Optimization. A first course of mathematics for economists

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III.1 Economic Dynamics - Differential equations

Economic dynamics - Motivation

- Economic phenomena develop along a time dimension.
- Static analysis simplifies the analysis and provides "correct" intuitions on the properties of equilibrium...
- ... but can not assess
  - (i) the path to the equilibrium
  - (ii) alternative policies to reach the desired equilibrium
  - (iii) the social welfare consequences of the different paths
  - ... etc, etc.
- This is the realm of the economic dynamics.

Economic dynamics - Definitions

- **Def.**[Dynamic system]: A system is called dynamic if its behavior along time is determined by functional equations containing variables representing different moments in time.
- **Def.**[Functional equation]: A functional equation is a function in which the unknown is an equation.
  - An example of a functional equation is $y'(x) - y(x) = 0$. The solution to this functional equation is $y(x) = Ae^x$, because $y'(x) = Ae^x$ and the equation is satisfied $\forall x$. When x represent time, we will denote it by t.
  - Note that we are looking for a function verifying $y'(x) = y(x)$. Also, remember that $y(x) = \int y'(x)$. Thus we are looking for a function satisfying $y'(x) = \int y'(x)$, i.e. the solution of a differential equation is an integral.
- **Def.**[Differential equation]: A differential equation is a functional equation containing derivatives of the unknown.

Economic dynamics - Definitions (2)

- **Def.**[Ordinary differential equation]: An ordinary differential equation (ODE) is a differential equation containing a function of one independent variable and its derivatives.
- **Def.**[Partial differential equation]: A partial differentiable equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives.
- **Def.**[Order of a differential equation]: The order of a differential equation is given by the highest order derivative defining it.

ODE vs PDE

ODE

- Consider a function $y = y(x)$
- General formulation of an ODE:
  $F(x, y(x), y'(x), \ldots, y^n(x)) = 0$
- Unknown: $y(x)$
- example: $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + y^2 = \cos x$

PDE

- Consider a function $y = y(x, t, u)$
- General formulation of a PDE:
  $F(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial u}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 y}{\partial u^2}, \ldots, \frac{\partial^n y}{x^n}, \frac{\partial^n y}{t^n}, \frac{\partial^n y}{u^n}, x, t, u) = 0$
- Unknown: $y(x, t, u)$
- example: $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial u^2} - y$

Economic dynamics - Definitions (3)

- **Def.**[Solution of a differential equation]: The solution of a differentiable function is a function verifying the equation.
  - Usually, a differential equation admits as solution an infinite family of solutions characterized by the values of some parameters. This family of functions is called the general solution of the differential equation.
  - The constants identifying the different elements of this family of solutions are called constants of integration.
  - Each element of the general solution is called a particular solution of the differential equation.
  - The most common differential equation used in economic dynamics are the so-called Constant-Coefficient Linear Ordinary Differential Equations

Two simple examples

Example 1: Solve the differential equation $y'(t) = a, \ a \in \mathbf{R}$.

- Its general solution is a function $y(t)$ verifying $y'(t) = a, \forall t$.
  $y(t) = \int y'(t) = \int a \, dt = at + C, \ C \in \mathbf{R}$
- This is the general solution. Each value of C constitutes a particular solution of the differential equation.

Example 2: Solve the differential equation $y''(t) = a, \ a \in \mathbf{R}$.

- Integrating two successive times we obtain

$$y'(t) = \int a \, dt = at + b, \quad b \in \mathbf{R}$$

$$y(t) = \int (at + b) dt = \frac{1}{2} at^2 + bt + c, \quad (b, c) \in \mathbf{R}$$

- This is the general solution. Each value of (b, c) constitutes a particular solution of the differential equation.

General formulation

- The general Constant-Coefficient Linear Ordinary Differential Equation (that we will refer to as differential equation) is
  $a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = g(t) \quad [\alpha]$
  where
  - $y^{(n)}(t), y^{(n-1)}(t), \ldots$ represent the derivatives of order $n, n-1, \ldots$ of $y(t)$;
  - $a_j, \; j = 0, 1, 2, \ldots n$ are given real constants
  - $g(t)$ is a known function.
- Note that this differential equation is linear, its only independent variable is t, and its coefficients are constant. Accordingly it describes a Constant-Coefficient Linear Ordinary Differential Equation.
- Equation $[\alpha]$ is called the non-homogeneous equation.
- Its corresponding homogeneous equation is given by
  $a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y(t) = 0 \quad [\beta]$

Solving a differential equation - Three theorems

Theorem 1:

- Let $y_1(t)$ be a solution of $[\beta]$.
  Then, $Ay_1(t)$ where A is an arbitrary constant, is also a solution of $[\beta]$.

Theorem 2:

- Let $y_1(t)$ and $y_2(t)$ be two (linearly independent) solutions of $[\beta]$.
  Then, $A_1y_1(t) + A_2y_2(t)$ is also a solution of $[\beta]$ for any arbitrary constants A_1, A_2.

Theorem 3:

- Let $f(t; A_1, A_2, \ldots, A_n)$ be the general solution of $[\beta]$ where $A_j, j = 1, \ldots n$ are arbitrary constants. Let $\overline{y}(t)$ be a particular solution of $[\alpha]$.
  Then, $\overline{y}(t) + f(t; A_1, A_2, \ldots, A_n)$ is the general solution of $[\alpha]$.

Solving a differential equation - Remarks

- Proofs are left as exercises
- The particular solution $\overline{y}(t)$ will depend *ceteris paribus* of the functional form of $g(t)$
- Thus, a (general) two-step procedure to identify the solution is proposed
  1. propose $\overline{y}(t)$ with the same functional form as $g(t)$ with undetermined coefficients.
  2. substitute that function in the non-homogeneous equation and determine the coefficients allowing for satisfying the differential equation.

First-order differential equations

The differential equation

- The general formulation is $a_0 y'(t) + a_1 y(t) = g(t), \quad a_0 \neq 0$

Case 1: $a_1 = 0$

- The equation reduces to $a_0 y'(t) = g(t)$
- Thus, $y'(t) = \frac{1}{a_0} g(t)$
- and $y(t) = \int y'(t) = \int (\frac{1}{a_0} g(t)) dt = \frac{1}{a_0} \int g(t) dt + C$

First-order differential equations (2)

Case 2: $a_1 \neq 0$

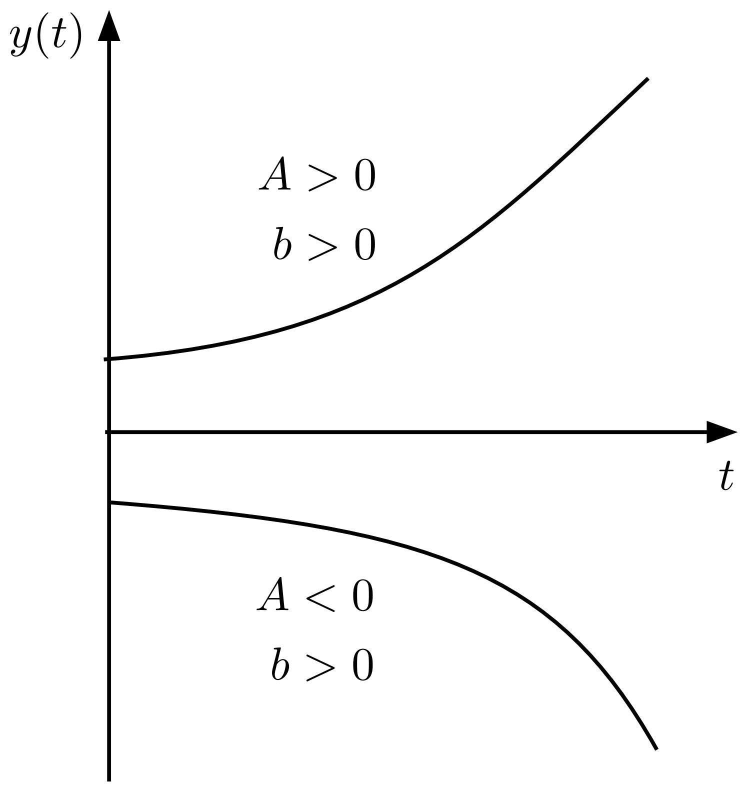
1. Consider the corresponding homogeneous equation
   $a_0 y'(t) + a_1 y(t) = 0$
2. Rewrite it as $y'(t) + by(t) = 0$ with $b = a_1/a_0$ $\qquad [\gamma]$
3. Equivalently, $\frac{y'(t)}{y(t)} = -b$
4. Recall that $\frac{y'(t)}{y(t)} = \frac{d \log y(t)}{dt}$
5. Integrating on both sides,
   - $\int \left(\frac{d \log y(t)}{dt} \right) dt = \log y(t)$
   - $\int -b dt = -bt + C$
   - Thus, $\log y(t) = -bt + C$
6. Taking antilogs $y^h(t) = e^{-bt+C} = e^C e^{-bt} \equiv Ae^{-bt}$

First-order differential equations (3)

Case 2: $a_1 \neq 0$ (cont'd)

- $y^h(t) = Ae^{-bt}$ is the general solution of the homogeneous equation.
- To determine the value of A we need an additional condition. For example the value of the function at the initial period ($t = 0$). Denote it as $y(0) = y_0$. Then $A = y_0$.
- The behavior of $y(t)$ depends on the sign of A and b:
  - If $b < 0$ then $-b > 0$ and e^{-bt} increases monotonically in t.
  - Therefore, If $A > 0$, then $Ae^{-bt} \to +\infty$
  - while if $A < 0$, then $Ae^{-bt} \to -\infty$
  - If $b > 0$ then $-b < 0$ and e^{-bt} decreases monotonically in t.
  - Therefore, $Ae^{-bt} \to 0$
- see figure.

First-order differential equations (4)



First-order differential equations (5)

Case 2: $a_1 \neq 0$ (cont'd)

1. Find a particular solution for the non-homogeneous equation $a_0 y'(t) + a_1 y(t) = g(t)$
2. Case (i): $g(t)$ is a constant function: $g(t) = a, \;\; a \in \mathbf{R}$
   - The non-homogeneous equation is now $a_0 y'(t) + a_1 y(t) = a \qquad [\delta]$
   - As described before, try a particular solution with the structure of $g(t)$. Thus, $\overline{y}(t) = \mu, \quad \mu \in \mathbf{R} \qquad [\varepsilon]$
   - Substituting $[\varepsilon]$ into $[\delta]$ we have $0 + a_1\mu = a$ so that $\mu = a/a_1$ and $\overline{y}(t) = a/a_1$.
   - Solution of the equation: $y(t) = Ae^{-bt} + \frac{a}{a_1}$

     Remark: If $a_1 = 0$, take $\overline{y}(t) = \mu t$ and substitute as before. Then, $a_0\mu = a, \mu = a/a_0, \overline{y}(t) = at/a_0$ and $y(t) = Ae^{-bt} + at/a_0$.

First-order differential equations (6)

Case 2i: Example

- Solve $y'(t) + 2y(t) = 8$
- Solve the homogeneous equation: $y'(t) + 2y(t) = 0$
  - $y'(t)/y(t) = -2 \to \int \frac{d \log y(t)}{dt} = \int -2dt \to \log y(t) = -2t + C$
  - taking antilogs, $y^h(t) = e^{-2t+C} = e^C e^{-2t} = Ae^{-2t}$
- Find a particular solution of the non-homogeneous equation
  - Let $\overline{y}(t) = \mu$
  - Substituting, $2\mu = 8 \to \mu = 4$ so that $\overline{y}(t) = 4$
- Solution of the differential equation: $y(t) = Ae^{-2t} + 4$
- Note trajectory of $y(t) \to 0 \; \forall A$

First-order differential equations (7)

Case 2: $a_1 \neq 0$ (cont'd)

3. Case (ii): $g(t)$ is exponential: $g(t) = Be^{st}, \ (B, s) \in \mathbf{R}$
   - The non-homogeneous equation is now
     $a_0 y'(t) + a_1 y(t) = Be^{st} \qquad [\delta']$
   - Try a particular solution $\overline{y}(t) = Ce^{st}, \quad C \in \mathbf{R} \qquad [\varepsilon']$
   - Substituting $[\varepsilon']$ into $[\delta']$ we have
     $a_0 sCe^{st} + a_1 Ce^{st} = Be^{st}$ or $e^{st}\Big(C(a_o s + a_1) - B\Big) = 0$
   - This expression will be verified $\forall t$ iff
     $\Big(C(a_o s + a_1) - B\Big) = 0$ or $C = \frac{B}{a_0 s + a_1}$; thus, $\overline{y}(t) = \frac{Be^{st}}{a_0 s + a_1}$
   - Solution of the differential equation: $y(t) = Ae^{-2t} + \frac{Be^{st}}{a_0 s + a_1}$

     Remark: If $a_0 s + a_1 = 0$ take $\overline{y}(t) = tCe^{st}$ so that
     $C = B/a_0, \overline{y}(t) = \frac{tBe^{st}}{a_0}$ and $y(t) = Ae^{-2t} + \frac{tBe^{st}}{a_0}$

First-order differential equations (8)

Case 2ii: Example

- Solve $y'(t) + 2y(t) = \frac{1}{2}e^{3t}$
- Solve the homogeneous equation: $y'(t) + 2y(t) = 0$
  - $y'(t)/y(t) = -2 \rightarrow \int \frac{d \log y(t)}{dt} = \int -2dt \rightarrow \log y(t) = -2t + C$
  - taking antilogs, $y^h(t) = e^{-2t+C} = e^C e^{-2t} = Ae^{-2t}$
- Find a particular solution of the non-homogeneous equation
  - Let $\overline{y}(t) = Ce^{3t}$
  - Substituting, $3Ce^{3t} + 2Ce^{3t} = \frac{1}{2}e^{3t}$ so that $C = \frac{1}{10}$ and $\overline{y}(t) = \frac{1}{10}e^{3t}$
- Solution of the differential equation: $y(t) = Ae^{-2t} + \frac{1}{10}e^{3t}$

First-order differential equations (9)

Case 2: $a_1 \neq 0$ (cont'd)

4. Case (iii): $g(t)$ is polynomial (illustration for degree 1): $g(t) = c_0 + c_1 t,\ (c_0, c_1) \in \mathbf{R}$
   - The non-homogeneous equation is now $a_0 y'(t) + a_1 y(t) = c_0 + c_1 t \qquad [\delta'']$
   - Try a particular solution $\overline{y}(t) = \alpha + \beta t \qquad [\varepsilon'']$
   - Substituting $[\varepsilon'']$ into $[\delta'']$ we have $a_0\beta + a_1(\alpha + \beta t) = c_0 + c_1 t$ or $(a_1\beta - c_1)t + a_0\beta + a_1\alpha - c_0 = 0$
   - This expression will be verified $\forall t$ iff $a_1\beta - c_1 = 0$ and $a_0\beta + a_1\alpha - c_0 = 0$. That is iff $\alpha = \frac{a_1c_0 - a_0c_1}{a_1^2}$ and $\beta = \frac{c_1}{a_1}$
   - Solution of the differential equation: $y(t) = Ae^{-2t} + \left(\frac{a_1c_0 - a_0c_1}{a_1^2} + \frac{tc_1}{a_1}\right)$
   - Remark If $a_1 = 0$, take $\overline{y}(t) = \alpha t + \beta t^2$ so that $\alpha = c_0/a_0$ and $\beta = c_1/2a_0$

First-order differential equations (10)

Case 2iii: Example

- Solve $y'(t) + 2y(t) = 1 - 4t$
- Solve the homogeneous equation: $y'(t) + 2y(t) = 0$
  - $y'(t)/y(t) = -2 \rightarrow \int \frac{d \log y(t)}{dt} = \int -2dt \rightarrow \log y(t) = -2t + C$
  - taking antilogs, $y^h(t) = e^{-2t+C} = e^C e^{-2t} = Ae^{-2t}$
- Find a particular solution of the non-homogeneous equation
  - Let $\overline{y}(t) = \alpha + \beta t$
  - Substituting, $\beta + 2(\alpha + \beta t) = 1 - 4t$ so that $\alpha = 3/2, \ \beta = -2, \ \overline{y}(t) = \frac{3}{2} - 2t$
- Solution of the differential equation: $y(t) = Ae^{-2t} + \frac{3}{2} - 2t$

First-order differential equations (11)

Case 2: $a_1 \neq 0$ (cont'd)

5. Case (iv): $g(t)$ is trigonometric:
   $g(t) = B_1 \cos wt + B_2 \sin wt, \;\; B_1, B_2, w \in \mathbf{R}$
   - The non-homogeneous equation is now
     $a_0 y'(t) + a_1 y(t) = B_1 \cos wt + B_2 \sin wt \qquad [\delta''']$
   - Try a particular solution $\overline{y}(t) = \alpha \cos wt + \beta \sin wt \qquad [\varepsilon''']$
   - Substituting $[\varepsilon''']$ into $[\delta''']$ we have
     $\cos wt(a_0 \beta w + a_1 \alpha - B_1) + \sin wt(a_1 \beta - a_0 \alpha w - B_2) = 0$
   - This expression will be verified $\forall t$ iff $\alpha = \frac{a_1 B_1 - a_0 w B_2}{a_1^2 + a_0^2 w^2}$ and
     $\beta = \frac{a_1 B_2 + a_0 w B_1}{a_1^2 + a_0^2 w^2}$
   - Solution of the differential equation:
     $y(t) = Ae^{-2t} + \left(\frac{a_1 B_1 - a_0 w B_2}{a_1^2 + a_0^2 w^2}\right) \cos wt + \left(\frac{a_1 B_2 + a_0 w B_1}{a_1^2 + a_0^2 w^2}\right) \sin wt$
   - Remark If $a_1^2 + a_0^2 w^2 = 0$ take $\overline{y}(t) = \alpha t \cos wt + \beta t \sin wt$ so that $\alpha = B_1/a_0$ and $\beta = B_2/a_0$.

First-order differential equations (12)

Case 2iv: Example

- Solve $y'(t) + 2y(t) = 3\sin 2t - 2\cos 2t$
- Solve the homogeneous equation: $y'(t) + 2y(t) = 0$
  - $y'(t)/y(t) = -2 \to \int \frac{d\log y(t)}{dt} = \int -2dt \to \log y(t) = -2t + C$
  - taking antilogs, $y^h(t) = e^{-2t+C} = e^C e^{-2t} = Ae^{-2t}$
- Find a particular solution of the non-homogeneous equation
  - Let $\overline{y}(t) = \alpha \sin 2t + \beta \cos 2t$
  - Substituting, $\cos 2t(2\alpha + 2\beta + 2) + \sin 2t(2\alpha - 2\beta - 3) = 0$ so that $\alpha = -\frac{1}{4},\ \beta = -\frac{3}{4},\ \overline{y}(t) = -\frac{1}{4}\sin 2t - \frac{3}{4}\cos 2t$
- Solution of the differential equation: $y(t) = Ae^{-2t} - \frac{1}{4}\sin 2t - \frac{3}{4}\cos 2t$

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First-order differential equations (13)

- The four specifications of $g(t)$ cover most economic applications.
- The general solution of the differential equation is $y(t) = Ae^{-bt} + \overline{y}(t)$ (recall Theorem 3).
- What is left is to determine the value of A.
- We need an additional condition. Let it be $y(t) = \tilde{y}$, for $t = \tilde{t}$.
- Then, $\tilde{y} = Ae^{-b\tilde{t}} + \overline{y}(\tilde{t})$ or
- $A = e^{-b\tilde{t}}[\tilde{y} - \overline{y}(\tilde{t})]$
- Often in economic applications we take the value of y at $t = 0$.
- Let $y(0) = y_0$. Then, $A = y_0 - \overline{y}_0$.

First-order differential equations (14)

Interpretation of the solution in economic applications

1. General solution: $y(t) = (y_0 - \overline{y}_0)e^{-bt} + \overline{y}(t)$
2. $\overline{y}(t)$ is the equilibrium value of y:
   - if $\overline{y}(t)$ is constant $\to$ stationary equilibrium
   - if $\overline{y}(t)$ is a function of t $\to$ dynamic equilibrium
3. $(y_0 - \overline{y}_0)e^{-bt} = y(t) - \overline{y}(t)$ represent the evolution along time of the value of y from $t = 0$ until reaching the equilibrium value.

Stability of the perfectly competitive equilibrium

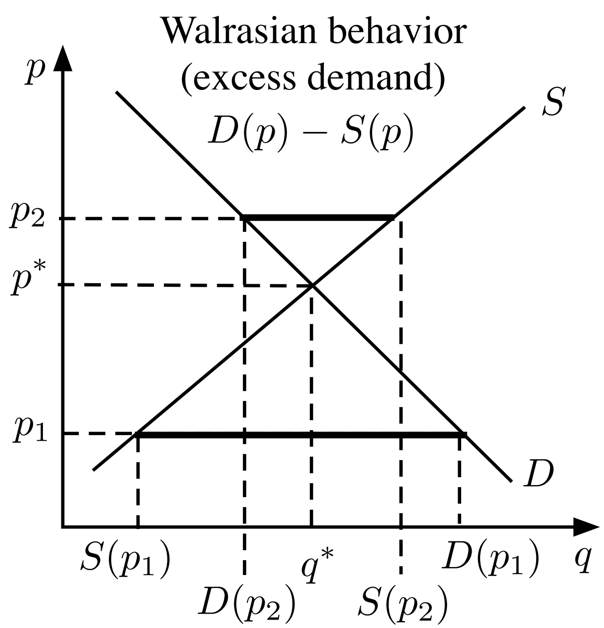
Set-up

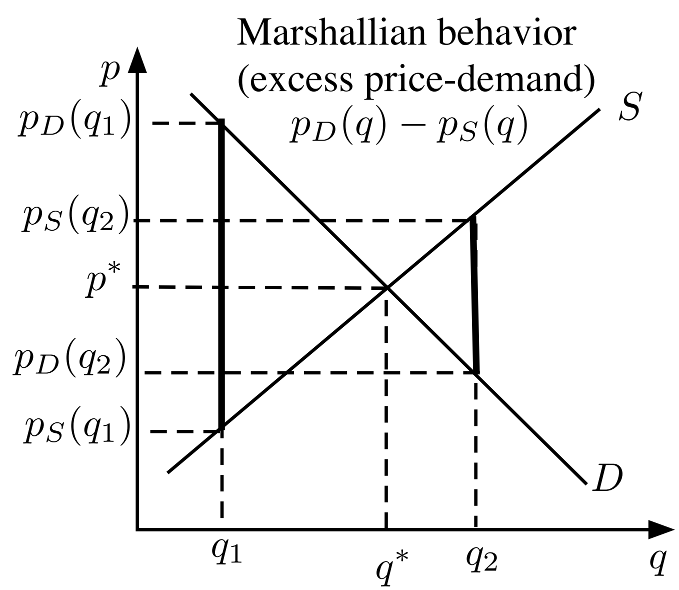
- Market equilibrium as interaction of demand and supply
- Initial state: market in equilibrium; then a shock $\rightarrow$ away from equilibrium
- Question 1: will the forces acting on demand and supply drive the market back to an equilibrium?
- Question 2: If so, will it be the previous equilibrium? (stability)
- Two concepts of stability
  - Static stability: tells us whether market forces will recover the equilibrium; but not the path the market will follow until recovering the equilibrium;
  - Dynamic stability describes the path from the initial out-of-equilibrium situation towards the equilibrium.

Static Stability

Behavioral assumption

- walrasian behavior: price tends to increase (decrease) under excess demand (supply): $\frac{dED(p)}{dp} = \frac{d[D(p)-S(p)]}{dp} > 0.$
- marshallian behavior: quantity tends to $\uparrow$ $(\downarrow)$ under excess price-demand (supply): $\frac{dEpD(q)}{dq} = \frac{d[p_D(q)-p_S(q)]}{dq} < 0.$





Dynamic Stability

Walrasian behavioral assumption

- Assume the price of a market evolves with time, $p \equiv p(t)$ with $p(0) \equiv p_0$.
- Assume price is the only variable adjusting over time. At each instant demand and supply adjust immediately their values according to the information provided by the price (i.e. only price adjusts in time, not quantities).
- Let $ED(p)$ denote the excess demand function at a price p
- Assume $\exists p^*$ s.t. $ED(p^*) = 0$, so that p^* is the market equilibrium price.
- Question: how price evolves in time when deviates from p^*?
- Assume that at $t = 0, ED(p_0) \neq 0$.

Dynamic Stability (2)

Walrasian behavioral assumption (cont'd)

- Assume the greater the excess demand, the greater will be the change of price.
- Formally, the price adjustment process is modeled with a differential equation $p'(t) = f(ED(p(t)))$ with $f(0) = 0$ and $f' > 0$.
- The stability result depends on the particular properties of the demand and supply functions. To illustrate, assume,

$$
\begin{aligned}
D(p(t)) =& a + bp \\
S(p(t)) =& a_1 + b_1 p, \text{ so that} \\
ED(p(t)) =& (a - a_1) + (b - b_1)p(t)
\end{aligned}
$$

Then, $p^* = \frac{a-a_1}{b_1-b} > 0$ (by assumption)

Dynamic Stability - Walrasian behavior

The differential equation

- Assume the differential equation describing the evolution of the price is linear, i.e. $p'(t) = cED(p(t))$. As c is a constant it does not have any impact on the stability analysis, so that we take $c = 1$ and
  $p'(t) = (a - a_1) + (b - b_1)p(t) = p^*(b_1 - b) + (b - b_1)p(t)$. Hence,
  $p'(t) = (b - b_1)(p(t) - p^*)$
- This is a first-order linear differential equation.
- Write it as
  $p'(t) - (b - b_1)p(t) = -(b - b_1)p^*$
  so that $g(t) = -(b - b_1)p^*$ is a constant function.

Dynamic Stability - Walrasian behavior (2)

Solving the differential equation

- General solution of the homogeneous equation $p'(t) - (b - b_1)p(t) = 0$ is given by $p^h(t) = Ae^{(b-b_1)t}$
- Particular solution of the non-homogeneous equation $p'(t) - (b - b_1)p(t) = -(b - b_1)p^*$ is given by $\overline{p}(t) = p^*$
- General solution of the non-homogeneous equation is thus $p(t) = Ae^{(b-b_1)t} + p^*$
- The equilibrium will be stable if as $t \to \infty$, then $p(t) \to p^*$.
- Equivalently, stability requires $Ae^{(b-b_1)t} \to 0$. In turn, this condition will hold whenever $(b - b_1) < 0$.
- Remark: Note that $b = \frac{dD(p)}{dp}$ and $b_1 = \frac{dS(p)}{dp}$. If demand is downward sloping ($b < 0$) and supply is upward sloping ($b_1 > 0$), it follows that $(b - b_1) < 0$ and the equilibrium will always be stable.

Second-order differential equations

The differential equation

- The general formulation is
$a_0 y''(t) + a_1 y'(t) + a_2 y(t) = g(t), \quad a_0 \neq 0$

The homogeneous equation

- The associated homogeneous equation is:
$a_0 y''(t) + a_1 y'(t) + a_2 y(t) = 0$
- Re-write it as
$y''(t) + b_1 y'(t) + b_2 y(t) = 0$, where $b_1 = a_1/a_0$ and $b_2 = a_2/a_0$
- To solve the homogeneous equation, we follow an analogous argument as in the 1st-order differential equations.
- Thus, we conjecture that the solution will be of the type
$y^h(t) = e^{\lambda t}$ where λ is a constant to be determined.
- Accordingly, $y'(t) = \lambda e^{\lambda t}$ and $y''(t) = \lambda^2 e^{\lambda t}$.

Second-order differential equations (2)

The homogeneous equation (cont'd)

- Substituting, the homogeneous equation reads now,
  $\lambda^2 e^{\lambda t} + b_1 \lambda e^{\lambda t} + b_2 e^{\lambda t} = 0$ or
  $e^{\lambda t}(\lambda^2 + b_1 \lambda + b_2) = 0$

Solving the homogeneous equation

- The proposed solution $y(t) = e^{\lambda t}, \forall t$ requires that it will also solve the homogeneous equation.
- Accordingly, it must be the case that $\lambda^2 + b_1 \lambda + b_2 = 0$
- The values of λ satisfying this equation are
  $(\lambda_1, \lambda_2) = \frac{1}{2}\left(-b_1 \pm \sqrt{b_1^2 - 4b_2}\right)$
- The solution of the homogeneous equation depends on the properties of these roots.
- Let $\Delta \equiv b_1^2 - 4b_2$. Three cases: $\Delta \gtreqless 0$.

Second-order differential equations (3)

Solving the homogeneous equation. Case 1: $\Delta > 0$

- In this case we have two real roots. Thus, both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ satisfy $y''(t) + b_1 y'(t) + b_2 y(t) = 0$
- Accordingly, the general solution of the homogeneous equation is
  $y^h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ where A_1, A_2 are arbitrary constants.
- The evolution of $y(t)$ as $t \to \infty$ is monotonic. The stability of the solution depends on the sign of the roots.
- To assess the sign of the roots we appeal to Descartes' rule of signs:

  Let $P(x)$ be a polynomial with real coefficients and terms in descending powers of x. (a) The number of positive real zeros of $P(x)$ is smaller than or equal to the number of variations in sign occurring in the coefficients of $P(x)$. (b) The number of negative real zeros of $P(x)$ is smaller than or equal to the number of continuations in sign occurring in the coefficients of $P(x)$.

Second-order differential equations (4)

Solving the homogeneous equation. Case 1: $\Delta > 0$ (cont'd)

- Recall the quadratic equation we are studying is $\lambda^2 + b_1\lambda + b_2 = 0$
- If $b_1 > 0$ and $b_2 > 0$, there are two continuations of sign. Therefore, the two roots $\lambda_1 < 0$ and $\lambda_2 < 0$. Accordingly, $y(t) \to 0$ as $t \to \infty$. In words, $y(t)$ follows a monotone convergent trajectory towards zero.
- If $b_1 < 0$ and $b_2 < 0$, there is one continuation and one variation of sign. Therefore, one root will be positive and the other negative. The term with the positive root will diverge as $t \to \infty$. Accordingly, $y(t)$ follows a monotone divergent trajectory. The same argument holds if $b_1 > 0$ and $b_2 < 0$.
- If $b_1 < 0$ and $b_2 > 0$, there are two variations of sign. Therefore, the two roots $\lambda_1 > 0$ and $\lambda_2 > 0$. Accordingly, $y(t)$ follows a monotone divergent trajectory.

Second-order differential equations (5)

Solving the homogeneous equation. Case 1: $\Delta > 0$ (cont'd)

- If $b_1 \neq 0$ and $b_2 = 0$, one root is zero. The other root $= -b_1$. Accordingly, $y(t)$ follows a monotone divergent trajectory if $b_1 < 0$ and monotone convergent towards A_1 if $b_1 > 0$.
- If $b_1 = 0$ and $b_2 < 0$, both roots are equal but with different sign. In this case $y(t)$ follows a monotone divergent trajectory (see Case 2).

Δ	b_1	b_2	$y(t)$
+	+	+	convergent,monotone,0
	-	-	divergent,monotone
	+	-	divergent,monotone
	-	+	divergent,monotone
	+	0	convergent, monot, A_1
	-	0	divergent, monotone
	0	-	divergent, monotone

Second-order differential equations (6)

Solving the homogeneous equation. Case 2: $\Delta = 0$

- In this case $\lambda_1 = \lambda_2 = \hat{\lambda} = -\frac{1}{2}b_1$
- $y^h(t) = e^{\hat{\lambda}t}$ is a general solution of the homogeneous equation.
- Another general solution of the homogeneous equation is $te^{\hat{\lambda}t}$. To verify it compute,
  - $y'(t) = e^{\hat{\lambda}t} + t\hat{\lambda}e^{\hat{\lambda}t}$
  - $y''(t) = 2\hat{\lambda}e^{\hat{\lambda}t} + t\hat{\lambda}^2e^{\hat{\lambda}t}$
  - Substituting in the homogeneous equation, we obtain $(2\hat{\lambda}e^{\hat{\lambda}t} + t\hat{\lambda}^2e^{\hat{\lambda}t}) + b_1(e^{\hat{\lambda}t} + t\hat{\lambda}e^{\hat{\lambda}t}) + b_2te^{\hat{\lambda}t} = 0$
  - Rewrite it as $(2\hat{\lambda} + b_1)e^{\hat{\lambda}t} + (\hat{\lambda}^2 + b_1\hat{\lambda} + b_2)te^{\hat{\lambda}t} = 0$

Second-order differential equations (7)

Solving the homogeneous equation. Case 2: $\Delta = 0$ (cont'd)

- If $te^{\hat{\lambda}t}$ solves the equation, given that $e^{\hat{\lambda}t} \neq 0$, it must be the case that

$$\begin{aligned} 2\hat{\lambda} + b_1 &= 0 \\ \hat{\lambda}^2 + b_1\hat{\lambda} + b_2 &= 0 \end{aligned}$$

  - From the first condition we obtain $\hat{\lambda} = -\frac{1}{2}b_2$ (as we already know)
  - The second equation tells us that $\hat{\lambda}$ solves the quadratic equation we are studying:$\lambda^2 + b_1\lambda + b_2 = 0$
- Therefore, $y^h(t) = te^{\hat{\lambda}t}$ is a general solution of the homogeneous equation

Second-order differential equations (8)

Solving the homogeneous equation. Case 2: $\Delta = 0$ (cont'd)

- The two general solutions of the homogeneous equation allow us to write the general solution of $y(t)$ as
  $y^h(t) = A_1 e^{\hat{\lambda} t} + A_2 t e^{\hat{\lambda} t} = (A_1 + tA_2) e^{\hat{\lambda} t}$
  where A_1 and A_2 are arbitrary constants.
- Clearly, $(A_1 + tA_2)$ diverges as $t \to \infty$. Therefore,
  - If $e^{\hat{\lambda} t}$ diverges (i.e. $\hat{\lambda} > 0$ implying $b_1 < 0$), then $y(t)$ will also diverge.
  - If $\hat{\lambda} < 0$ ($b_1 > 0$), then $e^{\hat{\lambda} t} \to 0$ as $t \to \infty$. The trajectory of $y(t)$ depends on whether the diverging force dominates to or is dominated by the convergent force. It turns out that the convergent force dominates and therefore $y(t)$ converges to zero.

Second-order differential equations (9)

Solving the homogeneous equation. Case 2: $\Delta = 0$ (cont'd)

- To verify the last trajectory, compute $\lim_{t\to\infty}(A_1 e^{\hat{\lambda}t} + A_2 t e^{\hat{\lambda}t}) = A_1 \lim_{t\to\infty} e^{\hat{\lambda}t} + A_2 \lim_{t\to\infty} t e^{\hat{\lambda}t} = 0 + A_2 \lim_{t\to\infty} t e^{\hat{\lambda}t} = A_2 \lim_{t\to\infty} \frac{t}{e^{-\hat{\lambda}t}} =$ (L'Hôpital) $= \lim_{t\to\infty}(-\hat{\lambda}e^{-\hat{\lambda}t})^{-1} = 0$
- Finally if $b_1 = 0$, then $\hat{\lambda} = 0$, and $y(t)$ diverges.

Δ	b_1	b_2	$y(t)$
0	+	$\pm$	convergent
	-	$\pm$	divergent
	0	$\pm$	divergent

Second-order differential equations (10)

Solving the homogeneous equation. Case 3: $\Delta < 0$

- Imaginary roots:
  $\alpha \pm i\theta,\ i = \sqrt{-1},\ \alpha = -\frac{1}{2}b_1, \theta = \frac{1}{2}(|b_1^2 - 4b_2|)^{1/2}$.
- The solution is of the type:
  $y^h(t) = A_1 e^{(\alpha+i\theta)t} + A_2 e^{(\alpha-i\theta)t} = e^{\alpha t}(A_1 e^{i\theta t} + A_2 e^{-i\theta t})$
- Use Euler's formula: $e^{\pm i\theta t} = \cos\theta t \pm i \sin\theta t$ to obtain
  $y^h(t) = e^{\alpha t}[(A_1 + A_2)\cos\theta t + (A_1 - A_2) i \sin\theta t)]$
- Given that A_1 and A_2 are of the type $c \pm id,\ c, d \in$ **R**, it follows that
  $A_1 + A_2 = 2c$ and $A_1 - A_2 = -2di$ so that
  $y(t) = e^{\alpha t} 2[(c\cos\theta t + d\sin\theta t)]$

Second-order differential equations (11)

Solving the homogeneous equation. Case 3: $\Delta < 0$ (cont'd)

- The trajectory of $y^h(t)$ will be cyclical consistently with the $\sin$ and $\cos$ functions with period $2\Pi/\theta$.
- The amplitude of the cycle depends on $\alpha(= -\frac{1}{2}b_1)$:
  - If $\alpha > 0$ (or $b_1 < 0$), the amplitude is increasing and $y(t) \to \infty$ as $t \to \infty$.
  - If $\alpha < 0$ (or $b_1 > 0$), the amplitude is decreasing and $y(t)$ converges as $t \to \infty$.
  - If $\alpha = 0$ (or $b_1 = 0$), the amplitude is constant.

Δ	b_1	b_2	$y(t)$
-	+	$\pm$	cycle,convergent
	-	$\pm$	cycle,divergent
	0	$\pm$	cycle,constant

Second-order differential equations (12)

Solving the homogeneous equation. Summary

- $b_1 > 0$ is necessary condition for convergence
- $b_1 < 0$ is sufficient condition for divergence
- $b_1 = 0, b_2 \neq 0$
  - $\Delta > 0$ divergent trajectory
  - $\Delta = 0$ divergent trajectory
  - $\Delta < 0$ constant cycle

Second-order differential equations (13)

Particular solution of the non-homogeneous equation

- $a_0 y''(t) + a_1 y'(t) + a_2 y(t) = g(t) \qquad [\delta]$
- Solution depends on the structure of $g(t)$

Case 1: $g(t)$ constant

- Let $g(t) = k,\ k \in \mathbf{R}$
- Try as solution $\overline{y}(t) = s,\ s \in \mathbf{R}$
- Then $y'' = y' = 0$ and substituting in $[\delta]$ we obtain $a_2 s = k$ so that $\overline{y}(t) = k/a_2$ is a particular solution.
- If $a_2 = 0$, then try $\overline{y}(t) = st$. In this case, substituting in $[\delta],\ a_1 s = k$ so that $\overline{y}(t)|_{a_2=0} = kt/a_1$ is a particular solution.
- If $a_1 = a_2 = 0$, try $\overline{y}(t) = st^2$ to obtain $\overline{y}(t)|_{a_1=a_2=0} = kt^2/2a_0$ as a particular solution.

Second-order differential equations (14)

Example

- Solve $y''(t) + y'(t) - 6y(t) = 4$
- General solution of homogeneous equation
- $y''(t) + y'(t) - 6y(t) = 0$
- Let $y(t) = e^{\lambda t}$, so that $y' = \lambda e^{\lambda t},\ y'' = \lambda^2 e^{\lambda t}$
- Substituting $\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 6e^{\lambda t} = 0$ or
- $e^{\lambda t}(\lambda^2 + \lambda - 6) = 0$
- Remark: 1 continuation, 1 change of sign $\rightarrow \lambda_1 < 0, \lambda_2 > 0$
- Roots: $\lambda_1 = 2,\ \lambda_2 = -3$
- $y^h(t) = A_1 e^{2t} + A_2 e^{-3t}$

Second-order differential equations (15)

Example (cont'd)

- Particular solution of non-homogeneous equation
- $y''(t) + y'(t) - 6y(t) = 4$
- Let $\overline{y}(t) = \mu$
- Substituting $-6\mu = 4$ or $\mu = -2/3$,
- so that $\overline{y}(t) = -\frac{2}{3}$
- Solution of the differential equation:
  $y(t) = A_1 e^{2t} + A_2 e^{-3t} - \frac{2}{3}$

Second-order differential equations (16)

Particular solution of the non-homogeneous equation

- The particular solution is interpreted as the equilibrium value (stationary or dynamic) of $y(t)$.

Computing the values of A_1 and A_2

- 2variables $\rightarrow$ 2 conditions
  - (i) the value of the function $y(t)$ at a particular t^*, e.g. $t = 0, y(0) = y_0$
  - (ii) the value of the first derivative at that point in time $y'(t^*)$
- Substituting $t^*, y(t^*), y'(t^*)$ in the general solution we obtain two linear equations in two unknowns (A_1 and A_2) that can be solved.

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Second-order differential equations (17)

Example (cont'd)

- Computing A_1, A_2
- Solution of the differential equation:
  $y(t) = A_1 e^{2t} + A_2 e^{-3t} - \frac{2}{3}$
- Then $y'(t) = 2A_1 e^{2t} - 3A_2 e^{-3t}$
- Consider $t = 0$ and suppose $y(0) = \frac{1}{3}$. Then, $y'(0) = 0$
- Substituting,

$$\begin{aligned} \frac{1}{3} &= A_1 + A_2 - \frac{2}{3} \\ 0 &= 2A_1 - 3A_2 \end{aligned}$$

so that $A_1 = \frac{3}{5},\ A_2 = \frac{2}{5}$

Second-order differential equations (18)

Particular solution of the non-homogeneous equation

- Case 2: $g(t)$ polynomial: $g(t) = c_0 + c_1 t + c_2 t^2$
- Case 3: $g(t)$ exponential: $g(t) = Be^{dt}$

Same logic.

An illustrative example

A market model

- Consider the following variant of the market model presented under the first order differential equations:

$$
\begin{aligned}
D(p(t)) =& a + bp(t) \\
S(p(t)) =& a_1 + b_1 p(t) + mp'(t) + np''(t)
\end{aligned}
$$

  where by assumption $b_1 > 0, b < 0$, and $(a - a_1) > 0$.

- Assume the market is perfect in every period, so that $D(p(t)) = S(p(T)), \forall t$.
- Thus, the equilibrium condition is a second order differential equation:
  $p''(t) + \frac{m}{n} p'(t) + \frac{b_1 - b}{n} p(t) = \frac{a - a_1}{n} \quad [\phi]$
- The equilibrium intertemporal price trajectory is the solution of this equation.

An illustrative example (2)

Particular solution of $[\phi]$

- A particular solution of the differential equation $[\phi]$ is $\overline{p}(t) = \frac{a-a_1}{b_1-b} > 0$
- As usual (given that it is a constant), we interpret this solution as a stationary equilibrium.

General solution of the homogeneous equation

- The homogeneous equation is $p''(t) + \frac{m}{n}p'(t) + \frac{b_1-b}{n}p(t) = 0$
- The roots of the characteristic equation $\lambda^2 + \frac{m}{n}\lambda + \frac{b_1-b}{n} = 0$ are:

$$\lambda_1, \lambda_2 = \frac{-\frac{m}{n} \pm \sqrt{(\frac{m}{n})^2 - 4\frac{b_1-b}{n}}}{2} = \frac{-\frac{m}{n} \pm \sqrt{\Delta}}{2}$$

- There are three cases: $\Delta \gtreqless 0$

An illustrative example (3)

Case 1: $\Delta > 0$

- When $\Delta > 0$, there are two real roots λ_1, λ_2.
- Then, the general solution of the homogeneous equation is $p^h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

Case 2: $\Delta = 0$

- When $\Delta = 0$, there is a single real root $\lambda_1 = \lambda_2 = \lambda$.
- Then, the general solution of the homogeneous equation is $p^h(t) = A_1 e^{\lambda t} + A_2 t e^{\lambda t} = e^{\lambda t}(A_1 + A_2 t)$ where $\lambda = -\frac{m}{n}$

Case 3: $\Delta < 0$

- When $\Delta < 0,\ \lambda_1, \lambda_2 = \alpha \pm \beta i$, where $\alpha = -\frac{m}{2n}$ and $\beta = \sqrt{(\frac{m}{n})^2 - 4\frac{b_1 - b}{n}}$

An illustrative example (4)

Equilibrium trajectory

- The trajectory of $p^h(t)$ is characterized by the signs of the coefficients of the characteristic equation.
- Applying Descarte's rule of signs, the following table summarizes all the posible situations

Case 1: $\Delta > 0$

n	m	signs	roots	trajectory
+	+	+++	2 negative	convergent
+	-	+-+	2 positive	divergent
-	+	+- -	1 pos, 1 neg	divergent
-	-	++-	1 pos, 1 neg	divergent

An illustrative example (5)

Case 2: $\Delta = 0$

- Recall that the general solution of the homogeneous equation is
$p^h(t) = e^{\lambda t}(A_1 + A_2 t)$ where $\lambda = -\frac{m}{n}$
- The following table summarizes all the posible situations

n	m	λ	trajectory	comments
+	+	-	convergent	L'Hôpital rule
+	-	+	divergent	
-	+	+	divergent	
-	-	-	convergent	L'Hôpital rule

An illustrative example (6)

Case 3: $\Delta < 0$

- Recall the general solution of the homogeneous equation is $p^h(t) = e^{\alpha t}(A_1 \cos\theta t + A_2 \sin\theta t)$ where $\alpha = -\frac{m}{2n}$ and $\beta = \sqrt{(\frac{m}{n})^2 - 4\frac{b_1 - b}{n}}$
- Trajectory is oscillating with period $2\pi/\theta$. The amplitude depends on the sign of α.
- The following table summarizes all the posible situations

n	m	α	trajectory
+	+	-	osc. decreasing
+	-	+	osc. increasing
-	+	+	osc. increasing
-	-	-	osc. decreasing

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An illustrative example (7)

Solution of the 2nd-degree differential equation

- The solution of the 2nd-degree differential equation $[\phi]$ is $p(t) = p^h(t) + \overline{p}(t)$ where $g(t) = \frac{a-a_1}{n}$
- What is left is to determine the values of the constants A_1, A_2.
- To do it we need two additional assumptions. Tipically, these refer to the values of $p(t)$ and $p'(t)$ at $t = 0$.
- To illustrate, assume $\Delta > 0$, so that the the general solution of the homogeneous equation is $p^h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$
  - Then, $p'(t) = -A_1 \lambda_1 e^{\lambda_1 t} - A_2 \lambda_2 e^{\lambda_2 t}$, so that
  - $p'(0) = -A_1 \lambda_1 - A_2 \lambda_2$

An illustrative example (8)

Solution of the 2nd-degree differential equation (cont'd)

- We have a system of two equations in A_1 and A_2 as follows

$$p(0) = A_1 + A_2$$
$$p'(0) = -A_1\lambda_1 - A_2\lambda_2$$

whose solution is

$$A_1 = \frac{-1}{\lambda_1 - \lambda_2}\Big(\lambda_2 p_0 + p'(0)\Big)$$
$$A_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}\Big(p_0 + \frac{1}{\lambda_1}p'(0)\Big)$$

Simultaneous systems of differential equations

Definition

- A simultaneous system is composed of at least two differential equations with at least two unknown functions.
- Solvable if (i) the system has as many equations as unknowns and (ii) equations are independent and consistent.

First-order 2×2 systems in normal form

$$
\begin{aligned}
y'(t) =& a_{11}y(t) + a_{12}z(t) + g_1(t) \\
z'(t) =& a_{21}y(t) + a_{22}z(t) + g_2(t)
\end{aligned}
$$

- a_{ij} are given constants, $g_s(t)$ are known functions
- Strategy to solve the system
  - Find a general solution of the homogeneous system
  - Find a particular solution of the non-homogenous system
  - Solution of the system: sum of the previous two solutions

Simultaneous systems of differential equations (2)

Solution of the homogeneous system

$$\left.\begin{aligned} y'(t) =& a_{11}y(t) + a_{12}z(t) \\ z'(t) =& a_{21}y(t) + a_{22}z(t) \end{aligned}\right\} \qquad [\mu]$$

- Following the same logic as with the equations, let us try as general solution the functions

$$\left.\begin{aligned} y(t) &= \alpha_1 e^{\lambda t} \\ z(t) &= \alpha_2 e^{\lambda t} \end{aligned}\right\} \qquad [\nu]$$

  where α_1, α_2 are constant and at least one of them is not zero.

- Substituting $[\nu]$ in $[\mu]$ we obtain:

$$\left.\begin{aligned} \lambda\alpha_1 e^{\lambda t} =& a_{11}y(t) + a_{12}z(t) \\ \lambda\alpha_2 e^{\lambda t} =& a_{21}y(t) + a_{22}z(t) \end{aligned}\right\}$$

Simultaneous systems of differential equations (3)

Solution of the homogeneous system (cont'd)

- That can be rewritten as

$$\left.\begin{aligned} e^{\lambda t}[(a_{11}-\lambda)\alpha_1 + a_{12}\alpha_2] &= 0 \\ e^{\lambda t}[a_{21}\alpha_1 + (a_{22}-\lambda)\alpha_2] &= 0 \end{aligned}\right\} \qquad [\psi]$$

- The functions $[\nu]$ will be solutions of the system $[\mu]$ iff $[\psi]$ is satisfied $\forall t$, namely iff

$$\left.\begin{aligned} (a_{11}-\lambda)\alpha_1 + a_{12}\alpha_2 &= 0 \\ a_{21}\alpha_1 + (a_{22}-\lambda)\alpha_2 &= 0 \end{aligned}\right\} \qquad [\psi 2]$$

or in matrix form $\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Simultaneous systems of differential equations (4)

Solution of the homogeneous system (cont'd)

- The system $[\psi 2]$ has a solution if its determinant is zero:
  $\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$, or
  $\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \; [\psi 3]$
- This is the characteristic equation of the system $[\mu]$
- The solution of $[\psi 3]$ yields two values λ_1 and λ_2
- Three possible cases:
  - Case 1: $\lambda_1 \neq \lambda_2, \lambda_i \in \mathbf{R}, i = 1, 2$
  - Case 2: $\lambda_1 = \lambda_2 = \widetilde{\lambda}, \widetilde{\lambda} \in \mathbf{R}$
  - Case 3: $\lambda_1 = \alpha + \theta i; \;\; \lambda_2 = \alpha - \theta i$

Case 1: two real and different roots

Consider first λ_1

- Let $\lambda = \lambda_1$ and substitute it in $[\psi 2]$.
- Assume (α'_1, α'_2) is a solution of $[\psi 2]$, and rewrite it as

$$\left.\begin{array}{l}(a_{11} - \lambda_1)\alpha'_1 + a_{12}\alpha'_2 = 0 \\ a_{21}\alpha'_1 + (a_{22} - \lambda_1)\alpha'_2 = 0\end{array}\right\} \qquad [C11]$$

  where we already know that its determinant, $[\psi 3]$ is zero.
- Accordingly, the equations are NOT linearly independent, and we can only solve for α'_1/α'_2
- Thus, let's fix $\alpha'_1 = 1$ and let's determine the value of α'_2.

Case 1: two real and different roots (2)

Consider first λ_1 (cont'd)

- Then $[C11]$ becomes

$$\left.\begin{aligned}(a_{11}-\lambda_1)+a_{12}\alpha_2' &= 0\\ a_{21}+(a_{22}-\lambda_1)\alpha_2' &= 0\end{aligned}\right\}$$

- Solving for α_2' we obtain

$$\alpha_2' = \frac{\lambda_1 - a_{11}}{a_{12}} = \frac{a_{21}}{\lambda_1 - a_{22}}$$

Case 1: two real and different roots (3)

Consider now λ_2

- Let $\lambda = \lambda_2$ and substitute it in $[\psi 2]$.
- Assume (α_1'', α_2'') is a solution of $[\psi 2]$
- Following a parallel argument as before, and letting $\alpha_1'' = 1$, we will conclude

$$\alpha_2'' = \frac{\lambda_2 - a_{11}}{a_{12}} = \frac{a_{21}}{\lambda_2 - a_{22}}$$

Case 1: two real and different roots (4)

Conclusion

- We have two solutions of the homogeneous system $[\mu]$

$$\left.\begin{aligned} y(t) &= \alpha_1' e^{\lambda_1 t} \\ z(t) &= \alpha_2' e^{\lambda_1 t} \end{aligned}\right\}$$

and

$$\left.\begin{aligned} y(t) &= \alpha_1'' e^{\lambda_2 t} \\ z(t) &= \alpha_2'' e^{\lambda_2 t} \end{aligned}\right\}$$

Case 1: two real and different roots (5)

Conclusion (cont'd)

- Applying theorem 2 (see p.9), we can combine both solutions introducing two arbitrary constants A_1 and A_2 to obtain as general solution of the homogeneous system $[\mu]$

$$\left.\begin{aligned} y(t) &= A_1\alpha_1' e^{\lambda_1 t} + A_2\alpha_1'' e^{\lambda_2 t} \\ z(t) &= A_1\alpha_2' e^{\lambda_1 t} + A_2\alpha_2'' e^{\lambda_2 t} \end{aligned}\right\}$$

- Substituting the values of $\alpha_1', \alpha_2', \alpha_1'', \alpha_2''$, the solution reads

$$\left.\begin{aligned} y(t) &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ z(t) &= A_1 \frac{\lambda_1 - a_{11}}{a_{12}} e^{\lambda_1 t} + A_2 \frac{\lambda_2 - a_{11}}{a_{12}} e^{\lambda_2 t} \end{aligned}\right\}$$

Case 1: two real and different roots (6)

Conclusion (cont'd)

- Alternatively, we can write the solution as

$$\left.\begin{aligned} y(t) &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ z(t) &= A_1' e^{\lambda_1 t} + A_2' e^{\lambda_2 t} \end{aligned}\right\}$$

  where

$$\frac{A_1'}{A_1} = \alpha_2'$$
$$\frac{A_2'}{A_2} = \alpha_2''$$

Case 2: two real and equal roots

- Let $\lambda_1 = \lambda_2 = \tilde{\lambda}$
- Let's propose as solution of the homogeneous equation

$$\left.\begin{aligned} y(t) &= (A_1 + A_2 t)e^{\tilde{\lambda} t} \\ z(t) &= (A_1' + A_2' t)e^{\tilde{\lambda} t} \end{aligned}\right\} \qquad [C21]$$

where A_1, A_1', A_2, A_2' are constants.

- Differentiating,

$$\left.\begin{aligned} y'(t) &= \tilde{\lambda}(A_1 + A_2 t)e^{\tilde{\lambda} t} \\ z'(t) &= \tilde{\lambda}(A_1' + A_2' t)e^{\tilde{\lambda} t} \end{aligned}\right\} \qquad [C22]$$

Case 2: two real and equal roots (2)

- Substituting $[C21]$ and $[C22]$ into μ we obtain

$$\left.\begin{aligned}\tilde{\lambda}(A_1 + A_2 t)e^{\tilde{\lambda}t} =& a_{11}(A_1 + A_2 t)e^{\tilde{\lambda}t} + a_{12}(A_1' + A_2' t)e^{\tilde{\lambda}t} \\ \tilde{\lambda}(A_1' + A_2' t)e^{\tilde{\lambda}t} =& a_{21}(A_1 + A_2 t)e^{\tilde{\lambda}t} + a_{22}(A_1' + A_2' t)e^{\tilde{\lambda}t}\end{aligned}\right\} \qquad [C23]$$

- Simplifying reduces to

$$\left.\begin{aligned}\Big[(\tilde{\lambda} - a_{11})A_2 - a_{12}A_2'\Big]t + \Big[(\tilde{\lambda} - a_{11})A_1 + A_2 - a_{12}A_1'\Big] = 0 \\ \Big[(\tilde{\lambda} - a_{22})A_2' - a_{21}A_2\Big]t + \Big[(\tilde{\lambda} - a_{22})A_1' + A_2' - a_{21}A_1\Big] = 0\end{aligned}\right\} \qquad [C24]$$

Case 2: two real and equal roots (3)

- Candidate $[C21]$ will be a solution if it satisfies $[C24]$ $\forall t$:

$$\left.\begin{aligned} (\tilde{\lambda} - a_{11})A_2 - a_{12}A_2' = 0 \\ (\tilde{\lambda} - a_{11})A_1 + A_2 - a_{12}A_1' = 0 \\ (\tilde{\lambda} - a_{22})A_2' - a_{21}A_2 = 0 \\ (\tilde{\lambda} - a_{22})A_1' + A_2' - a_{21}A_1 = 0 \end{aligned}\right\}$$

- Recall that by definition, $\tilde{\lambda}$ satisfies $[\psi 3]$ so that

$$\frac{\tilde{\lambda} - a_{11}}{a_{12}} = \frac{a_{21}}{\tilde{\lambda} - a_{22}} \quad [C25]$$

Case 2: two real and equal roots (4)

- Solving for (A_1, A_1', A_2, A_2') we obtain

$$\left.\begin{array}{ll}
A_2' = \dfrac{\tilde{\lambda} - a_{11}}{a_{12}} A_2 & [C26] \\[2ex]
A_2' = \dfrac{a_{21}}{\tilde{\lambda} - a_{22}} A_2 & [C27] \\[2ex]
A_1' = \dfrac{\tilde{\lambda} - a_{11}}{a_{12}} A_1 + \dfrac{1}{a_{12}} A_2 & [C28] \\[2ex]
A_1' = \dfrac{a_{21}}{\tilde{\lambda} - a_{22}} A_1 - \dfrac{1}{\tilde{\lambda} - a_{22}} A_2' & [C29]
\end{array}\right\}$$

- Observe that from $[C25]$ the two expressions of A_2' are the same.

Case 2: two real and equal roots (5)

- Substituting $[C26]$ into $[C29]$ and using $[C25]$ we obtain

$$A_1' = \frac{\tilde{\lambda} - a_{11}}{a_{12}} A_1 - \frac{\tilde{\lambda} - a_{11}}{a_{12}(\tilde{\lambda} - a_{22})} A_2 \quad [C210]$$

- Comparing $[C28]$ and $[C210]$, it follows that they are compatible iff

$$\frac{1}{a_{12}} = -\frac{\tilde{\lambda} - a_{11}}{a_{12}(\tilde{\lambda} - a_{22})}$$

  or

$$\tilde{\lambda} = \frac{a_{11} + a_{22}}{2}$$

- in which case $[C21]$ will actually be the solution of the homogeneous equation $[\mu]$

Case 3: complex roots

- we neglect this case

Particular solution of the non-homogeneous system

- Assume $g_1(t) = b_1$ and $g_2(t) = b_2$, with $b_1, b_2 \in \mathbf{R}$
- Then the system of differential equations becomes

$$\left.\begin{aligned} y'(t) =& a_{11}y(t) + a_{12}z(t) + b_1 \\ z'(t) =& a_{21}y(t) + a_{22}z(t) + b_2 \end{aligned}\right\} \qquad [G1]$$

- Consider the following particular solution

$$\left.\begin{aligned} \overline{y}(t) =& B_1 \\ \overline{z}(t) =& B_2 \end{aligned}\right\} \qquad [G2]$$

where $B_1, B_2 \in \mathbf{R}$

Particular solution of the non-homogeneous system (2)

- Substituting $[G2]$ into $[G1]$ and simplifying yields

$$\left.\begin{aligned} a_{11}B_1 + a_{12}B_2 &= -b_1 \\ a_{21}B_1 + a_{22}B_2 &= -b_2 \end{aligned}\right\} \qquad [G3]$$

- Solving for B_1 and B_2 we obtain,

$$B_1 = \frac{-b_1 a_{22} + b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$B_2 = \frac{-b_2 a_{11} + b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

- Finally to determine A_1 and A_2, we need to introduce some "initial" conditions $y(0), z(0)$.

An illustrative example

- Consider a firm setting the price of its product $p(t)$ dynamically to reflect demand. Also, it looks after maintaining a low level of inventory $I(t)$ (to reduce overall cost).
- Think of the state of demand as being approximated by the level of inventory.
- the level of inventory is determined be the excess of production.
- Assue that we can summarize all this information in the following system:

$$\left.\begin{aligned} p'(t) &= 3p(t) - 4I(t) \\ I'(t) &= 4p(t) - 7I(t) \end{aligned}\right\}$$

- Note that this is directly a homogeneous system of linear first-order differential equations:

An illustrative example (2)

$$\left.\begin{aligned} p'(t) - 3p(t) + 4I(t) = 0 \\ I'(t) - 4p(t) + 7I(t) = 0 \end{aligned}\right\} \qquad [Ex1]$$

- Let's consider the following solution candidate

$$\left.\begin{aligned} p(t) = \alpha_1 e^{\lambda t} \\ I(t) = \alpha_2 e^{\lambda t} \end{aligned}\right\} \qquad [Ex2]$$

- so that

$$\left.\begin{aligned} p'(t) = \alpha_1 \lambda e^{\lambda t} \\ I'(t) = \alpha_2 \lambda e^{\lambda t} \end{aligned}\right\} \qquad [Ex3]$$

An illustrative example (3)

- Substituting $[Ex2]$ and $[Ex3]$ into $[Ex1]$ and simplifying, we obtain

$$\left.\begin{aligned} e^{\lambda t}(\alpha_1\lambda - 3\alpha_1 + 4\alpha_2) = 0 \\ e^{\lambda t}(\alpha_2\lambda - 4\alpha_1 + 7\alpha_2) = 0 \end{aligned}\right\}$$

- implying

$$\left.\begin{aligned} \alpha_1(\lambda - 3) + 4\alpha_2 = 0 \\ -4\alpha_1 + \alpha_2(\lambda + 7) = 0 \end{aligned}\right\} \qquad [Ex4]$$

- or in matrix form

$$\begin{pmatrix} \lambda - 3 & 4 \\ -4 & \lambda + 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad [Ex5]$$

An illustrative example (4)

- Compute

$$\begin{vmatrix} \lambda - 3 & 4 \\ -4 & \lambda + 7 \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 4\lambda - 5 = 0$$

- This is the characteristic equation that has as solutions $\lambda_1 = 1$ and $\lambda_2 = -5$
- Note that at these values λ_1 and λ_2 the equations in $[Ex4]$ are linearly dependent.

An illustrative example (5)

- Consider $\lambda = \lambda_1$
- Substitute λ_1 in $[Ex4]$ so that

$$\left.\begin{aligned} -2\alpha_1' + 4\alpha_2' = 0 \\ -4\alpha_1' + 8\alpha_2' = 0 \end{aligned}\right\}$$

- As they are linearly dependent, we can only solve for α_2'/α_1'.
- Thus, fix $\alpha_1' = 1$ and solve for α_2'.
- We obtain $\alpha_2' = 1/2$

An illustrative example (6)

- Consider $\lambda = \lambda_2$
- Substitute λ_2 in $[Ex4]$ so that

$$\left.\begin{aligned} -8\alpha_1'' + 4\alpha_2'' &= 0 \\ -4\alpha_1'' + 2\alpha_2'' &= 0 \end{aligned}\right\}$$

- As they are linearly dependent, we can only solve for α_2'/α_1'.
- Thus, fix $\alpha_1' = 1$ and solve for α_2'.
- We obtain $\alpha_2'' = 2$

An illustrative example (7)

- Recovering $[Ex2]$ we have two solutions

$$\left.\begin{aligned} p(t) &= \alpha_1' e^{\lambda_1 t} \to \; p(t) = e^{t} \\ I(t) &= \alpha_2' e^{\lambda_1 t} \to \; I(t) = \frac{1}{2} e^{t} \end{aligned}\right\}$$

and

$$\left.\begin{aligned} p(t) &= \alpha_1'' e^{\lambda_2 t} \to \; p(t) = e^{-5t} \\ I(t) &= \alpha_2'' e^{\lambda_2 t} \to \; I(t) = 2e^{-5t} \end{aligned}\right\}$$

- combining both solutions (recall Thm 2)

$$\left.\begin{aligned} p(t) &= A_1 e^{t} + A_2 e^{-5t} \\ I(t) &= A_1 \frac{1}{2} e^{t} + A_2 2 e^{-5t} \end{aligned}\right\} \qquad [Ex6]$$

An illustrative example (8)

- To solve for A_1 and A_2 assume

$$p(0) = I(0) = 1 \qquad [Ex7]$$

- Evaluating $[Ex6]$ at $t = 0$ and substituting $[Ex7]$ in, we obtain

$$\left.\begin{aligned} 1 &= A_1 + A_2 \\ 1 &= A_1 \frac{1}{2} + A_2 2 \end{aligned}\right\}$$

- so that $A_1 = 2/3$ and $A_2 = 1/3$.
- Finally,

$$\left.\begin{aligned} p(t) &= \frac{2}{3} e^{t} + \frac{1}{3} e^{-5t} \\ I(t) &= \frac{1}{3} e^{t} + \frac{2}{3} e^{-5t} \end{aligned}\right\}$$

Phase diagrams

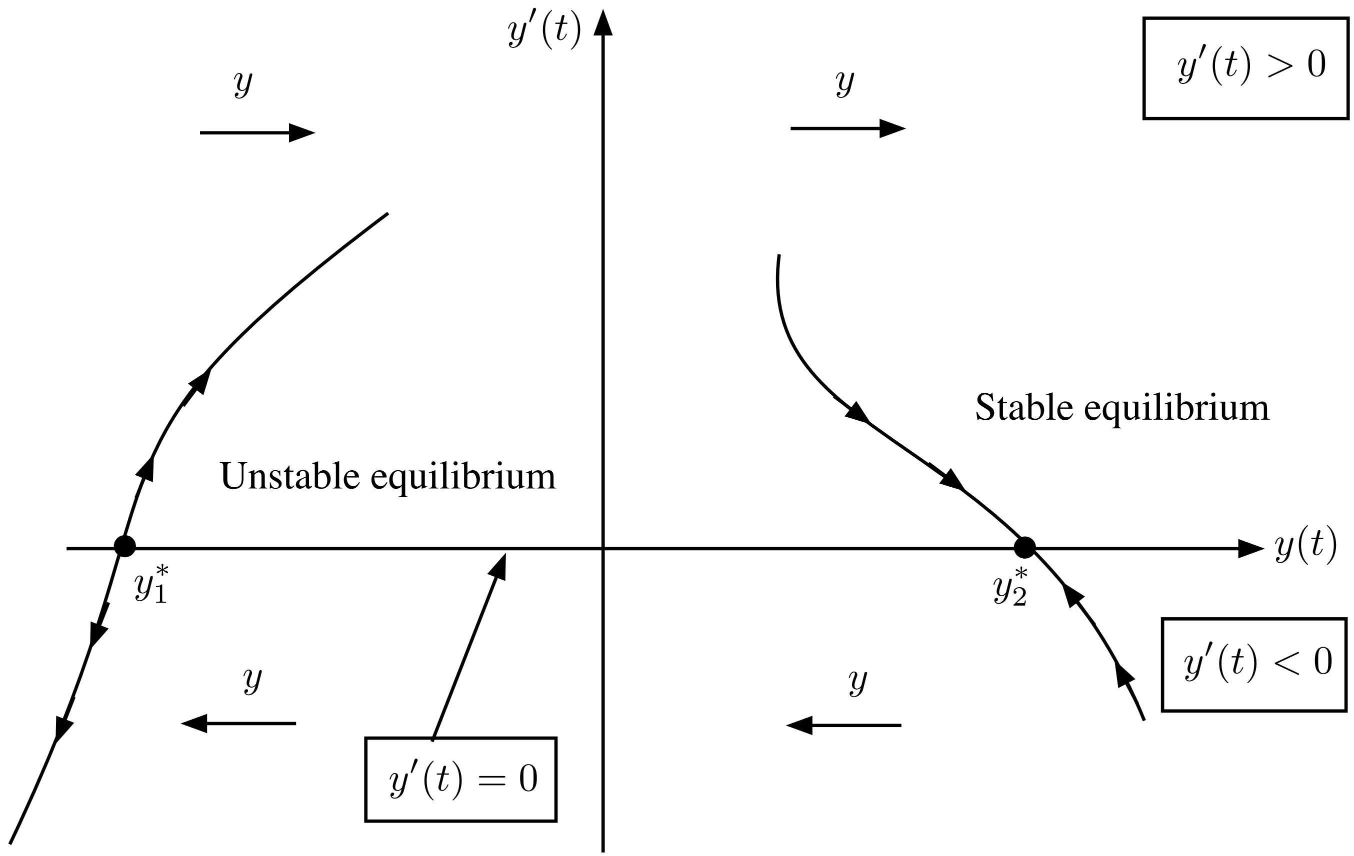
Introduction

- Qualitative info of trajectory of $y(t)$
- First-order differential equations
- Two cases: $y'(t) = f(y)$, and $y'(t) = f(x, y)$
- where f need not be a linear function

Case 1: $y'(t) = f(y)$

- Plot $y'(t)$ in the space y', y
- Above the horizontal axis $y'(t) > 0$. That is, y increases with time. Graphically, y moves from left to right
- Below the horizontal axis $y'(t) < 0$. That is, y decreases with time. Graphically, y moves from right to left
- Existence of solution: $y'(t) = 0$ locus of potential solution values of $y(t)$.
- Stability (convergence): examine whether forces on a point where $y'(t) \neq 0$ point towards y^*

Phase diagrams (2)



- Positive slope phase curve: unstability
- Negative slope phase curve: stability

Phase diagrams (3)

Illustration

(a) $y' = y - 3$

- $y' = 0 \Leftrightarrow y = 3$
- slope phase curve: $dy'/dy = 1 > 0 \rightarrow$ unstable (divergent)
- Solution of the differential equation (with $y(0) = 0$):

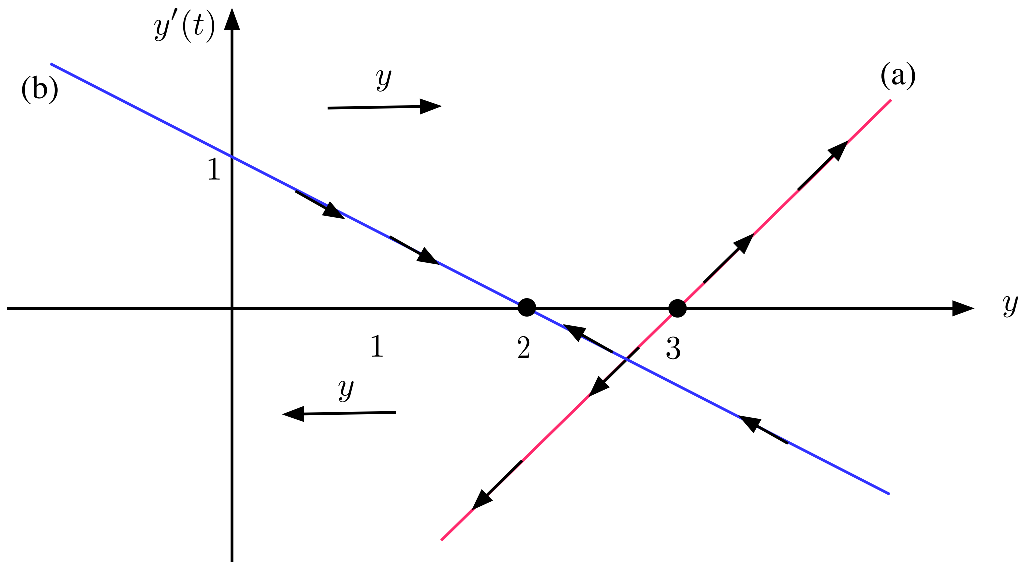
  $y(t) = 3(1 - e^t) \xrightarrow{t \to \infty} -\infty$

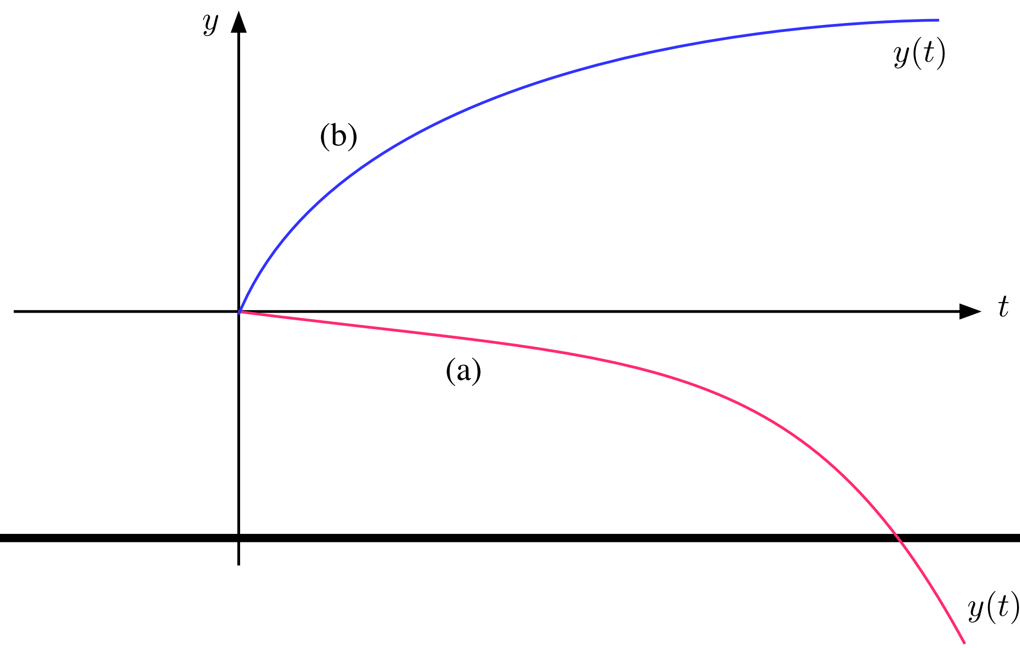
(b) $y' = 1 - \frac{1}{2}y$

- $y' = 0 \Leftrightarrow y = 2$
- slope phase curve: $dy'/dy = -\frac{1}{2} < 0 \rightarrow$ stable (convergent)
- Solution of the differential equation (with $y(0) = 0$):

  $y(t) = 2(1 - e^{-t/2}) \xrightarrow{t \to \infty} 2$

Phase diagrams (4)





Phase diagrams (5)

Case 2: $y'(t) = f(x, y)$

- Consider the system $x'(t) = f(x, y), \;\; y'(t) = g(x, t)$
- Find locus of potential equilibria: $x'(t) = 0, y'(t) = 0$

$$x'(t) = 0 \Leftrightarrow f(x, y) = 0 \rightarrow y = \hat{f}(x)$$

$$y'(t) = 0 \Leftrightarrow g(x, y) = 0 \rightarrow y = \hat{g}(x)$$

- if $\hat{f}, \hat{g}$ cannot be obtained; apply implicit function theorem
  - slope of $x'(t) = 0$ is $\frac{dy}{dx}|_{x'=0} = -\frac{f_x}{f_y}$
  - slope of $y'(t) = 0$ is $\frac{dy}{dx}|_{y'=0} = -\frac{g_x}{g_y}$

Phase diagrams (6)

Illustration

- Suppose $f_x < 0, f_y > 0, g_x > 0, g_y < 0$
- Then $x'(t)$ and $y'(t)$ have positive slope
- Suppose also $-\frac{f_x}{f_y} > -\frac{g_x}{g_y}$
- Then phase curves $x'(t) = 0$ and $y'(t) = 0$ will intersect (at least) once.
- Note $f_x = \partial f / \partial x = \partial x' / \partial x < 0$. Then,
  - $x' < 0$ below the phase curve $x' = 0$
  - $x' > 0$ above the phase curve $x' = 0$
- Note $g_y = \partial g / \partial y = \partial y' / \partial y > 0$. Then,
  - $y' < 0$ above the phase curve $y' = 0$
  - $y' > 0$ below the phase curve $y' = 0$
- Four regions. See figure

Phase diagrams (7)