Optimization. A first course of mathematics for economists

Xavier Martinez-Giralt
Universitat Autònoma de Barcelona
xavier.martinez.giralt@uab.eu

II.4 Static optimization - Linear programming

Linear programming - Definition

Linear inequality restrictions

- $g_i(\mathbf{x}) = \sum_{i=1}^{m} a_{ij} x_j \leq b_i, \;\; i = 1, \ldots, m; \;\; j = 1, \ldots, n$
- $g_i(\mathbf{x})$ continuous, continuously differentiable; $b_i \in \mathbf{R}$
- $\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$

Non-negativity restrictions: $x_j \geq 0, \;\; j = 1, \ldots, n$

Linear objective function

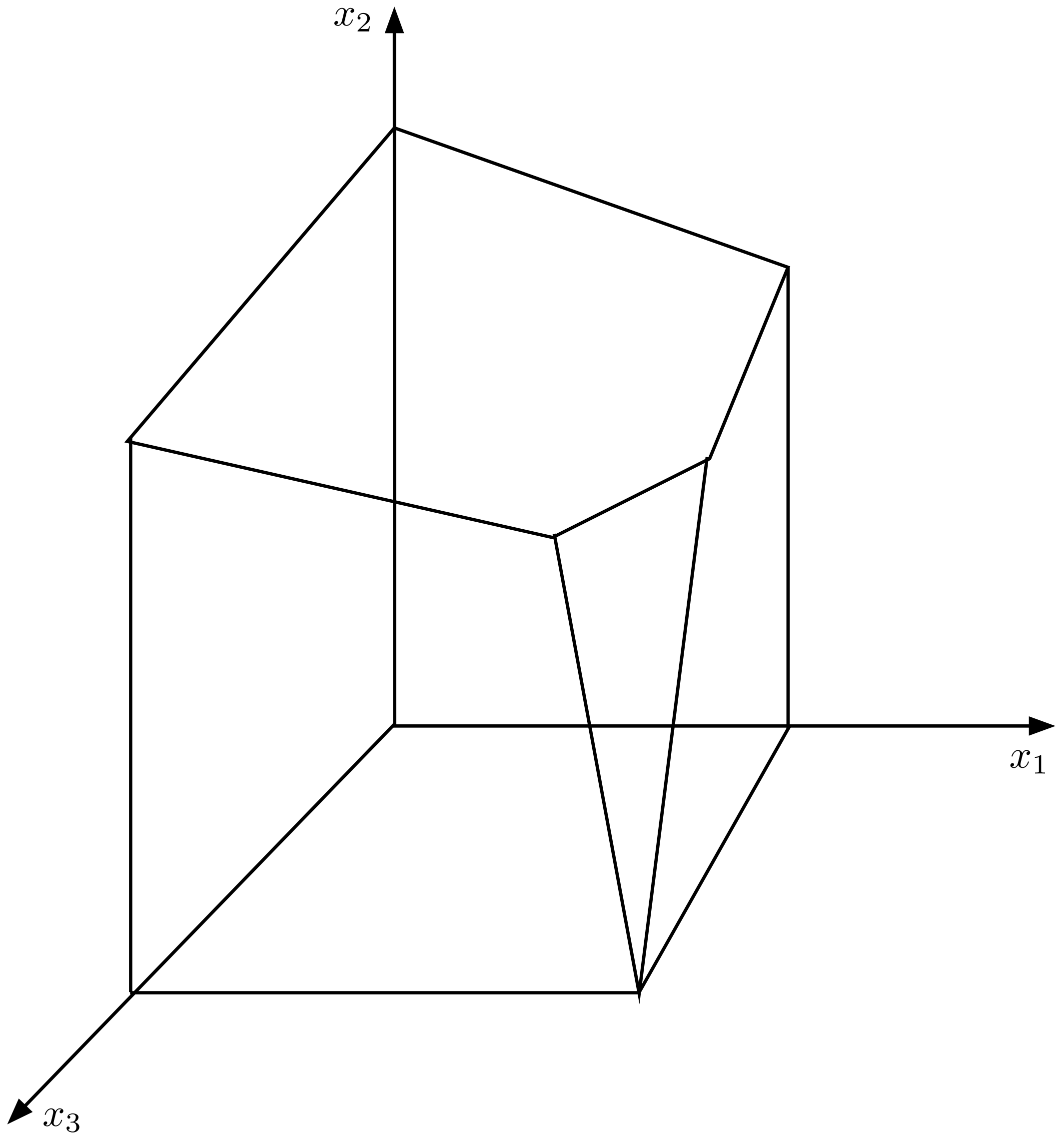
- $f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j = \mathbf{cx}, \;\; j = 1, \ldots, n$
- $\mathbf{c} = (c_1, \ldots, c_n), \;\; c_j \in \mathbf{R}$

Problem: $\max_{\mathbf{x}} \mathbf{cx}$ s.t. $\mathbf{Ax} \leq \mathbf{b}, \;\; \mathbf{x} \geq 0$

Linear programming - Description

- Particular case of non-linear programming.
- Linear restrictions $g_i, \rightarrow$ closed half-space.
- $\cap_i g_i, \rightarrow$ *convex polyhedral set*; if bounded, *convex polyhedron*.
- Opportunity set: closed convex polyhedral set in non-negative orthant of $\mathbf{R}^n$.
  - faces of polyhedral $\rightarrow$ bounding faces $\rightarrow$ hyperplanes.
  - vertices: points of intersection of n (or more) bounding faces.
  - edge: intersection of 2 bounding faces (hyperplanes).
  - vertices connected by edges.
- Figure (Intriligator, 2002), $[n = 3, m = 4]$: 7 bounding faces; 14 edges; 9 vertices (8 as intersection of 3 faces; 1 as intersection of 4 faces).

Linear programming - Description (2)



Linear programming - Description (3)

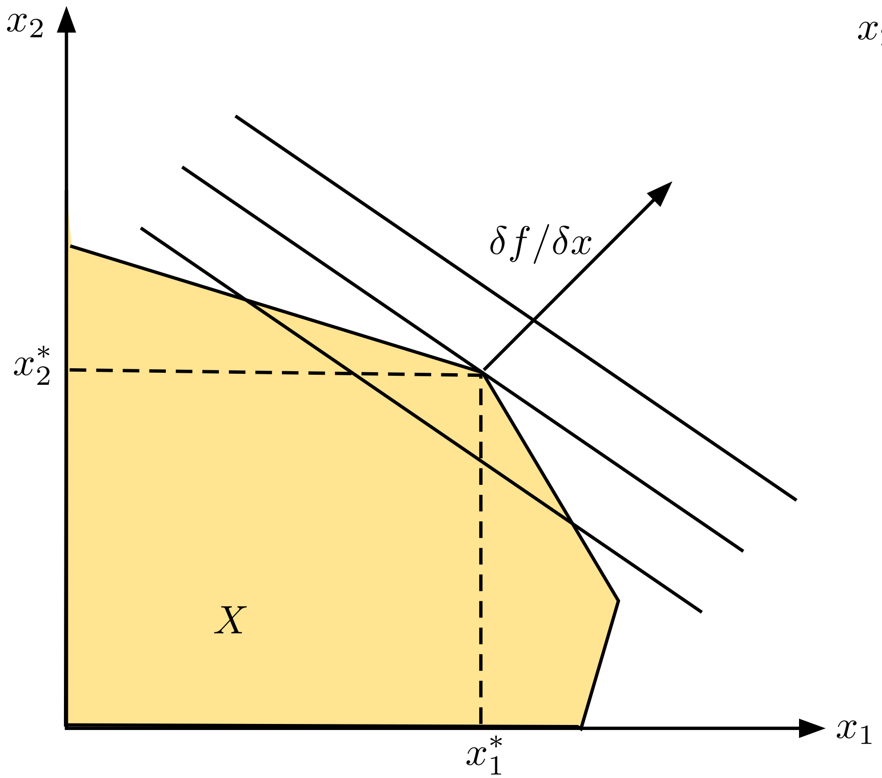
Linear objective function

- Contours $\rightarrow$ hyperplanes
- Contour map $\rightarrow$ parallel hyperplanes
- preference direction $\rightarrow$ gradient vector of f orthogonal to contours: $\nabla f \equiv \frac{\partial f}{\partial \mathbf{x}} = \mathbf{c}$

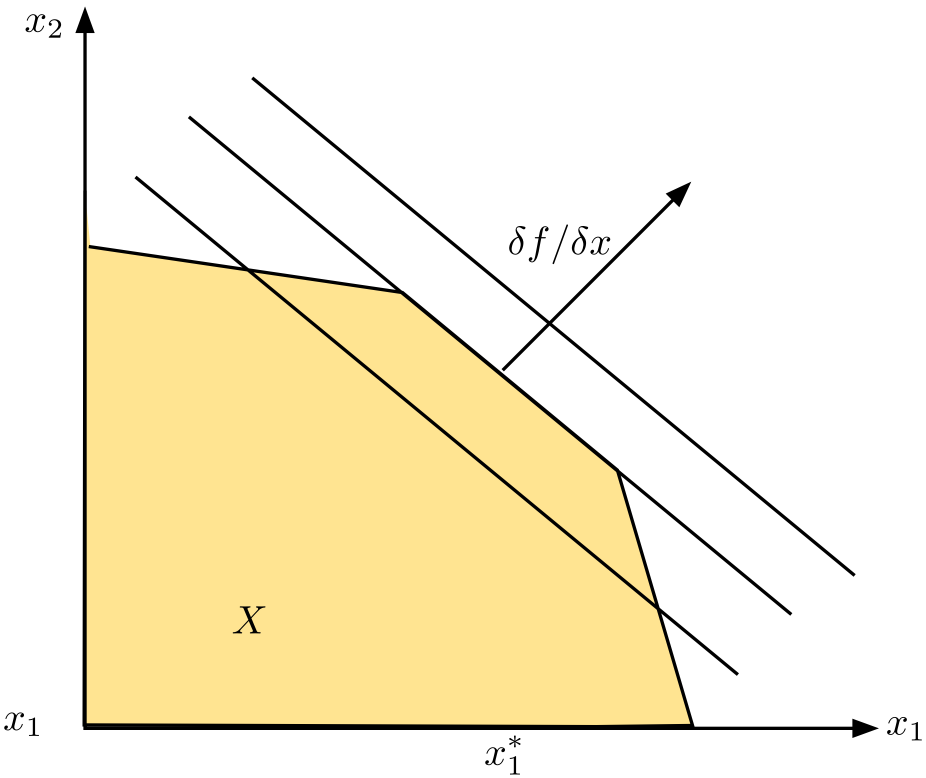
Solution

- Find a (set of) point(s) on the highest contour of f in the opportunity set
- If solution exists, must lie on the boundary of opportunity set
- Solution at point(s) where contour hyperplane is supporting hyperplane of the convex polyhedral opportunity set.
- In $\mathbf{R}^3$ solution at vertex, or on edge, or on bounding face

Linear programming - Description (4)



(a) Solution at a vertex



(b) Continuum of solutions

Linear programming - Description (5)

Solution (cont'd)

- At a solution (unique or multiple), value of f is unique.
- Convex opprotunity set $+$ linear objective function $\rightarrow$ local-global theorem: a local solution is global.
- Objective function continuous $+$ opportunity set closed $\rightarrow$ Weierstrasse theorem: solution exists if opportunity set is also bounded.
- In general 3 possible solutions:
  - unique at a vertex;
  - continuum on edge, boundary face, ...
  - no solution if opportunity set is unbounded or empty.

Economic interpretation

- The linearity of the objective function f and of the restrictions g_i imply that prices of inputs and outputs are taken as given. Equivalently, the firm faces perfectly competitive markets for inputs and outputs.
- Also, the linearity of the technology imply constant returns to scale.
- Therefore, in formulating an economic problem as a linear-programming model, we are assuming that the linearity assumptions are valid over the full range of values of the decision variables being considered in the problem.
- ... otherwise, the solution of the linear-programming model will not be an optimal solution to the economic problem.

An illustrative example

The set-up

- A firm producing chairs (good 1) and tables (good 2) in a given time period.
- Production require two inputs: wood and machine time.
- Technology:
  - Production 1 unit of good 1 requires
    - 20 units of wood
    - 5 hours of machine time
  - Production 1 unit of good 2 requires
    - 40 units of wood
    - 2 hours of machine time
- During the time period considered there are,
  - 400 units available of wood
  - 40 hours available of time machine

An illustrative example (2)

The set-up (cont'd)

- The capacity of production during the time period is
  - 6 units of good 1
  - 9 units of good 2
- The contribution to profit (price-cost) of the goods is
  - 100 € per unit of good 1
  - 60 € per unit of good 2

The problem

- Find the quantities of good 1, x_1, and of good 2, x_2, that maximize the profit of the firm

The linear programming formulation

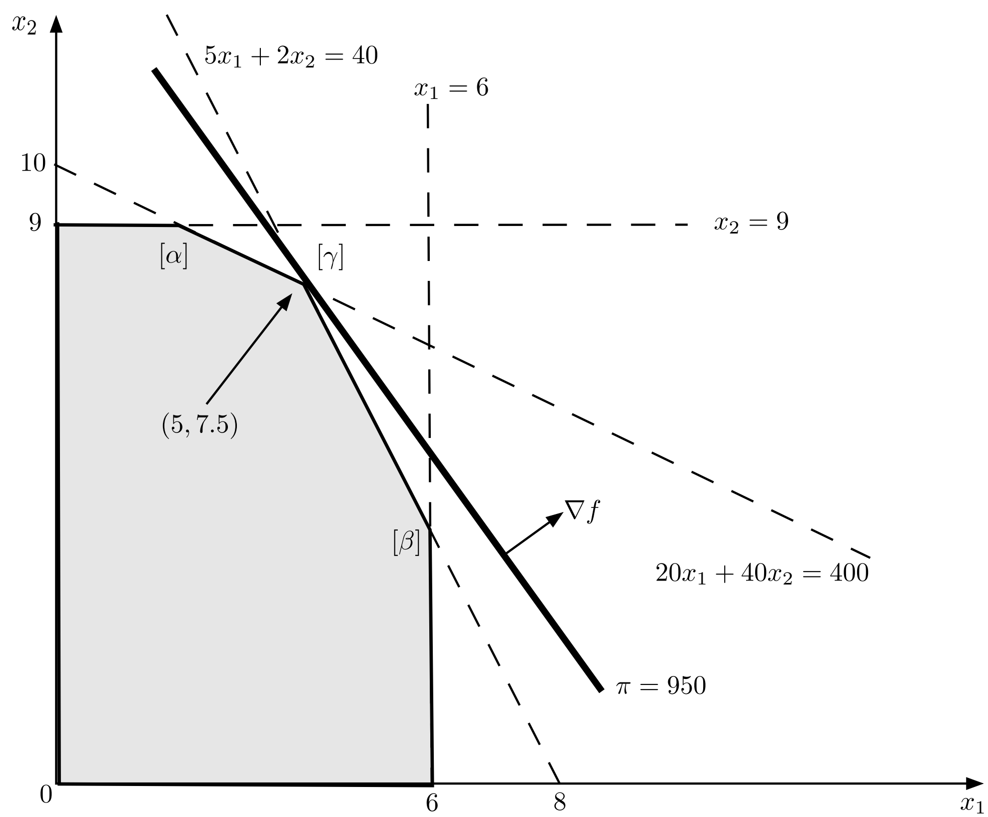
$$\max_{x_1,x_2} \Pi = 100x_1 + 60x_2 \quad \text{s.t.}$$

$$
\begin{aligned}
&20x_1 + 40x_2 \leq 400 && [1]\\
&5x_1 + 2x_2 \leq 40 && [2]\\
&x_1 \leq 6, \qquad x_1 \geq 0 && [3]\\
&x_2 \leq 9, \qquad x_2 \geq 0 && [4]
\end{aligned}
$$

- Solution: $x_1^* = 5;\ x_2^* = 7.5;\ \Pi^* = 950.$
- Note that [1] and [2] are binding; [3] and [4] are not binding.

Solution

- Kuhn-Tucker conditions
- Simplex methods $\rightarrow$ software applications
- In 2-dimensional problems: graphical analysis



Solution - Kuhn-Tucker

$$L = 100x_1 + 60x_2 + \lambda_1(400 - 20x_1 - 40x_2) + \lambda_2(40 - 5x_1 - 2x_2)$$

$$\frac{\partial L}{\partial x_1} = 100 - 20\lambda_1 - 5\lambda_2 \leq 0 \tag{1}$$

$$x_1\frac{\partial L}{\partial x_1} = x_1(100 - 20\lambda_1 - 5\lambda_2) = 0 \tag{2}$$

$$\frac{\partial L}{\partial x_2} = 60 - 40\lambda_1 - 2\lambda_2 \leq 0 \tag{3}$$

$$x_2\frac{\partial L}{\partial x_2} = x_2(60 - 40\lambda_1 - 2\lambda_2) = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda_1} = 400 - 20x_1 - 40x_2 \geq 0 \tag{5}$$

$$\lambda_1\frac{\partial L}{\partial \lambda_1} = \lambda_1(400 - 20x_1 - 40x_2) = 0 \tag{6}$$

$$\frac{\partial L}{\partial \lambda_2} = 40 - 5x_1 - 2x_2 \geq 0 \tag{7}$$

$$\lambda_2\frac{\partial L}{\partial \lambda_2} = \lambda_2(40 - 5x_1 - 2x_2) = 0 \tag{8}$$

Solution - Kuhn-Tucker (2)

$$x_1 \geq 0, x_2 \geq 0, \lambda_1 \geq 0, \lambda_2 \geq 0 \tag{9}$$

- Just consider an interior solution (for simplicity), so that $x_1^* > 0, x_2^* > 0, \lambda_1^* > 0, \lambda_2^* > 0$
- Take (5) and divide both sides by -10 to obtain

$$-40 + 2x_1 + 4x_2 = 0 \tag{10}$$

- Sum (7) and (10) to obtain

$$x_1 = \frac{2}{3}x_2 \tag{11}$$

Substitute (11) into e.g. (8) to obtain $x_2^* = 15/2$, which in turn gives $x_1^* = 5$

Solution - graphic analysis

- First we need to identify the feasible set and verify its convexity.
- Identify the vertices of the feasible set
  - $(x_1, x_2) = (6, 0)$; $(x_1, x_2) = (0, 9)$; $(x_1, x_2) = (0, 0)$
  - $g_1 \cap (x_2 = 9) \rightarrow (x_1, x_2) = (2, 9) \quad [\alpha]$
  - $g_2 \cap (x_2 = 9) \rightarrow (x_1, x_2) = (22/5, 9)$
    - Hence, $(x_1, x_2) = (2, 9)$ is a vertex of the feasible set.
  - $g_1 \cap (x_1 = 6) \rightarrow (x_1, x_2) = (6, 7)$
  - $g_2 \cap (x_1 = 6) \rightarrow (x_1, x_2) = (6, 5) \quad [\beta]$
    - Hence, $(x_1, x_2) = (6, 5)$ is a vertex of the feasible set.
  - $g_1 \cap g_2 \rightarrow (x_1, x_2) = (5, \frac{15}{2}) \quad [\gamma]$
- Therefore the feasible set has six vertices: $(0, 0), (0, 9), (2, 9), (5, \frac{15}{2}), (6, 5), (6, 0)$
- and the feasible set is convex.

Solution - graphic analysis (2)

- Note that $\nabla \Pi = (100, 60)$, so f increases north-eastwards.
- Next, identify if the solution(s) is (are) at a vertex or along an edge.
  - Slope of $g_1 : 400 = 20x_1 + 40x_2 \rightarrow \frac{dx_2}{dx_1} = -\frac{1}{2}$
  - Slope of $g_2 : 40 = 5x_1 + 2x_2 \rightarrow \frac{dx_2}{dx_1} = -\frac{5}{2}$
  - Slope of $\Pi : \overline{\Pi} = 100x_1 + 60x_2 \rightarrow \frac{dx_2}{dx_1} = -\frac{5}{3}$
- Thus, $\frac{dx_2}{dx_1}|_{g_1} > \frac{dx_2}{dx_1}|_{\overline{\Pi}} > \frac{dx_2}{dx_1}|_{g_2}$ so that the solution is located at a vertex.
- Evaluating $\Pi(x_1, x_2)$ at each vertex yields that the maximum value of Π is reached at $(x_1^*, x_2^*) = (5, 15/2)$ and $\Pi^* = 950$

Another illustrative example

- The problem. Let $f(x_1, x_2) = 3x_1 + 2x_2$. Solve,

$$\max_{x_1,x_2} 3x_1 + 2x_2 \text{ s.t.}$$
$$2x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 8$$
$$x_1 \ge 0, x_2 \ge 0$$

- or in matrix form

$$\max_{x_1,x_2} \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ s.t. } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

Another illustrative example (2)

- Remarks
  - slope restriction 1 = -2
  - slope restriction 2 = -1/2
  - slope level sets of f =-3/2
  - solution (if it exists) at a vertex
  - $\nabla f = (3, 2)$, increases north-eastwards
  - Vertices of feasible set: $\{(0, 0), (0, 4), (\frac{1}{3}, \frac{10}{3}), (3, 0)\}$

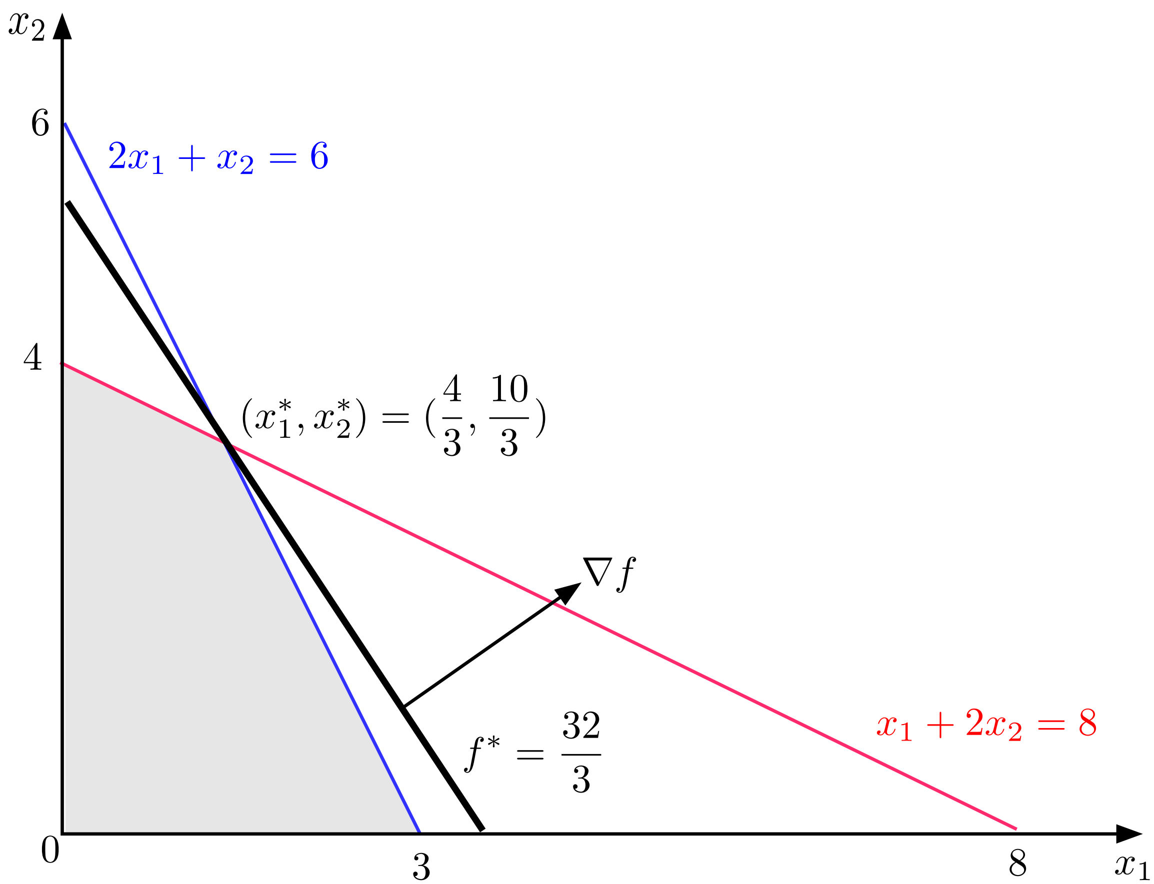
$$(x_1 \geq 0) \cap (x_2 \geq 0) = (0, 0), \quad g_1 \cap g_2 = (4/3, 10/3)$$

$$\min\{g_1(0, x_2) = (0, 6), g_2(0, x_2) = (0, 4)\} = (0, 4)$$

$$\min\{g_1(x_1, 0) = (3, 0), g_2(x_1, 0) = (8, 0)\} = (3, 0)$$

- Solution: evaluate $f(x_1, x_2)$ at each vertex and choose max
- $(x_1^*, x_2^*) = (4/3, 10/3)$ and $f(x_1^*, x_2^*) = 32/3$.

Another illustrative example (3)



Sensitivity analysis

- Suppose

$$2x_1 + x_2 \le 6 + \varepsilon_1$$
$$x_1 + 2x_2 \le 8 + \varepsilon_2$$

  where $\varepsilon_i \in \mathbf{R}$ are small enough variations so that the set of binding constraints under (x_1^*, x_2^*) does not change.
- What is the impact on the solution?
- Let us write the problem as

$$\max_{x_1, x_2} \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{b}$$

  with $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

Sensitivity analysis (2)

- We already know that at $\mathbf{x}^*$ both constraints are binding.
- Also we can write $\mathbf{x}^* = A^{-1}\mathbf{b}$ where

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

- Finally, $f^* \equiv f(\mathbf{x}^*) = \mathbf{c}^T A^{-1}\mathbf{b}$.
- Define $\Delta = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$
- Following the same logic,

$$\mathbf{x}(\Delta) = A^{-1}(\mathbf{b}+\Delta) = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6+\varepsilon_1 \\ 8+\varepsilon_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} + \frac{2\varepsilon_1-\varepsilon_2}{3} \\ \frac{10}{3} + \frac{2\varepsilon_2-\varepsilon_1}{3} \end{pmatrix}$$

Sensitivity analysis (3)

- and

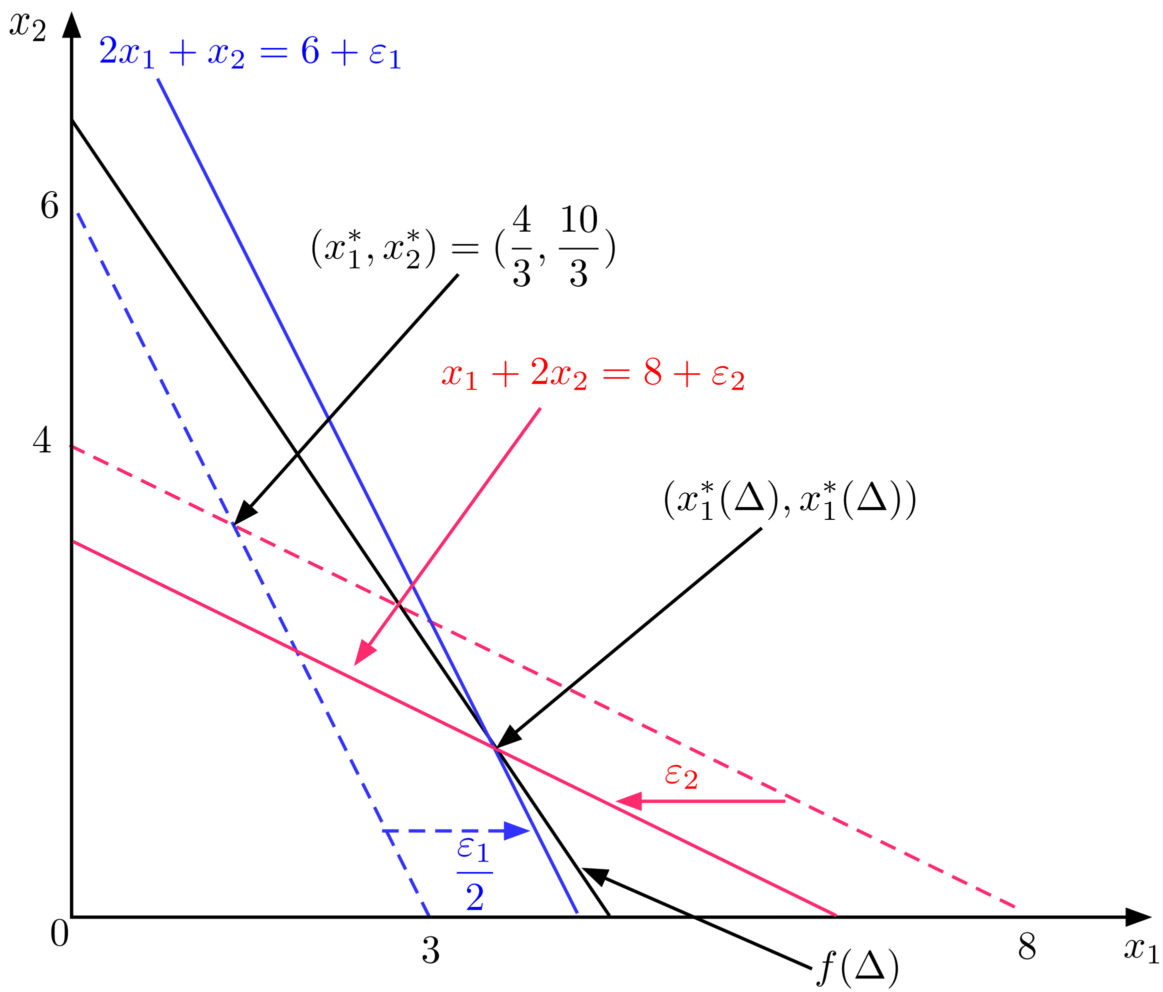
$$f(\Delta) = \mathbf{c}^T \mathbf{x}(\Delta) = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{4}{3} + \frac{2\varepsilon_1 - \varepsilon_2}{3} \\ \frac{10}{3} + \frac{2\varepsilon_2 - \varepsilon_1}{3} \end{pmatrix} = \frac{32}{3} + \frac{1}{3}(4\varepsilon_1 + \varepsilon_2)$$

- so that the overall variation in the value of the objective function is $f(\Delta) - f^* = \frac{1}{3}(4\varepsilon_1 + \varepsilon_2)$
- How is it distributed between x_1 and x_2?
- The contribution of $x_1 (x_2)$ to f is $3x_1 (2x_2)$. Given that the variation of x_i is $x_i(\Delta) - x_i^* = \frac{2\varepsilon_i - \varepsilon_j}{3}$, it follows

$$(f(\Delta) - f^*)_{x_1} = 3\frac{2\varepsilon_1 - \varepsilon_2}{3} = 2\varepsilon_1 - \varepsilon_2$$

$$(f(\Delta) - f^*)_{x_2} = 2\frac{2\varepsilon_2 - \varepsilon_1}{3} = \frac{2}{3}(2\varepsilon_2 - \varepsilon_1)$$

Sensitivity analysis (4)



Sensitivity analysis (5)

- From a different perspective, sensitivity analysis refers to the impact of a softening (tightenning) of the restrictions on the value of the objective function.
- We have obtained $f(\Delta) - f^* = \frac{1}{3}(4\varepsilon_1 + \varepsilon_2) \quad [\alpha]$
- Rewrite it as

$$f(\Delta) - f^* = \mathbf{y}^T \Delta = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

- y_i measures the sensitivity of f to the softening (tightenning) of the restriction g_i.
- $y_1 = \frac{4}{3}$ means that for each additional € available in resource 1, the value of f varies in 4/3€.
- $y_2 = \frac{1}{3}$ means that for each additional € available in resource 2, the value of f varies in 1/3€.

Shadow prices

- Recall $\mathbf{y}^T = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \end{pmatrix}$
- The sensitivities of f to changes in the restrictions are called shadow prices.
- Shadow prices represent
  - max price at which to buy an additional unit of the corresponding resource
  - min price at which to sell units of that input
- Therefore, shadow price of an input represent the unit value of that input.
- As a consequence, the stock of inputs of a company valued at their respective shadow prices gives the maximum value of f
- In our example, $(6)(\frac{4}{3}) + (8)(\frac{1}{3}) = \frac{32}{3} = f^*$
- or $f^* = \mathbf{y}^T\mathbf{b}$

Shadow prices (2)

- Why?
- Recall $f(\Delta) = f^* + \mathbf{y}^T\Delta \quad [\alpha]$
- Let $\widehat{\Delta} = \begin{pmatrix} -6 \\ -8 \end{pmatrix} = -\mathbf{b}$
- At $\widehat{\Delta}$ inputs are zero, $f = 0$, and the restrictions are still binding [so that we are not changing the nature of the problem]
- From $[\alpha]$

$$f(\widehat{\Delta}) = f^* + \mathbf{y}^T\widehat{\Delta} = \frac{32}{3} + \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -6 \\ -8 \end{pmatrix} = \frac{32}{3} - \frac{32}{3}$$

$$f(\widehat{\Delta}) = 0 = f^* + \mathbf{y}^T\widehat{\Delta} \Rightarrow f^* = -\mathbf{y}^T\widehat{\Delta}$$

$$f^* = \mathbf{y}^T(-\widehat{\Delta}) = \mathbf{y}^T\mathbf{b}$$

Duality - Introduction and motivation

- Shadow prices allow to solve another, related, linear program - the dual-.
- Suppose a buyer proposes to acquire all the assets (inputs) of the company
  - buyer aims at min the cost of acquisition
  - seller only sells if at least obtains as much as what can get by producing:
    - one unit of x_1 contributes $3x_1$ to f. Producing that unit requires 2 units of input 1 (of the available 6) and 1 unit of input 2 (of the available 8).
    - one unit of x_2 contributes $2x_2$ to f. Producing that unit requires 1 units of input 1 (of the available 6) and 2 units of input 2 (of the available 8).
  - thus seller requires unit prices (y_1, y_2) such that

$$2y_1 + y_3 \geq 3 \quad \text{and} \quad y_1 + 2y_2 \geq 2$$

Duality - Introduction and motivation (2)

- the company has 6 units of input 1 and 8 units of input 2.
- the buyer wants to set prices (y_1, y_2) that minimize the amount of money to pay for the production capacity of the company, namely $\min_{y_1,y_2} 6y_1 + 8y_2$ subject to the conditions of the seller. Formally,

$$\min_{y_1,y_2} 6y_1 + 8y_2 \text{ s.t.}$$
$$2y_1 + y_3 \geq 3$$
$$y_1 + 2y_2 \geq 2$$

- in matrix form, recall $\mathbf{b} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \;\text{ s.t. }\; A^T\mathbf{y} \geq \mathbf{c},\; \mathbf{y} \geq 0$$

Duality - Introduction and motivation (3)

- The solution to this program (see problem [6.2]) is

$$\mathbf{y}^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

  and $F(y_1^*, y_2^*) = \frac{32}{3}$

- Remark 1: $\mathbf{y}^*$ corresponds to the shadow prices of the primal problem.
- Remark 2: $F(y_1^*, y_2^*) = f(x_1^*, x_2^*)$
- Remark 3: constraints of F^* guarantee that seller receives as much money from selling than from producing $\Rightarrow$ prices $\mathbf{y}^*$ used by buyer to value inputs correspond to (min) prices at which seller is willing to sell. These are the shadow prices!

Duality

- To every linear programming problem (*primal problem*) there corresponds a dual problem.
- Example1:
  - Let the primal problem be a profit maximizations subject to resource constraints.
  - The dual problem is a minimization of the total cost of the resources subject to constraints that the value of the resources used in producing one unit of each output be at least as great as the profit received from the sale of that output.
- Variables of the dual problem are Lagrange multipliers for the primal problem. $\rightarrow$ interpretation as sensitivity of optimal value of objective function of primal problem wrt changes in frontier of constraints.
- i.e. dual variables are (shadow) prices.

Duality (2)

- Example $[n = 3, m = 2]$.
  - Let the *primal problem* be
    $\max_{\mathbf{x}} f(\mathbf{x}) = \mathbf{cx}$ s.t. $\mathbf{Ax} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$
    where
    $$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \ \mathbf{c} = (c_1, c_2, c_3), \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$ and
    $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$
  - or equivalently, for $i = 1, 2; \ j = 1, 2, 3$
    $\max_{\{x_1, x_2, x_3\}} \sum_{j=1}^{3} c_j x_j$ s.t. $\sum_{j=1}^{3} a_{ij} x_j \leq b_i, \ x_j \geq 0.$

Duality (3)

- Example $[n = 3, m = 2]$ (cont'd)
  - The corresponding *dual problem* is
    $\min_{\mathbf{y}} g(\mathbf{y}) = \mathbf{y}\mathbf{b}$ s.t. $\mathbf{y}\mathbf{A} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$
    where
    $\mathbf{y} = (y_1, y_2),\ y_i \geq 0$
  - or equivalently, for $i = 1, 2;\ j = 1, 2, 3$
    $\min_{\{y_1, y_2\}} \sum_{i=1}^{2} b_i y_i$ s.t. $\sum_{i=1}^{2} a_{ij} y_i \geq c_j,\ y_i \geq 0.$
- Remarks
  - both problems look for an extremum of a linear function s.t. linear inequality constraints
  - both problems use the same parameters $\mathbf{A}, \mathbf{b}, \mathbf{c}$
  - the dual to the dual problem recovers the original one.

Duality - theorem

- The maximum value of the primal problem equals the minimum value of the dual problem
- The constraints of the primal problem appear in the objective function of the dual problem

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The dual linear programming problem

- The primal problem has 4 constraints $\rightarrow$ the dual problem has 4 variables. Denote them as w_1, w_2, w_3, w_4.
- The primal problem has two variables $\rightarrow$ the dual problem has 2 constraints.
- Finally, the non-negativity of the variables of the dual problem is also required.

$$\min_{w_1,w_2,w_3,w_4} Z = 400w_1 + 40w_2 + 6w_3 + 9w_4 \quad \text{s.t.}$$

$$20w_1 + 5w_2 + w_3 \geq 100$$

$$40w_1 + 2w_2 + w_4 \geq 60$$

$$w_1 \geq 0,\ w_2 \geq 0,\ w_3 \geq 0,\ w_4 \geq 0$$

The dual linear programming problem (2)

Solution of the dual problem

- $w_1^* = 0.625$ €, $w_2^* = 17.5$ €, $w_3^* = 0$ €, $w_4^* = 0$ €, $\quad Z^* = 950$ €.

Economic interpretation of the dual variables

- Rate of change in total profits (marginal profit) if an additional unit of a given input is made available.
- $w_1^* = 0.625$ € means that profits could be increased by as much as 0.625 € if an extra unit of wood would be available in the production process.
- A dual variable =0 means that profits would not increase if additional resources were available. The restriction is not binding in the optimal solution.
- In this sense, the dual variables measure the *shadow prices* of each of the resources. They are associated to the lagrange multipliers λ_i.

Equivalence of primal and dual problems

- Primal problem:
  $\max_{\mathbf{x}} f(\mathbf{x}) = \mathbf{cx}$ s.t. $\mathbf{Ax} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$
- Lagrangean function:
  $L(\mathbf{x}, \lambda) = \mathbf{cx} + \lambda(\mathbf{b} - \mathbf{Ax})$
- K-T conditions:

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{c} - \lambda \mathbf{A} \leq \mathbf{0}$$

$$\frac{\partial L}{\partial \mathbf{x}}\mathbf{x} = (\mathbf{c} - \lambda \mathbf{A})\mathbf{x} \leq \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = \mathbf{b} - \mathbf{Ax} \geq \mathbf{0}$$

$$\lambda \frac{\partial L}{\partial \lambda} = \lambda(\mathbf{b} - \mathbf{Ax}) \geq \mathbf{0}$$

$$\mathbf{x} \geq \mathbf{0}, \quad \lambda \geq \mathbf{0}$$

Equivalence of primal and dual problems (2)

- Dual problem:
  $\min_{\mathbf{y}} g(\mathbf{y}) = \mathbf{yb}$ s.t. $\mathbf{yA} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$
- Lagrangean function:
  $L(\mathbf{y}, \mu) = \mathbf{yb} + (\mathbf{c} - \mathbf{yA})\mu$
- K-T conditions:

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{y}} &= \mathbf{b} - \mathbf{A}\mu \geq \mathbf{0} \\
\mathbf{y}\frac{\partial L}{\partial \mathbf{y}} &= \mathbf{y}(\mathbf{b} - \mathbf{A}\mu) = \mathbf{0} \\
\frac{\partial L}{\partial \mu} &= \mathbf{c} - \mathbf{yA} \geq \mathbf{0} \\
\frac{\partial L}{\partial \mu}\mu &= (\mathbf{c} - \mathbf{yA})\mu = \mathbf{0} \\
\mathbf{y} \geq \mathbf{0}, &\quad \mu \geq \mathbf{0}
\end{aligned}
$$

Equivalence of primal and dual problems (3)

Remarks

- If $\mathbf{x} = \mu$ and $\mathbf{y} = \lambda$, both sets of conditions are identical
- The decision variables of one problem are the lagrange multipliers of the dual problem.

Duality theorem

- A necessary and sufficient condition for a feasible vector $\mathbf{x}^*$ to represent a solution to a linear programming problem is that there exists a feasible vector $\mathbf{y}^*$ for the dual problem for which the values of the objective functions of both problems are equal.
- Formally,

$$f(\mathbf{x}^*) \geq f(\mathbf{x}), \forall \mathbf{x} \in X \iff \exists \mathbf{y}^* \in Y \text{ s.t. } \begin{cases} g(\mathbf{y}^*) \geq g(\mathbf{y}), \forall \mathbf{y} \in Y \\ f(\mathbf{x}^*) = g(\mathbf{y}^*) \end{cases}$$

Appendix. Illustrating the simplex algorithm

The problem

$$\max_{x_1,x_2} f(x_1,x_2) = 3x_1 + 2x_2 \text{ s.t.}$$

$$2x_1 + x_2 \le 6$$

$$x_1 + 2x_2 \le 8$$

$$x_1 \ge 0,\ x_2 \ge 0$$

Step 1: add slack variables in the constraints

$$2x_1 + x_2 + s_1 = 6$$

$$x_1 + 2x_2 + s_2 = 8$$

$$x_1 \ge 0,\ x_2 \ge 0,\ s_1 \ge 0,\ s_2 \ge 0$$

Appendix. Illustrating the simplex algorithm (2)

Step 2: Select a vertex of the feasible set and evaluate (s_1, s_2, f). If feasible, select $(x, y) = (0, 0)$

- At $(x, y) = (0, 0)$, it follows $(s_1, s_2) = (6, 8)$ and $f(0, 0) = 0$

Step 3: solve for the constraints and the objective function in terms of the variables equal to zero in the solution (*non-basic variables)*. [Variables different from zero are called *basic variables*]

$$s_1 = 6 - 2x_1 - x_2 \tag{12}$$

$$s_2 = 8 - x_1 - 2x_2 \tag{13}$$

$$f = 3x_1 + 2x_2 \tag{14}$$

Appendix. Illustrating the simplex algorithm (3)

Step 4: Move to a neighboring vertex

- For each non-basic variable, determine the maximum increased within the feasible set:
  - Consider x_1. According to (12), its maximum possible increase is 3.
  - Consider x_1. According to (13), its maximum possible increase is 8.
  - Thus, maximum feasible increase of x_1 is 3.
  - Consider x_2. According to (12), its maximum possible increase is 6.
  - Consider x_2. According to (13), its maximum possible increase is 4.
  - Thus, maximum feasible increase of x_2 is 4.

Appendix. Illustrating the simplex algorithm (4)

Step 4: Move to a neighboring vertex (cont'd)

- Compute the increase in f associated to the increase in x_1 and x_2 respectively
  - For $\Delta x_1 = 3$ it follows $\Delta f = 9$
  - For $\Delta x_2 = 4$ it follows $\Delta f = 8$
- move along the direction of maximum increase of f, namely $x_1 = 3$.
- For $(x_1, x_2) = (3, 0)$ substituting in (12) and (13) we obtain $(s_1, s_2) = (0, 5)$.
- In this way we generate a new basic solution $(x_1, x_2, s_1, s_2, f) = (3, 0, 0, 5, 9)$ with non-basic variables x_2, s_1.
- Remark: The movement from one basic solution to another is called *pivot transformation*. It is the central feature of the simplex algorithm.

Appendix. Illustrating the simplex algorithm (5)

Step 5: repeat step 3 using the new basic solution

- solve for the constraints and the objective function in terms of the non-basic variables:
  - From (12),

$$x_1 = 3 - \frac{1}{2}x_2 - \frac{1}{2}s_1 \tag{15}$$

  - Substituting (15) into (13),

$$s_2 = 5 - \frac{3}{2}x_2 + \frac{1}{2}s_1 \tag{16}$$

  - Substituting (15) into (14),

$$f = 9 + \frac{1}{2}x_2 - \frac{3}{2}s_1 \tag{17}$$

Appendix. Illustrating the simplex algorithm (6)

Step 6: repeat step 4.

- Move to a neighboring vertex. For each non-basic variable, determine the maximum increased within the feasible set:
  - Note that f can only increase in the direction x_2.
  - Consider x_2. According to (15), its maximum possible increase is 6.
  - Consider x_2. According to (16), its maximum possible increase is 10/3.
  - Thus, maximum feasible increase of x_2 is 10/3.

Appendix. Illustrating the simplex algorithm (7)

Step 6: (cont'd)

- Compute the increase in f associated to the increase in x_2 (using (17))
  - For $\Delta x_2 = 10/3$ it follows $\Delta f = 5/3$
- Move along this direction of maximum increase of f, namely $x_2 = 10/3$.
- For $(s_1, x_2) = (0, 10/3)$ substituting in (15) and (16) we obtain $(x_1, s_2) = (4/3, 0)$.
- In this way we generate a new basic solution $(x_1, x_2, s_1, s_2, f) = (4/3, 10/3, 0, 0, 32/3)$ with non-basic variables s_1 and s_2.

Appendix. Illustrating the simplex algorithm (8)

Step 7: repeat step 3 using the new basic solution

- solve for the constraints and the objective function in terms of the non-basic variables:

$$x_1 = \frac{4}{3} - \frac{2}{3}s_1 + \frac{1}{3}s_2 \tag{18}$$

$$s_2 = \frac{10}{3} + \frac{1}{3}s_1 - \frac{2}{3}s_2 \tag{19}$$

$$f = \frac{32}{3} - \frac{4}{3}s_1 - \frac{1}{3}s_2 \tag{20}$$

- Note $\frac{\partial f}{\partial s_1} < 0$ and $\frac{\partial f}{\partial s_2} < 0$. Hence, there is no direction in which f can increase.

Conclusion: The solution generated by the simplex algorithm is

$x_1^* = \frac{4}{3}, \quad x_2^* = \frac{10}{3}, \quad f(x_1^*, x_2^*) = \frac{32}{3}$

Appendix. Illustrating the simplex algorithm (9)