
  - Solution candidate: $y_t^h = b^t A$, A to be determined $[heq2]$
  - must satisfy $[heq1] \forall t$: $b^t A - b(b^{t-1})A = b^t A - b^t A = 0, \forall t$

First-order difference equations (2)

Homogeneous equation $y_t - by_{t-1} = 0$

Behavior of solution $y_t^h = b^t A$

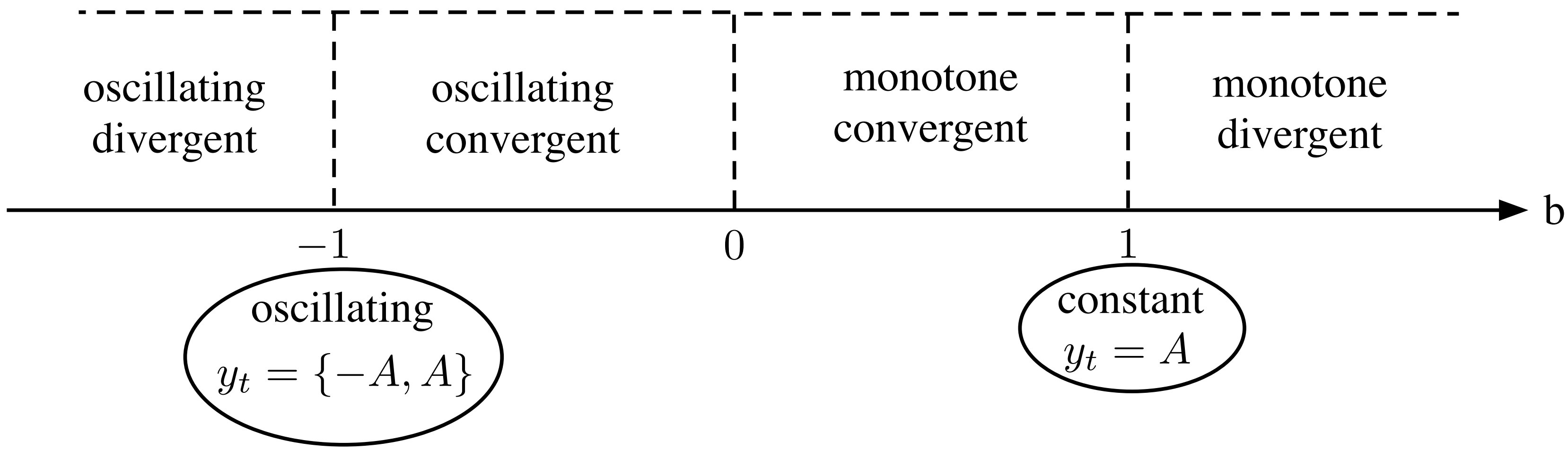
- Depends on the sign and (absolute) value of b:

$b > 0$	monotone
$b < 0$	oscillating
$\lvert b \rvert < 1$	convergent
$\lvert b \rvert > 1$	divergent
$b = 1$	constant at value A
$b = -1$	oscillating, constant amplitude $\{-A, A\}$

First-order difference equations (3)

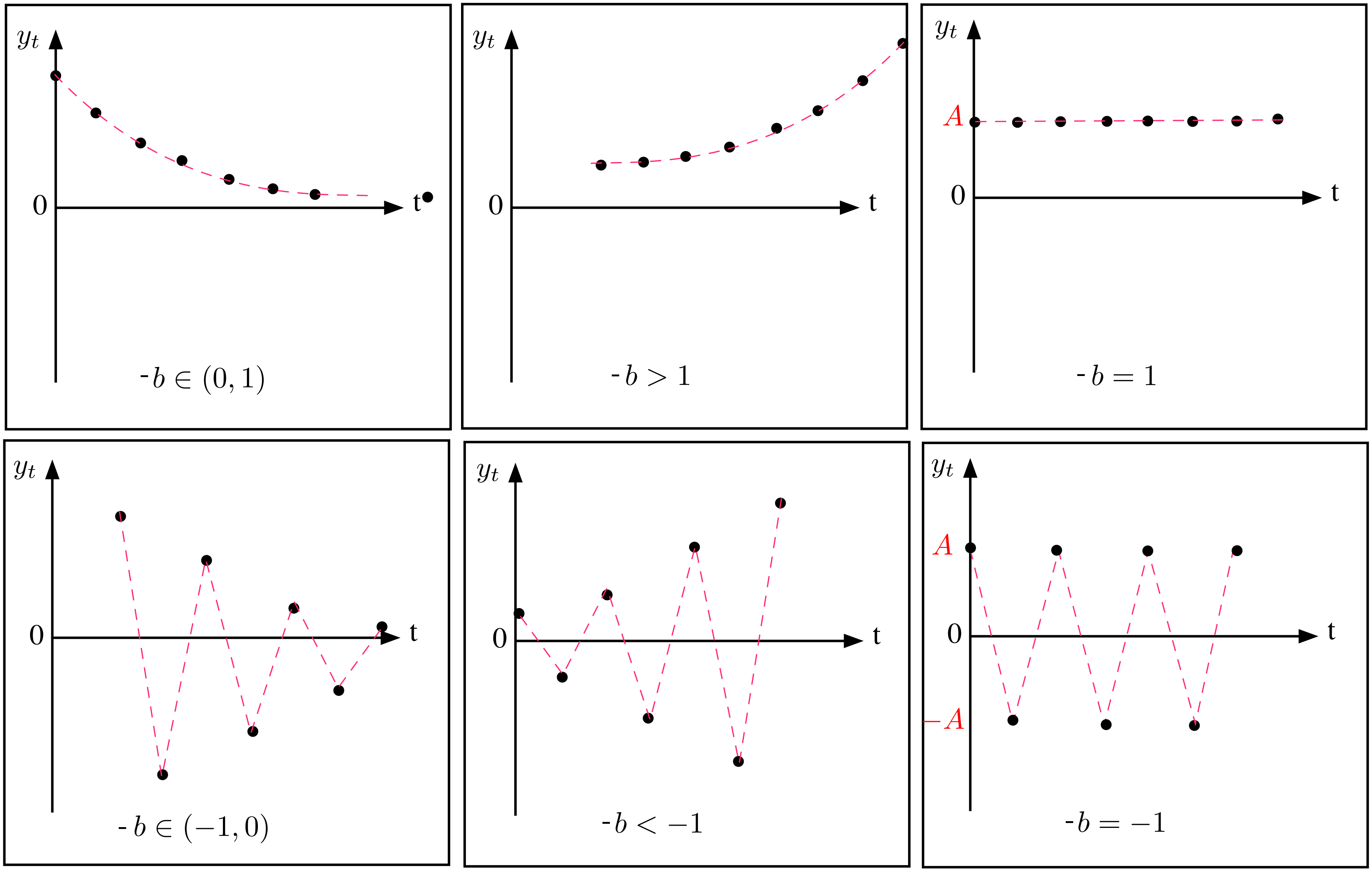
Behavior of solution $y_t^h = b^t A$ (cont'd)

- Six possible trajectories:



- more graphically

First-order difference equations (4)



First-order difference equations (5)

Particular solution of general equation - Case 1: $g(t) = a$

- Recall: $c_1 y_t + c_0 y_{t-1} = g(t) \qquad [heq3]$
- Solution has same structure as $g(t)$
- Let $\overline{y}_t = \mu, \forall t$
- Substitute it into $[heq3]$ to obtain $c_1\mu + c_0\mu = a$
- or $\mu = \frac{a}{c_1+c_0}$
- so that a particular solution of $[heq3]$ is

$$\overline{y}_t = \frac{a}{c_1 + c_0}$$

- Solution of the 1st-order difference equation: $y_t = -b^t A + \frac{a}{c_1+c_0}$
- Remark: If $c_1 + c_0 = 0$ use $\overline{y}_t = t\mu, \forall t$ to obtain $\overline{y}_t = \frac{-at}{c_0}$

First-order difference equations (6)

Determining A

- Additional condition. Let $y(0) = y_0$
- Then,

$$y_0 = A + \frac{a}{c_1 + c_0}$$

or

$$A = y_0 - \frac{a}{c_1 + c_0}$$

and solution of the difference equation is

$$y_t = -b^t \left[y_0 - \frac{a}{c_1 + c_0} \right] + \frac{a}{c_1 + c_0}$$

First-order difference equations (7)

Particular solution of general equation - Case 2: $g(t) = Bd^t$

- Recall: $c_1 y_t + c_0 y_{t-1} = g(t) \qquad [heq3]$
- Solution has same structure as $g(t)$
- Let $\overline{y}_t = \mu d^t, \forall t$
- Substitute it into $[heq3]$ to obtain $c_1 \mu d^t + c_0 \mu d^{t-1} = Bd^t$
- or $d^{t-1}(c_1 \mu d + c_0 \mu - Bd) = 0$ so that $\mu = \frac{Bd}{c_1 d + c_0}$
- so that a particular solution of $[heq3]$ is
  $\overline{y}_t = \frac{Bd}{c_1 d + c_0} d^t$
- Solution of the 1st-order difference equation:
  $y_t = -b^t A + \frac{Bd}{c_1 d + c_0} d^t$
- Remark: If $c_1 d + c_0 = 0$ use $\overline{y}_t = t\mu d^t, \forall t$ to obtain $\overline{y}_t = \frac{-Bd}{c_0} t d^t$

First-order difference equations (8)

Particular solution of general equation - Case 3: $g(t) = a_0 + a_1 t$

- Remark: Argument generalizes to a general polynomial of degree m.
- Recall: $c_1 y_t + c_0 y_{t-1} = g(t) \qquad [heq3]$
- Solution has same structure as $g(t)$
- Let $\overline{y}_t = \mu_0 + \mu_1 t, \forall t$
- Substitute it into $[heq3]$ to obtain
  $c_1(\mu_0 + \mu_1 t) + c_0(\mu_0 + \mu_1(t-1)) = a_0 + a_1 t$
- or $t(\mu_1(c_0 + c_1) - a_1) + (\mu_0(c_0 + c_1) - \mu_1 c_0 - a_0) = 0$
- that will be satisfied $\forall t$ if

$$\left.\begin{aligned} \mu_1(c_0 + c_1) - a_1 &= 0 \\ \mu_0(c_0 + c_1) - \mu_1 c_0 - a_0 &= 0 \end{aligned}\right\}$$

First-order difference equations (9)

Particular solution of general equation - Case 3 (cont'd)

- solving the system for μ_0 and μ_1, we obtain

$$\mu_1 = \frac{a_1}{c_0 + c_1}$$

$$\mu_0 = \frac{a_1 c_0 + a_0(c_0 + c_1)}{(c_0 + c_1)^2}$$

- so that a particular solution of $[heq3]$ is

$$\overline{y}_t = \frac{a_1 c_0 + a_0(c_0 + c_1)}{(c_0 + c_1)^2} + \frac{a_1}{c_0 + c_1} t$$

Solution of the 1st-order difference equation:

$$y_t = -b^t A + \frac{a_1 c_0 + a_0(c_0 + c_1)}{(c_0 + c_1)^2} + \frac{a_1}{c_0 + c_1} t$$

First-order difference equations (10)

Particular solution of general equation - Case 3 (cont'd)

- if $(c_0 + c_1) = 0$, use $\overline{y}_t = t(\mu_0 + \mu_1 t)$.
- Substituting it in $[heq3]$ we obtain $-2t(\mu_1 c_0 - a_1) + (\mu_1 c_0 - \mu_0 c_0 - a_0) = 0$
- that is verified $\forall t$ for

$$\mu_1 = \frac{a_1}{c_0}$$

$$\mu_0 = \frac{a_1 - a_0}{c_0}$$

- so that a particular solution of $[heq3]$ is

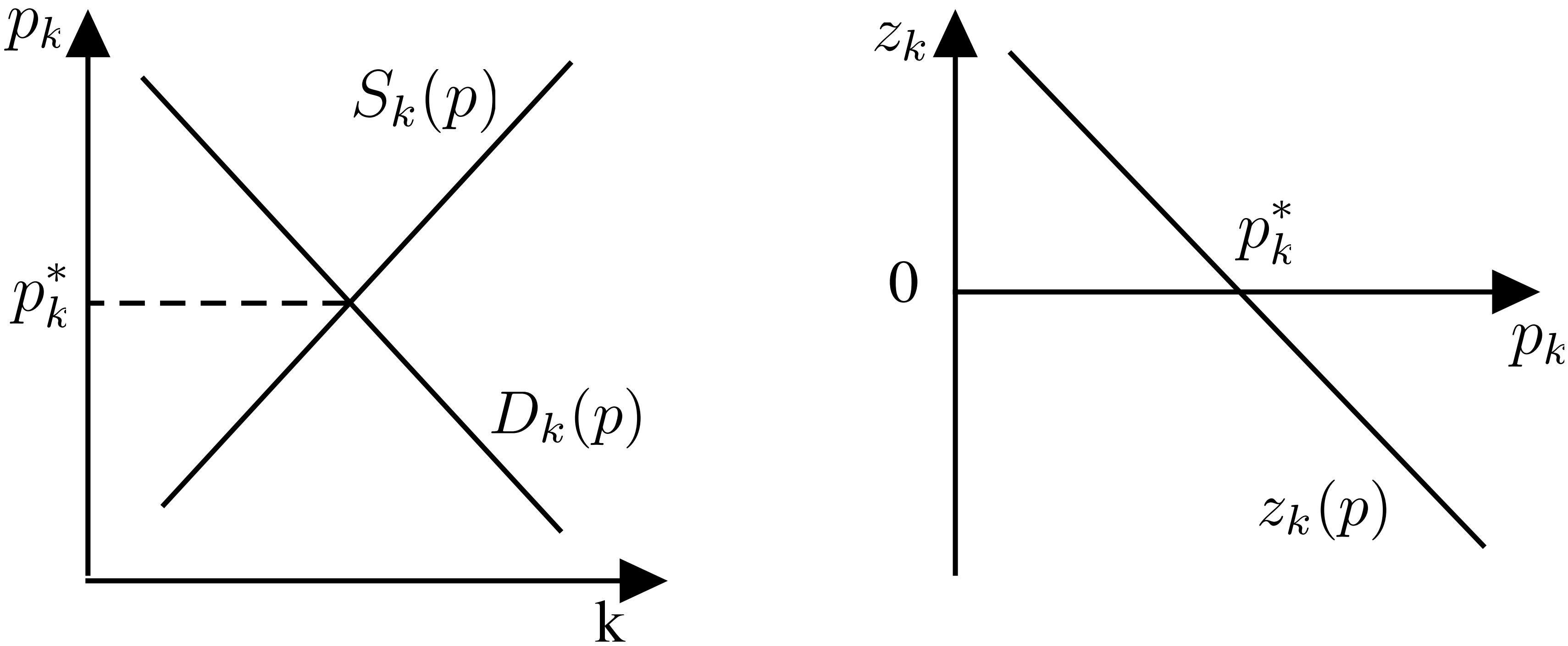
$$\overline{y}_t = t\Big(\frac{a_1 - a_0}{c_0} + \frac{a_1}{c_0}t\Big)$$

An application: Stability of the Walrasian equilibrium

Preliminary - Static stability

- Static stability $\Leftrightarrow$ Law of supply and demand:
  - define individual excess demand of good k as $e_{ik}(p) = x_{ik}(p) - w_{ik}$
  - define aggregate excess demand of good k as $z_k(p) = \sum_{i \in I} e_{ik}(p)$
  - define a walrasian price p^* as a price vector satisfying $z_k(p^*) = 0, \forall k$
  - rewrite $z_k(p) = D_k(p) - S_k(p)$
- p^* satisfies law of supply and demand iff $\frac{dz_k(p)}{dp_k} < 0, \forall k$.
- equivalently, $\frac{dD_k(p)}{dp_k} < \frac{dS_k(p)}{dp_k}, \forall k$
- Remark: always satisfied if $D_k' < 0$ and $S_k' > 0, \forall k$

An application: Stability of the Walrasian equilibrium (2)



Dynamic Stability of the Walrasian equilibrium

- General competitive model is static.
- Introduce a fictitious time schedule and a price formation mechanism.
- At $t = 1$ a random consumer makes an initial offer to all other consumers. Verify if $z_k(p) = 0$.
- At $t = 2$ another random consumer makes an offer. Prices adjust. Verify if $z_k(p) = 0$.
- Protocol goes on and on until the prices do not change from one period to the next (i.e. $z_k(p) = 0$).
- Formally, the price formation mechanism (in each market k) is described by:

$$p_t - p_{t-1} = rz(p_{t-1}) \qquad [Dyn1]$$

where $r > 0$ and the subindex k is avoided to simplify notation.

Dynamic Stability of the Walrasian equilibrium - Example

- For a representative market k, let

$$D_t(p_t) = ap_t + b$$
$$S_t(p_t) = Ap_t + B$$

- For future reference, the equilibrium price at any period t is

$$p_t = \frac{b-B}{A-a} = p^*$$

- Excess demand function in $t-1$ is:

$$z(p_{t-1}) = (a-A)p_{t-1} + (b-B) \qquad [Dyn2]$$

- Substituting $[Dyn2]$ in $[Dyn1]$ we obtain

$$p_t = p_{t-1}[1 + r(a-A)] + r(b-B)$$

Dynamic Stability of the Walrasian equilibrium - Example (2)

- This is a first-order difference equation.
- Assume price at $t = 0$ is p_0. The solution of this difference equation is

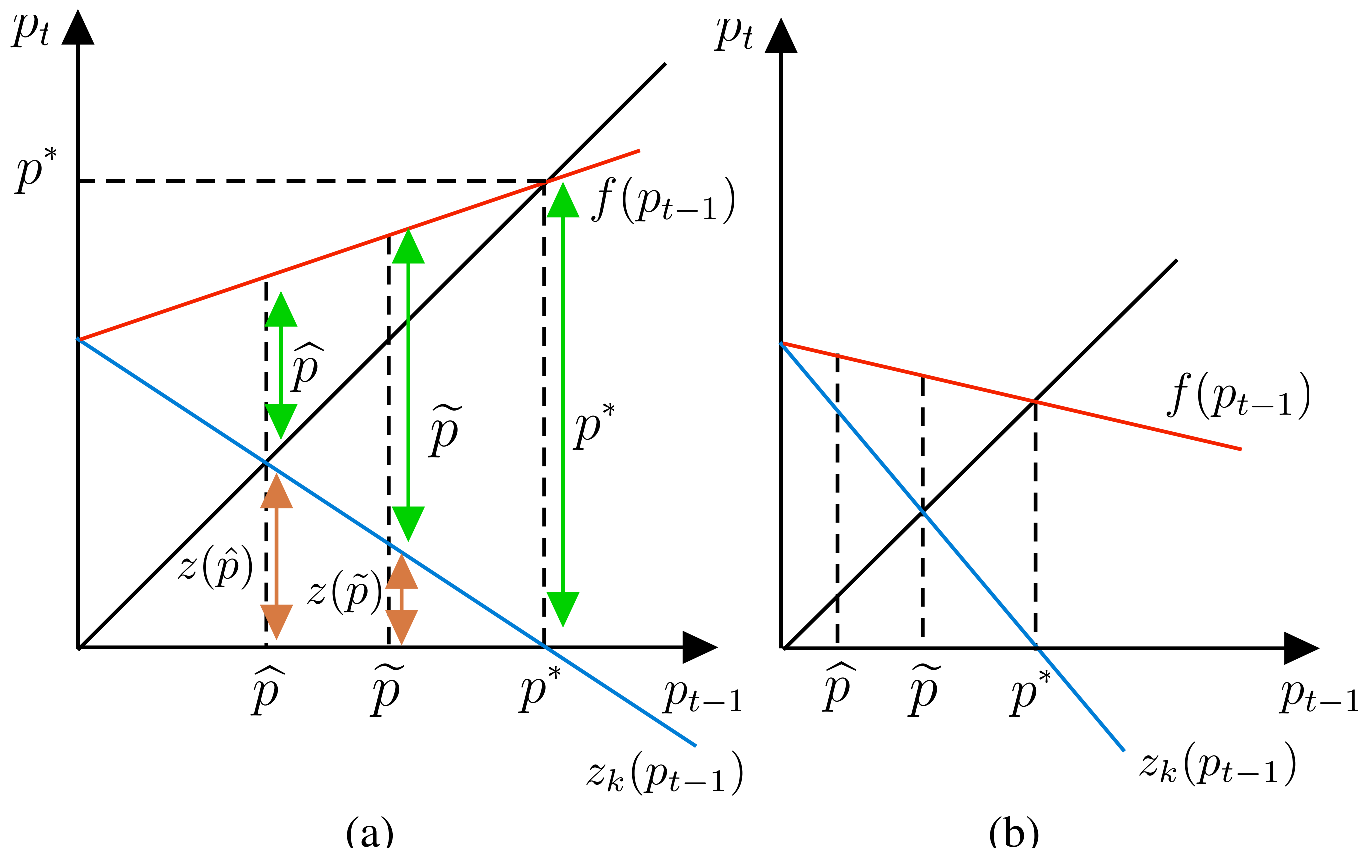
$$p_t = \left[p_0 - \frac{b-B}{A-a}\right](1 + r(a-A))^t + \frac{b-B}{A-a}, \text{ or}$$

$$p_t = (p_0 - p^*)(1 + r(a-A))^t + p^*$$

- The constant term is precisely p^*.
- The term $(p_0 - p^*)$ captures the shock driving the market away from equilibrium.
- The term $(1 + r(a-A))^t$ captures the adjustment process from p_0 to p^*
- Finally r captures the degree of the adjustment.

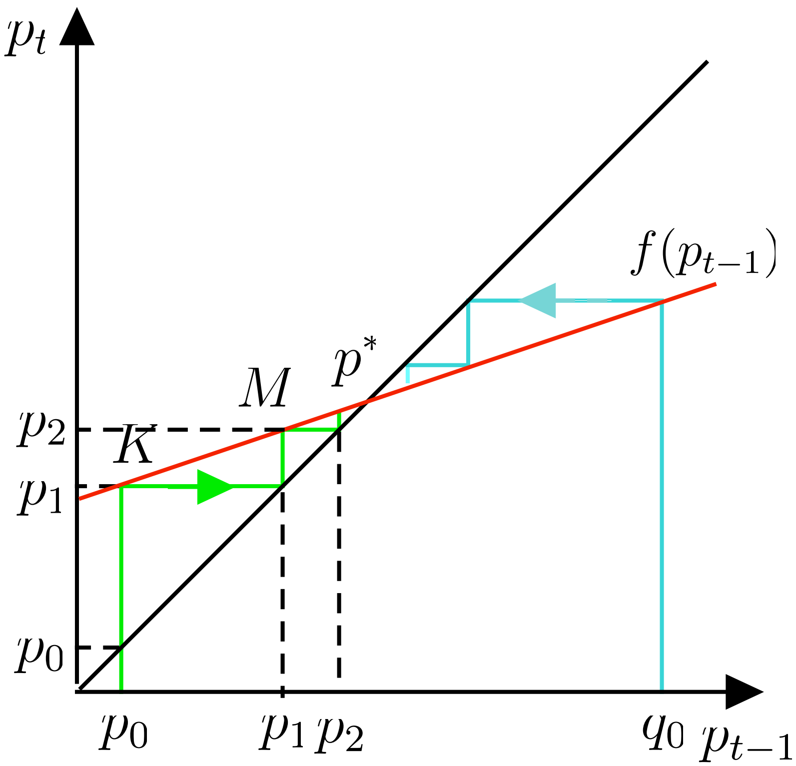
Dynamic Stability - Graphical analysis

- Recall $p_t = p_{t-1} + rz(p_{t-1}) \equiv f(p_{t-1})$
- The function $f(p_{t-1})$ may be increasing or decreasing
- The following figure shows its graphical derivation ($r = 1$):

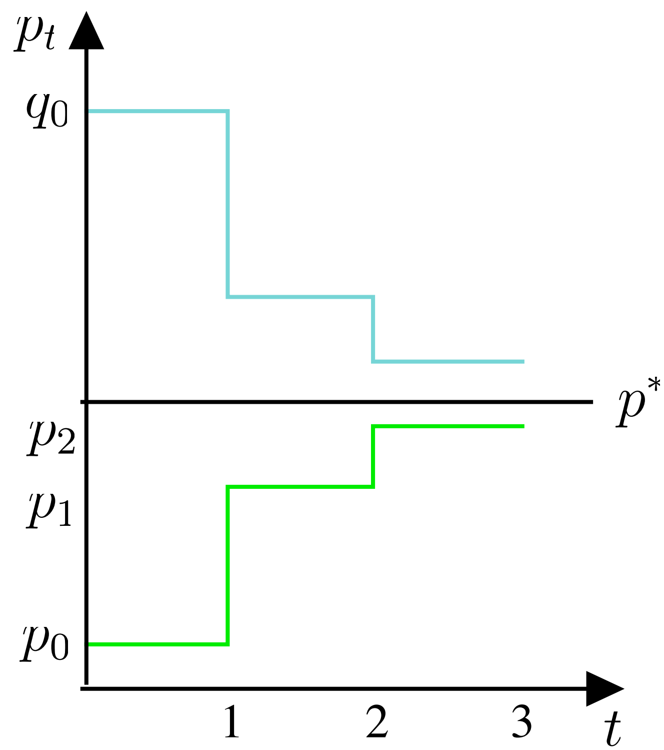


Dynamic Stability - Graphical analysis (2)

- Assume $f(p_{t-1})$ is increasing and $|f'| < 1$.
- The following figure describes the stability of p^*



(a)



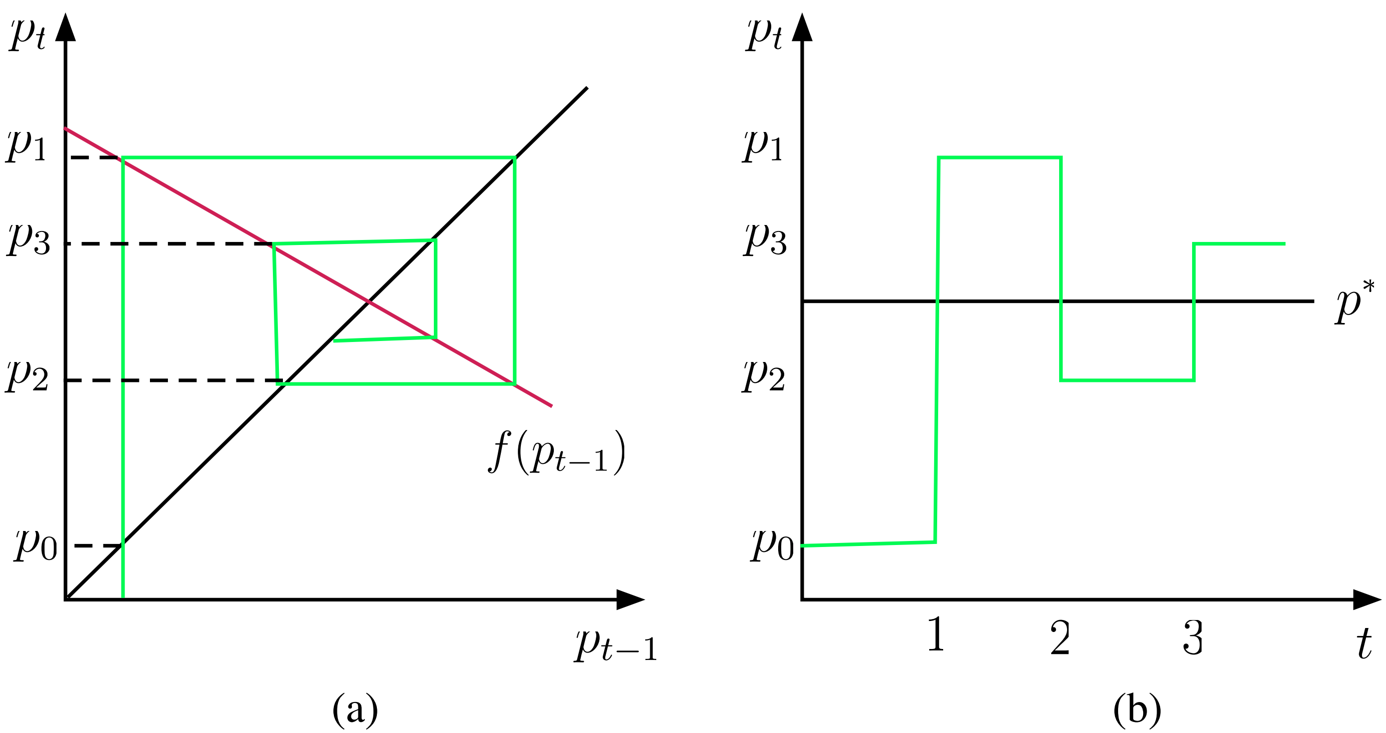
(b)

Dynamic Stability - Graphical analysis (3)

- Consider p_0 placing us at point K.
- Next period, $p_1 = f(p_0)$, placing us at point M
- Next period, $p_2 = f(p_1)$, and so on and so forth.
- The process converges to p^* located at the intersection of the function $f(p_{t-1})$ with the 45-degree line.
- A parallel argument develops if the initial price is q_0.
- The figure on the right hand side shows the “temporal” trajectory of the price.
- Note that f increasing and slope < 1 generate a monotonic convergent trajectory.

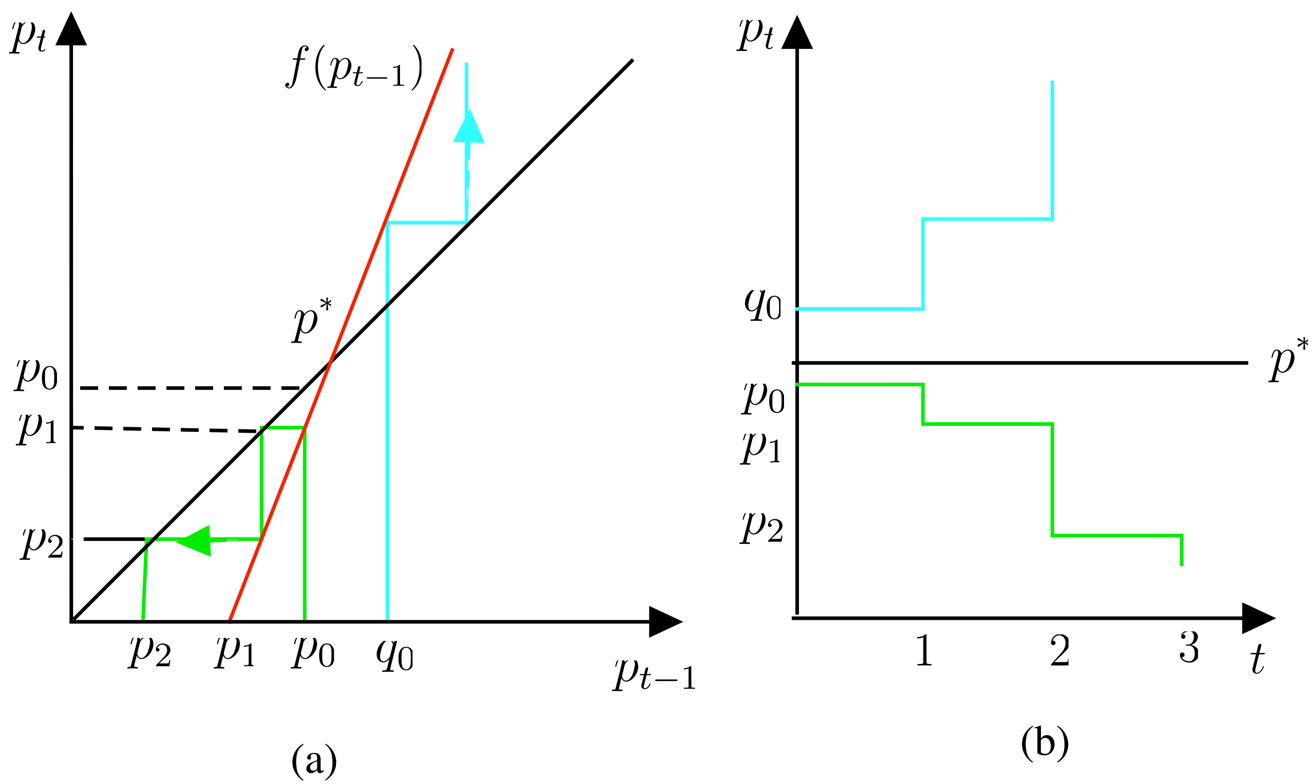
Dynamic Stability - Graphical analysis (3)

- Consider f decreasing and $|f'| < 1$
- the trajectory cyclically converges towards p^*



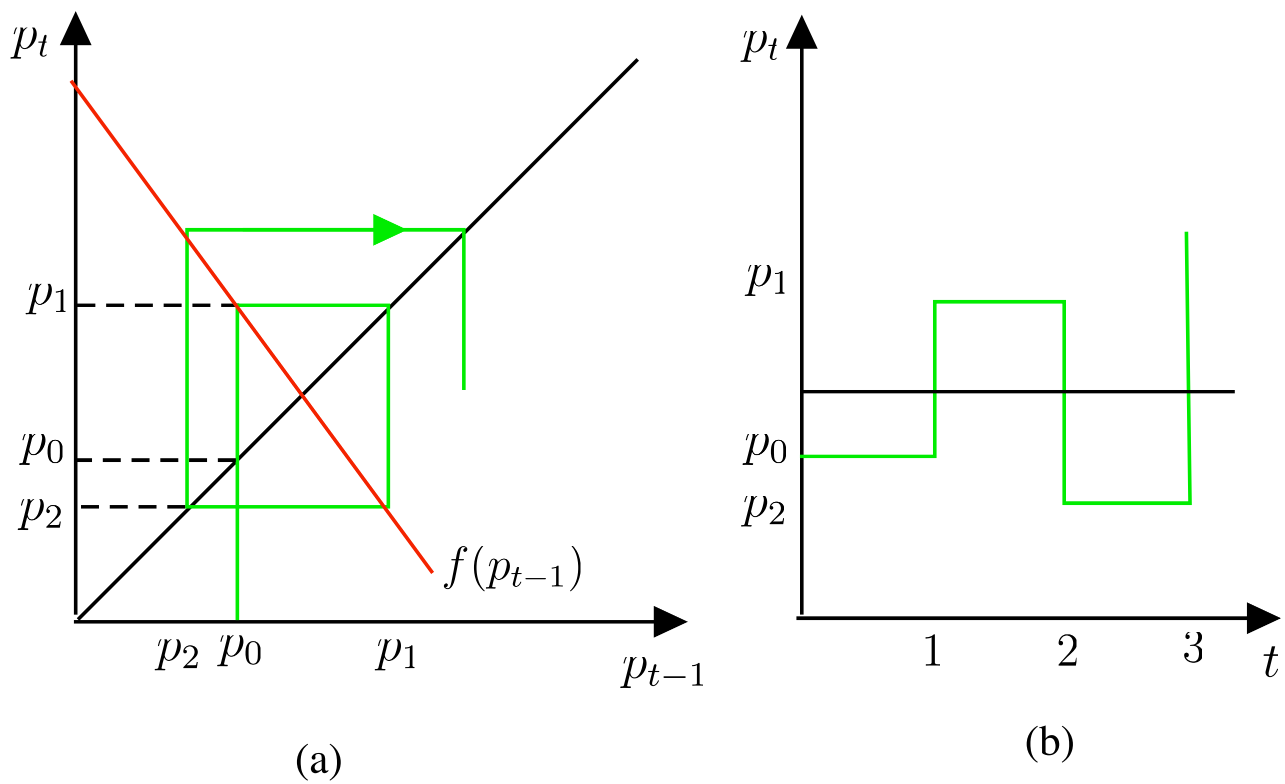
Dynamic Instability - Graphical analysis

- Consider f increasing and $|f'| > 1$
- the trajectory monotonically diverges from p^*



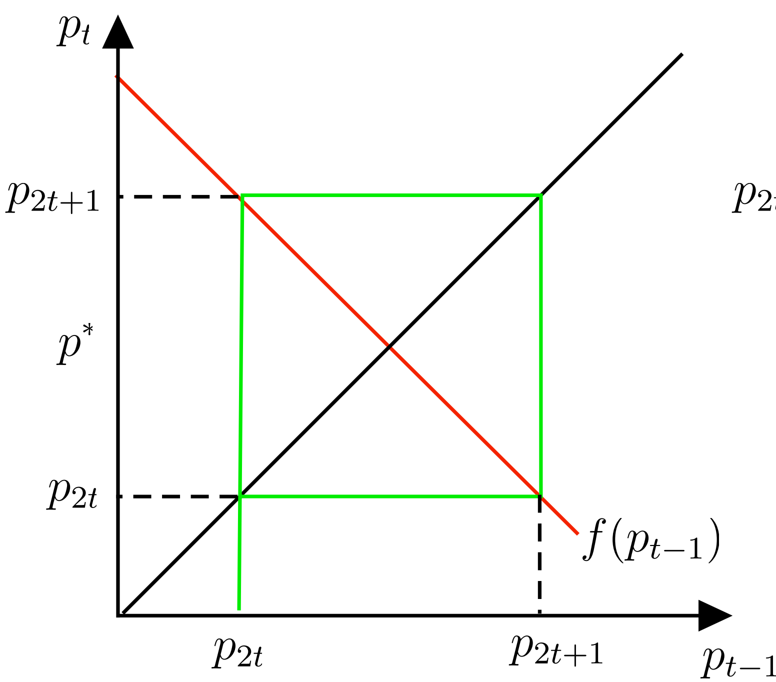
Dynamic Instability - Graphical analysis (2)

- Consider f decreasing and $|f'| > 1$
- the trajectory cyclically diverges from p^*

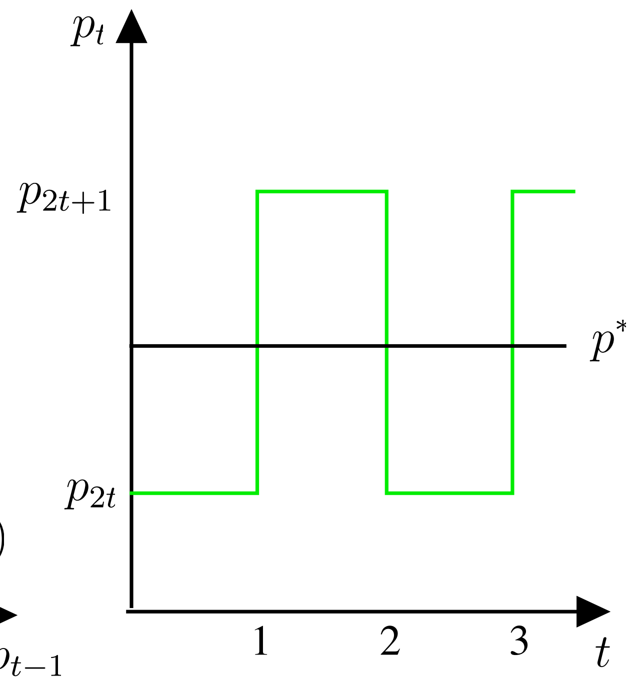


Dynamic Instability - Graphical analysis (3)

- Consider f decreasing and $|f'| = 1$
- the trajectory describes a constant cycle around p^*



(a)



(b)

Compound interest & Present Discounted Value

- An individual opens a bank account
  - Initial wealth w_0 in period $t = 0$
  - In each t individual deposits income y_t
  - In each t individual withdraws c_t for consumption
  - Interest rate r constant along time
- $w_t = (1 + r)w_{t-1} + (y_t - c_t), \forall t$
- Define $a = (1 + r), b_t = y_t - c_t$
- so $w_t = aw_{t-1} + b_t \qquad [PDV1]$
- Remark

  Generalization of Case 1. Constant varies every period

Compound interest & Present Discounted Value (2)

- Solution of $[PDV1]$. Algebraic argument

$$t = 1, w_1 = aw_0 + b_1$$

$$t = 2, w_2 = aw_1 + b_2 = a^2 w_0 + ab_1 + b_2$$

$$t = 3, w_3 = aw_2 + b_3 = a^3 w_0 + a^2 b_1 + ab_2 + b_3$$

$$\vdots$$

$$w_t = a^t w_0 + \sum_{k=1}^{t} a^{t-k} b_k$$

Compound interest & Present Discounted Value (3)

- Remark: If $b_t = b, \forall t$

$$\sum_{k=1}^{t} a^{t-k} b_k = b \sum_{k=1}^{t} a^{t-k} =$$

$$b(a^{t-1} + a^{t-2} + \cdots + a + 1) = b\frac{1-a^t}{1-a}$$

- Then,

$$w_t = a^y w_0 + b\frac{1-a^t}{1-a} = a^t\left(w_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$$

and we are back to Case 1.

Compound interest & Present Discounted Value (4)

- Solution: $w_t = a^t w_0 + \sum_{k=1}^{t} a^{t-k} b_k$ or $w_t = (1+r)^t w_0 + \sum_{k=1}^{t}(1+r)^{t-k}(y_k - c_k)$
- Multiply by $(1+r)^{-t}$ to obtain $(1+r)^{-t} w_t = w_0 + \sum_{k=1}^{t}(1+r)^{-k}(y_k - c_k)$
- Interpretation:
  - Individual at time $t = 0$
  - $(1+r)^{-t} w_t$ is the PDV of the assets at time t
  - PDV equals the sum of
    - initial wealth w_0
    - PDV of future deposits $\sum_{k=1}^{t}(1+r)^{-k} y_k$
    - PDV of future withdrawals $\sum_{k=1}^{t}(1+r)^{-k} c_k$

Compound interest & Present Discounted Value (5)

- Solution:
  $w_t = (1+r)^t w_0 + \sum_{k=1}^{t} (1+r)^{t-k} (y_k - c_k)$
- Interpretation - cont:
  - Individual at period t
  - assets w_t reflect
    - interest earned on initial deposit w_0
    - interest earned on all later deposits $\sum_{k=1}^{t} (1+r)^{t-k} y_k$
    - interest foregone from withdrawals $\sum_{k=1}^{t} (1+r)^{t-k} c_k$

Second-order difference equations

The difference equation

- The general formulation is
  $c_2 y_t + c_1 y_{t-1} + c_0 y_{t-2} = g(t)$
- solution follows same strategy as 1st-order difference equations and 2nd-order differential equations.
  - Find the general solution of the homogeneous equation, $f(t, A_1, \ldots, A_n)$
  - Fins a particular solution of the non-homogeneous equation, $\overline{y}(t)$
  - Solution of the difference equation is
    $y(t) = f(t, A_1, \ldots, A_n) + \overline{y}(t)$
  - Additional conditions allow for solving for $(A_1, \ldots, A_n)$

Second-order difference equations (2)

The homogeneous equation

- the associated homogeneous equation is: $c_2 y_t + c_1 y_{t-1} + c_0 y_{t-2} = 0$
- Rewrite it as $y_t + b_1 y_{t-1} + b_2 y_{t-2} = 0 \qquad [2heq1]$, where $b_1 = c_1/c_2$ and $b_2 = c_0/c_2$
- To solve it, follow a similar argument as in the 1st-order difference equations
- Solution candidate: $y_t = m^t, \; m \neq 0 \qquad [2heq2]$
- solution must satisfy $[2heq1] \forall t$:

$$m^t + b_1 m^{t-1} + b_2 m^{t-2} = 0 \quad \text{or}$$

$$m^t (1 + b_1 m^{-1} + b_2 m^{-2}) = 0 \quad \text{or}$$

$$m^{t+2} (m^2 + m b_1 + b_2) = 0$$

Second-order difference equations (3)

The homogeneous equation (cont'd)

- implying
$$m^2 + mb_1 + b_2 = 0$$
[characteristic equation].
- Roots of this polynomial are
$$(m_1, m_2) = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2}$$
- Let $\Delta \equiv b_1^2 - 4b_2$. Three cases $\Delta \gtreqless 0$

Second-order difference equations (4)

Solving the homogeneous equation. Case 1: $\Delta > 0$

- In this case we have two real roots. Thus, both m_1 and m_2 satisfy $y_t + b_1 y_{t-1} + b_2 y_{t-2} = 0$
- Applying theorem 2, the general solution of the homogeneous equation is
  $y(t) = A_1 m_1^t + A_2 m_2^t$ where A_1, A_2 are arbitrary constants.
- The evolution of $y(t)$ as $t \to \infty$ is monotonic. The stability of the solution depends on the sign of the roots.
- To assess the sign of the roots we appeal to Descartes' rule of signs:

  Let $P(x)$ be a polynomial with real coefficients and terms in descending powers of x. (a) The number of positive real zeros of $P(x)$ is smaller than or equal to the number of variations in sign occurring in the coefficients of $P(x)$. (b) The number of negative real zeros of $P(x)$ is smaller than or equal to the number of continuations in sign occurring in the coefficients of $P(x)$.

Second-order difference equations (5)

Solving the homogeneous equation. Case 1: $\Delta > 0$ (cont'd)

- Recall the quadratic equation we are studying is $m^2 + b_1 m + b_2 = 0$
- If $b_1 < 0$ and $b_2 > 0$, there are two variations of sign. Therefore, the two roots $m_1 > 0$ and $m_2 > 0$. Accordingly, $m_1^t > 0, \forall t$ and $m_2^t > 0, \forall t$ and y_t will show a monotone trajectory.
- In any other circumstance, cyclical trajectory.

Second-order difference equations (6)

Solving the homogeneous equation. Case 1: $\Delta > 0$ (cont'd)

- Namely,
  - If $b_1 > 0$ and $b_2 > 0$, there are two continuations of sign. Therefore, the two roots $m_1 < 0$ and $m_2 < 0$.
  - If $b_1 < 0$ and $b_2 < 0$ or $b_1 > 0$ and $b_2 < 0$, there is one continuation and one variation of sign. Therefore, one root will be positive and the other negative.
  - If $b_1 = 0$ and $b_2 < 0$, then, $m_1 = -m_2$.
  - If $b_1 \neq 0$ and $b_2 = 0$, then, $m_i = 0, m_j = -b_1$.
- Since the sign of each root may be positive or negative, great variety of trajectories. However,
- If $m_i < 0$ then sgn $m_i^t \begin{cases} > 0 \text{ if } t \text{ even} \\ < 0 \text{ if } t \text{ odd} \end{cases}$
- Any trajectory will converge iff $|m_1| < 1$ *and* $|m_2| < 1$.

Second-order difference equations (7)

Solving the homogeneous equation. Case 2: $\Delta = 0$

- In this case $m_1 = m_2 = \widehat{m} = -\frac{1}{2}b_1$
- $y(t) = \widehat{m}^t$ is a general solution of the homogeneous equation.
- Another general solution of the homogeneous equation is $t\widehat{m}^t$.
- To verify, substitute it in the homogeneous equation to obtain $\widehat{m}t + b_1(t-1)\widehat{m}^{t-1} + b_2(t-2)\widehat{m}^{t-2} = 0$
- Rewrite it as

$$\widehat{m}^{t-2}\Big(\widehat{m}^2 t + b_1(t-1)\widehat{m} + b_2(t-2)\Big) = 0$$

$$\widehat{m}^{t-2}t\Big(\widehat{m}^2 + b_1\widehat{m} + b_2\Big) - b_1\widehat{m} - 2b_2 = 0$$

Second-order difference equations (8)

Solving the homogeneous equation. Case 2: $\Delta = 0$ (cont'd)

- Note that the expression in brackets is the characteristic equation. Also recall $m \neq 0$. Hence, the previous reduces to

$$-b_1 \widehat{m} - 2b_2 = 0$$

- Substituting $\widehat{m}$ by its value, we obtain

$$\begin{aligned} -b_1\left(-\frac{1}{2}b_1\right) - 2b_2 &= 0 \\ \frac{1}{4}\left(b_1^2 - 4b_2\right) &= 0 \end{aligned}$$

- This is, $\Delta = 0$ as we are assuming.
- Therefore, $y_t = t\widehat{m}^t$ is also a general solution of the homogeneous equation.

Second-order difference equations (9)

Solving the homogeneous equation. Case 2: $\Delta = 0$ (cont'd)

- Applying theorem 2, the general solution of the homogeneous equation is:
  $y_t = A_1\widehat{m}^t + A_2t\widehat{m}^t = (A_1 + A_2t)\widehat{m}^t$ where A_1 and A_2 are arbitrary constants.
- When $|\widehat{m}| < 1$, the trajectory of y_t will converge because
  - the converging effect of m^t dominates
  - the divergent effect of t.

Second-order difference equations (10)

Solving the homogeneous equation. Case 3: $\Delta < 0$

- We skip this case.
- See 2nd-order differential equations for an intuition.

Second-order difference equations (11)

Particular solution of the non-homogeneous equation

- $c_2 y_t + c_1 y_{t-1} + c_0 y_{t-2} = g(t) \qquad [2heq3]$
- Solution depends on the structure of $g(t)$

Case 1: $g(t)$ constant

- Let $g(t) = k,\ k \in \mathbf{R}$
- Try as solution $\overline{y}_t = s,\ s \in \mathbf{R}$
- Then $y_t = y_{t-1} = y_{t-2} = s$. Substituting in $[2heq3]$ we obtain $s(c_0 + c_1 + c_2) = k$
- so that $\overline{y}_t = \frac{k}{c_0+c_1+c_2}$ is a particular solution.
- If $c_0 + c_1 + c_2 = 0$, then try $\overline{y}_t = st$. In this case, substituting in $[2heq3]$, we obtain $\overline{y}_t = \frac{-k}{c_1+2c_0}$ as a particular solution.
- If $c_1 + 2c_0 = 0$, try $\overline{y}_t = st^2$, to obtain $\overline{y}_t = \frac{k}{2c_0}$ as a particular solution.

Second-order difference equations (12)

Case 2: $g(t)$ exponential

- Let $g(t) = k^t, \; k \in \mathbf{R}$
- Try as solution $\overline{y}_t = sk^t, \; s \in \mathbf{R}$
- Substituting in $[2heq3]$ we obtain

$$c_2 sk^t + c_1 sk^{t-1} + c_0 sk^{t-2} = k^t$$

$$sk^{t-2}(c_0 + c_1 k + c_0 k^2) = k^t \quad \text{so that}$$

$$s = \frac{k^2}{c_0 + c_1 k + c_0 k^2} \quad \text{and}$$

$$\overline{y}_t = \frac{k^{t+2}}{c_0 + c_1 k + c_0 k^2}$$

- $\overline{y}_t$ is well-defined if $k \neq \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0}}{2c_0}$

Second-order difference equations (13)

Case 3: $g(t)$ polynomial

- Similar approach

Stability (convergence) of the solution path

- Let

$$y_t = A_1 m_1^t + A_2 m_2^t + \overline{y}_t$$

  be the the solution of

$$y_t + b_1 y_{t-1} + b_2 y_{t-2} = g(t)$$

- Then,
  - Any trajectory will converge iff $|m_1| < 1$ *and* $|m_2| < 1$, or equivalently
  - Any trajectory will converge iff $|b_1| < (1 + b_2)$ *and* $b_2 < 1$

Second-order difference equations (14)

Determining A_1 and A_2

- Two additional conditions needed.
- Usually, value of y_t at two moments in time.
- Often at $t = 0 \rightarrow y_0$ and $t = 1 \rightarrow y_1$

Illustration (Gandolfo, ch 4)

- Let $y_0 = 0,\ y_1 = -2,\ b_1 = 1.8,\ b_2 = 0.8$
- The equation to solve becomes: $y_t + 1.8y_{t-1} + 0.8y_{t-2} = 0$
- Characteristic equation: $m^2 + 1.8m + 0.8 = 0$
- Roots: $m_1 = -1,\ m_2 = -0.8$
- Remark: $m_i < 0, |m_2| < 1$ but $|m_1| = 1 \Rightarrow$ cyclical non-convergent trajectory.
- General solution of difference equation:
  $y_t = A_1(-1)^t + A_2(-0.8)^t$

Second-order difference equations (15)

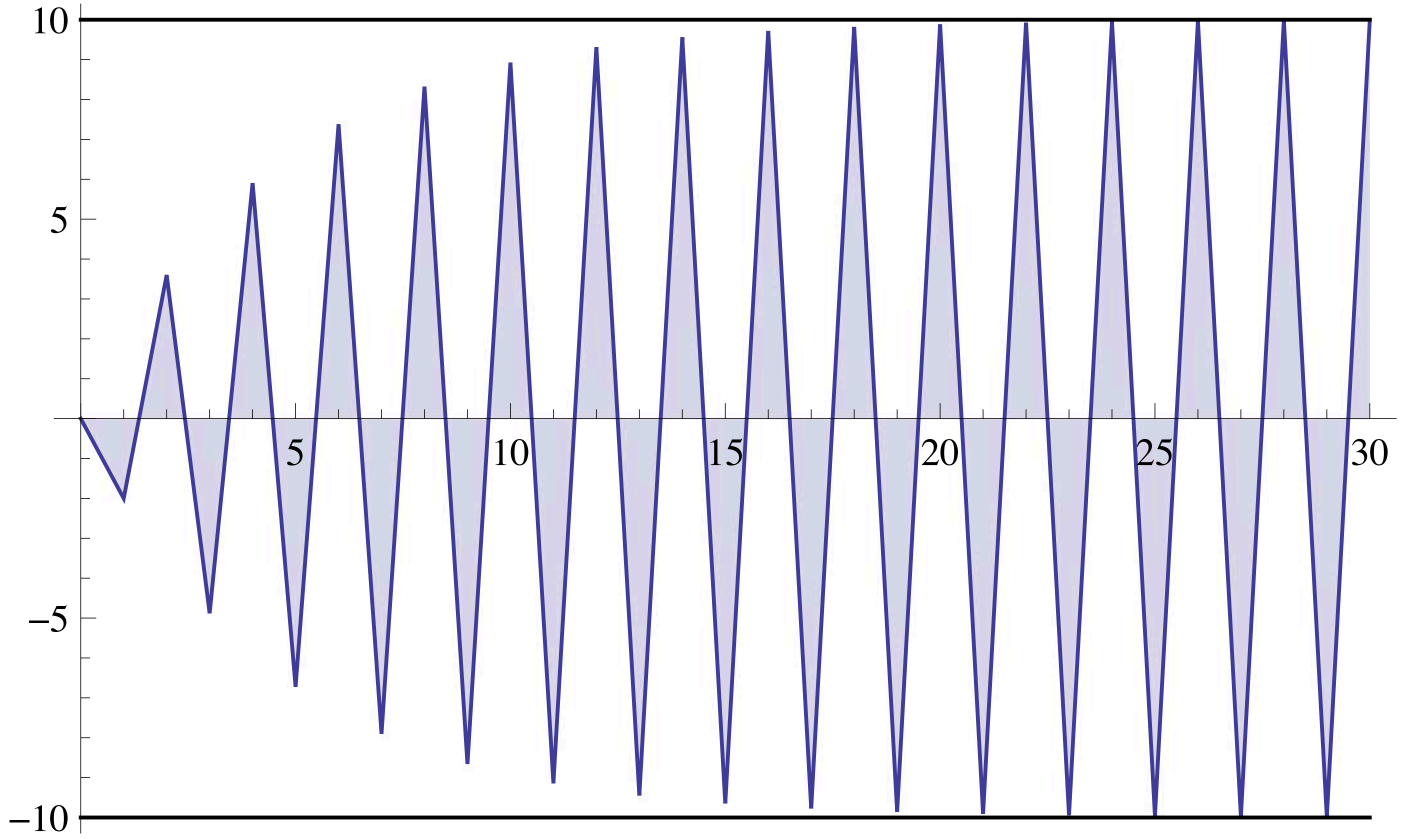
Illustration (Gandolfo, ch 4) (cont'd)

- Substitute values of y_0 and y_1 to obtain

$$\left.\begin{aligned} 0 &= A_1 + A_2 \\ -2 &= -A_1 - 0.8A_2 \end{aligned}\right\} \Rightarrow A_1 = 10,\ A_2 = -10$$

- Finally, $y_t = 10(-1)^t - 10(0.8)^t$
- Cyclical with increasing amplitude until reaching a limit cycle given by $(-10, 10)$:
  - $\lim_{t\to\infty}(-10(0.8)^t) = 0$
  - $10(-1)^t = \{-10, 10\}$
  - see figure and table of values

Second-order difference equations (16)



Second-order difference equations (16bis)

t	0	1	2	3	4	5	6	7	8	9
y(t)	0	-2	3.6	-4.88	5.90	-6.72	7.38	-7.90	8.32	-8.6
t	10	11	12	13	14	15	16	17	18	19
y(t)	8.93	-9.14	9.31	-9.45	9.56	-9.65	9.72	-9.77	9.81	-9.8
t	20	21	22	23	24	25	26	27	28	29
y(t)	9.88	-9.91	9.93	-9.94	9.95	-9.96	9.97	-9.97	9.98	-9.9

Second-order difference equations (17)

Determining A_1 and A_2 Illustration 2

- Let $y_0 = 0,\ y_1 = -2,\ b_1 = 1,\ b_2 = 1/4$
- The equation to solve becomes: $y_t + y_{t-1} + \frac{1}{4} y_{t-2} = 0$
- Characteristic equation: $m^2 + m + \frac{1}{4} = 0$
- Roots: $m_1 = m_2 = \widehat{m} = \frac{-1}{2}$
- Remark1: $\widehat{m} < 0$ and $|\widehat{m}| < 1 \Rightarrow$ cyclical convergent trajectory.
- Remark2: A second solution is $t\widehat{m}$.

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Second-order difference equations (18)

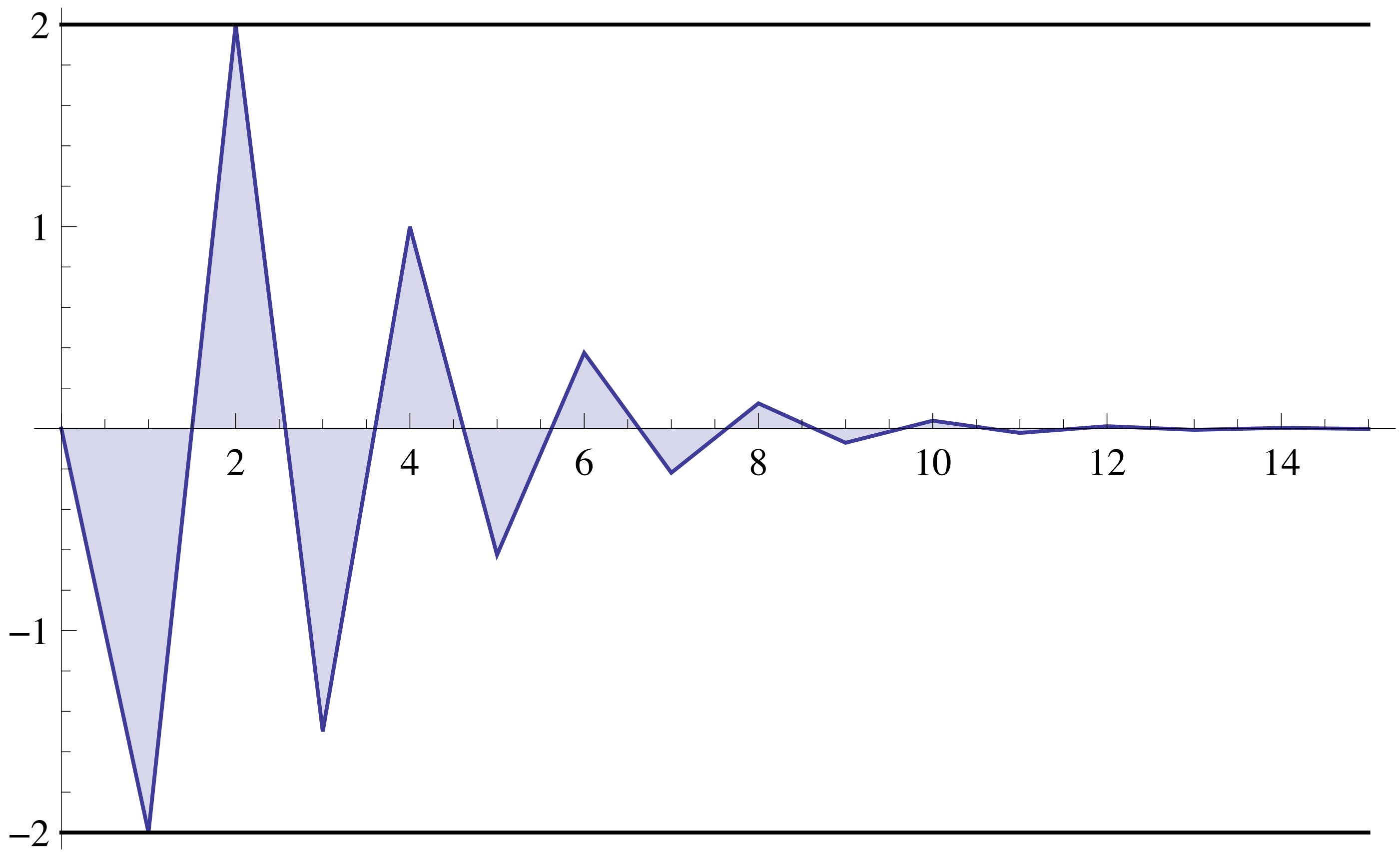
Illustration 2 (cont'd)

- General solution of homogeneous difference equation:
  $y_t = (A_1 + tA_2)\left(\frac{-1}{2}\right)^t$
- Substitute values of y_0 and y_1 to obtain

$$\left.\begin{aligned} 0 &= A_1 \\ -2 &= (A_1 + A_2)\left(\frac{-1}{2}\right) \end{aligned}\right\} \Rightarrow A_1 = 0, \;\; A_2 = 4$$

- Finally, $y_t = 4t(\frac{-1}{2})^t$

Second-order difference equations (19)



Second-order difference equations (20)

Determining A_1 and A_2 Illustration 3

- Let $y_0 = 0,\ y_1 = -2,\ b_1 = -2,\ b_2 = 3/4$
- The equation to solve becomes: $y_t - 2y_{t-1} + \frac{3}{4}y_{t-2} = 0$
- Characteristic equation: $m^2 - 2m + \frac{3}{4} = 0$
- Roots: $m_1 = 3/2,\ m_2 = 1/2$
- Remark: $m_i > 0$, therefore, monotonic trajectory. Also $m_1 > 1 \Rightarrow$ monotonic divergent trajectory.

Second-order difference equations (21)

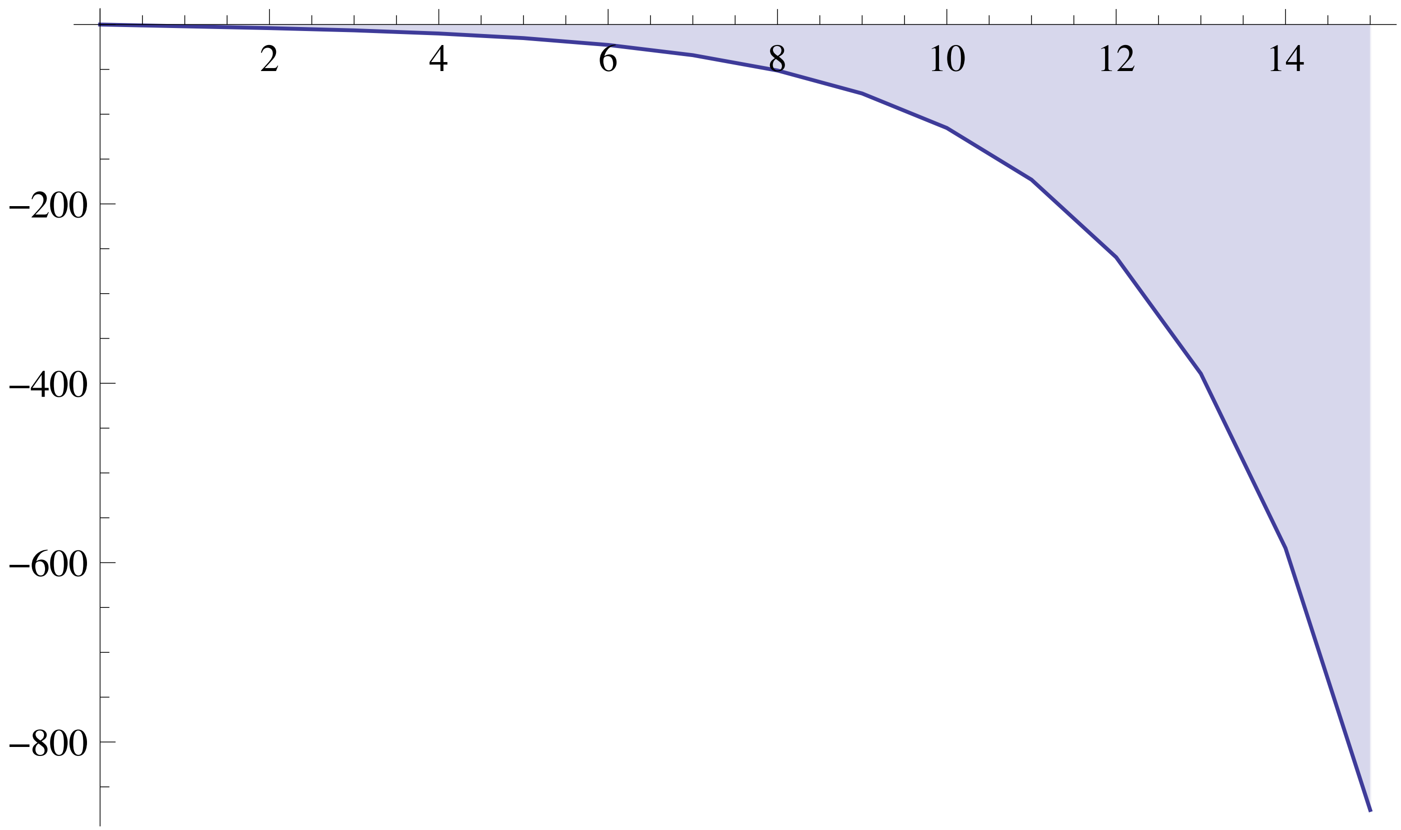
Illustration 2 (cont'd)

- General solution of homogeneous difference equation: $y_t = A_1\left(\frac{3}{2}\right)^t + A_2\left(\frac{1}{2}\right)^t$
- Substitute values of y_0 and y_1 to obtain

$$\left.\begin{aligned} 0 =& A_1 + A_2 \\ -2 =& A_1(3/2) + A_2(1/2) \end{aligned}\right\} \Rightarrow A_1 = -2,\ A_2 = 2$$

- Finally, $y_t = (-2)\left(\frac{3}{2}\right)^t + 2\left(\frac{1}{2}\right)^t$

Second-order difference equations (22)



Second-order difference equations (23)

Determining A_1 and A_2 Illustration 4

- Let $y_0 = 0,\ y_1 = -2,\ b_1 = -3/2,\ b_2 = 35/64$
- The equation to solve becomes: $y_t - \frac{3}{2}y_{t-1} + \frac{35}{64}y_{t-2} = 0$
- Characteristic equation: $m^2 - \frac{3}{2}m + \frac{35}{64} = 0$
- Roots: $m_1 = 7/8,\ m_2 = 5/8$
- Remark: $m_i > 0$, therefore, monotonic trajectory. Also $m_i < 1 \Rightarrow$ monotonic convergent trajectory.

Second-order difference equations (24)

Illustration 4 (cont'd)

- General solution of homogeneous difference equation:
  $y_t = A_1 \left(\frac{7}{8}\right)^t + A_2 \left(\frac{5}{8}\right)^t$
- Substitute values of y_0 and y_1 to obtain

$$\left.\begin{aligned} 0 &= A_1 + A_2 \\ -2 &= A_1(7/8) + A_2(5/8) \end{aligned}\right\} \Rightarrow A_1 = -8,\ A_2 = 8$$

- Finally, $y_t = (-8)\left(\frac{3}{2}\right)^t + 8\left(\frac{1}{2}\right)^t$

Second-order difference equations (25)