Optimization. A first course of mathematics for economists

Xavier Martinez-Giralt
Universitat Autònoma de Barcelona
xavier.martinez.giralt@uab.eu

I.2.- Continuity

Continuity

Intuition

- A function f is continuous when given any two arbitrarily close points of its domain generate images arbitrarily close.

Formal definition - preliminaries

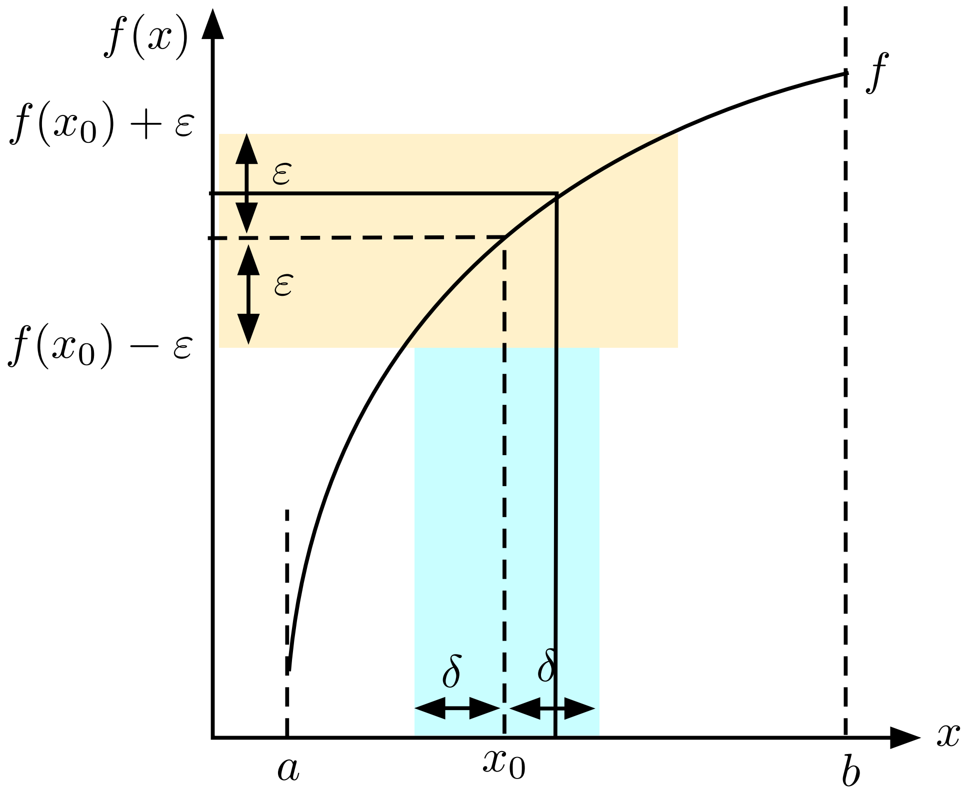
- Let $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^m$. Let x_0 be an accumulation point of A.
- We say that $b \in \mathbf{R}^m$ is the limit of f at the point $x_0 \in A$, $\lim_{x \to x_0} f(x) = b$, if given an arbitrary $\varepsilon > 0, \exists \delta > 0$ (dependent of f, x_0 and ε) such that $\forall x \in A, x \neq x_0, \|x - x_0\| < \delta$ implies $\|f(x) - b\| < \varepsilon$.
- Remark 1: If x_0 is not accumulation point, $\nexists x \neq x_0,\ x \in A$ close to x_0, and the definition is empty of content.
- Remark 2: It may happen that the limit of a function at a point does not exist. But whenever it exists, it is unique.

Continuity (2)

Formal definitions

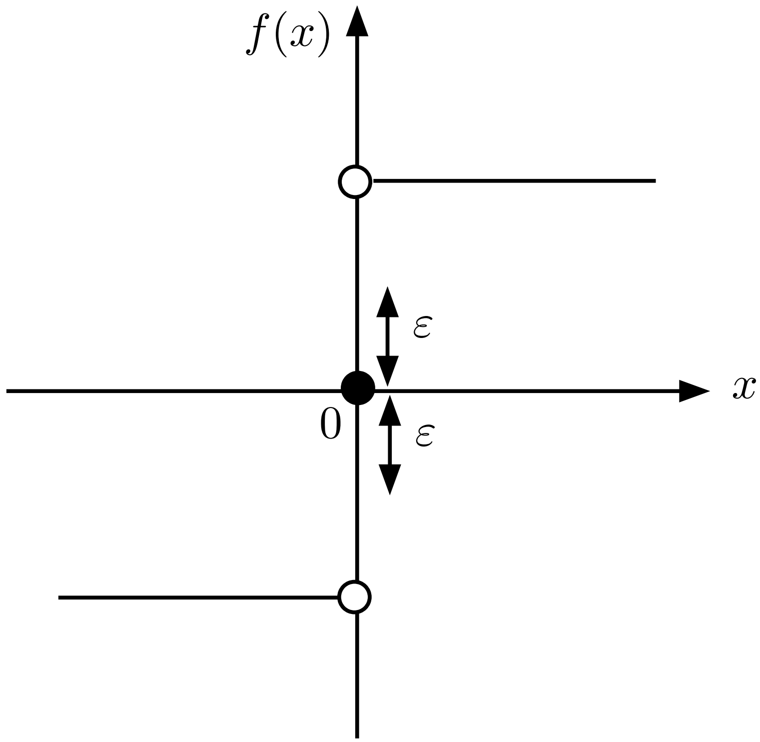
- Let $A \subset \mathbf{R}^n,\ f : A \to \mathbf{R}^m$. Let $x_0 \in A$. We say that f is continuous at a point $x_0 \in A$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in A, \|x - x_0\| < \delta$ implies $\|f(x) - f(x_0)\| < \varepsilon$.
- We say that f is continuous on $B \subset A$ if it is continuous $\forall x \in B$.
- When we say that "f is continuous" it means that f is continuous on its domain A.
- Continuity of f in $[a, b]$:
  - f continuous in (a, b), i.e. $\lim_{x \to x_0} f(x) = f(x_0)$ and
  - f right-continuous at a i.e. $\lim_{x \to a^+} f(x) = f(a)$ and
  - f left-continuous at b i.e. $\lim_{x \to b^-} f(x) = f(b)$

Continuity - Illustration



f is continuous at x_0

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$



f is not continuous at 0

Algebra with Continuous Functions

Preliminaries

- Let $f : A \to \mathbf{R}^m$ and $g : B \to \mathbf{R}^p$ be two functions such that $f(A) \subset B$. The composition of function g with function f, denoted as $g \circ f : A \to R^p$ is defined as $x \mapsto g(f(x))$.
- Let $f : A \to \mathbf{R}^m$ and $g : B \to \mathbf{R}^p$ be two continuous functions such that $f(A) \subset B$. Then, $g \circ f : A \to R^p$ is a continuous function.
- Let $A \subset \mathbf{R}^n$. Let x_0 be an accumulation point of A. Let $f : A \to R^m$ and $g : A \to R^m$ be two functions. Assume $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$.
    - Then, $\lim_{x \to x_0} (f + g)(x) = a + b$, where $f + g : A \to R^m$ is defined as $(f + g)(x) = f(x) + g(x)$.
    - Then, $\lim_{x \to x_0} (f \cdot g)(x) = ab$, where $f \cdot g : A \to R^m$ is defined as $(f \cdot g)(x) = f(x)g(x)$.

Algebra with Continuous Functions (2)

Preliminaries (cont'd)

- Assume $\lim_{x \to x_0} f(x) = a \neq 0$ and f is not zero in a neighborhood of x_0
  - Then, $\lim_{x \to x_0} (g/f)(x) = b/a$, where $g/f : A \to R^m$ is defined as $(g/f)(x) = g(x)/f(x)$.

Algebra with Continuous Functions

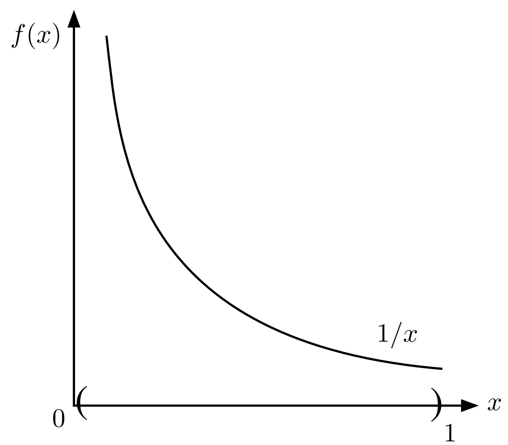
Let $A \subset \mathbf{R}^n$. Let x_0 be an accumulation point of A.

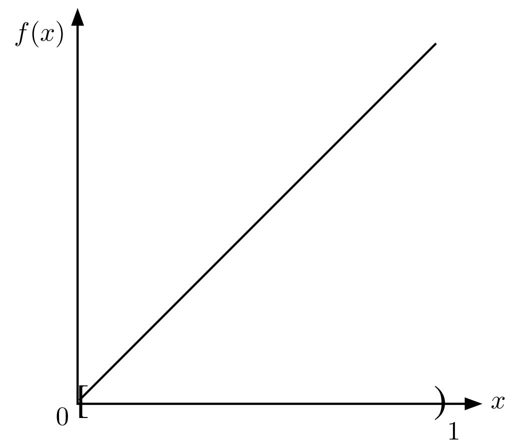
- Let $f : A \to R^m$ and $g : A \to R^m$ be two continuous functions at x_0. Then, $f + g : A \to R^m$ is continuous at x_0.
- Let $f : A \to R^m$ and $g : A \to R^m$ be two continuous functions at x_0. Then, $f \cdot g : A \to R^m$ is continuous at x_0.
- Let $f : A \to R^m$ and $g : A \to R^m$ be two continuous functions at x_0. Let $f(x_0) \neq 0$. Then, f is not zero in a neighborhood U of x_0, and $g/f : U \to R^m$ is continuous at x_0.

Boundedness Theorem

Intuition

- A continuous function defined on a compact set attains its maximum and minimum values at some point of the set.
- Remark 1: A continuous function need not be bounded. Example: $f(x) = 1/x, \; x \in (0, 1)$.
- Remark 2: A continuous and bounded function need not reach its maximum value at any point of its domain. Example: $f(x) = x, \; x \in [0, 1)$.





Boundedness Theorem (2)

Theorem

- Let $A \subset \mathbf{R}^n$ and $f : A \to \mathbf{R}$ be continuous.
- Let $K \subset A$ be a compact set.
- Then, f is bounded on K, that is $B = \{f(x)|x \in K\}$ is a bounded set.
- Furthermore, $\exists(x_0, x_1) \in K$ such that $f(x_0) = \inf(B)$ and $f(x_1) = \sup(B)$
- where $\sup(B)$ denotes the absolute maximum of f on K and $\inf(B)$ denotes the absolute minimum of f on K.

Intermediate Value Theorem

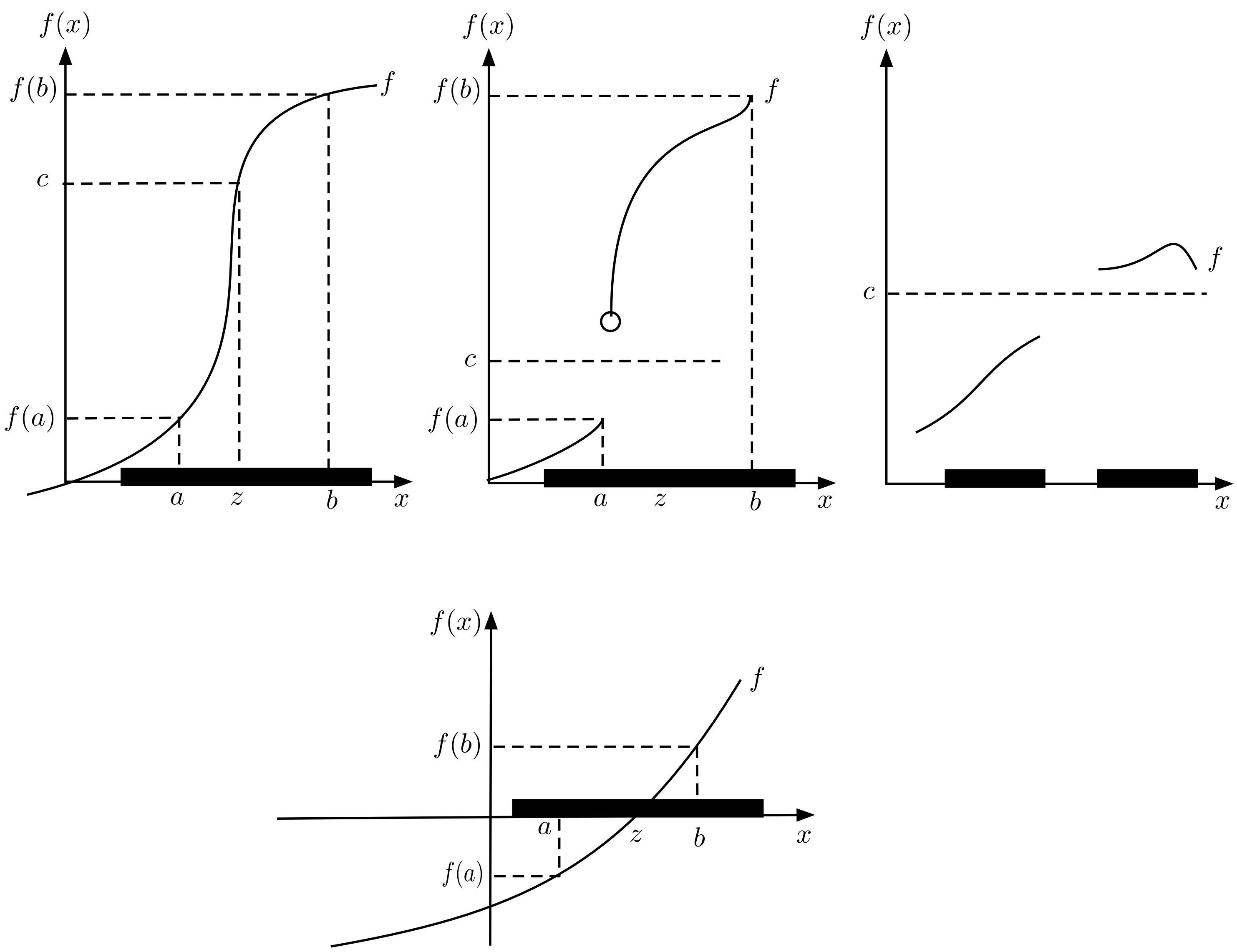
The Intermediate Value Theorem

- Let $A \subset \mathbf{R}^n$
- Let $f : A \to \mathbf{R}$ be a continuous function.
- Let $K \subset A$ be a connected set.
- Consider $a, b \in K$.
- Then, $\forall c \in [f(a), f(b)], \exists z \in K$ such that $f(z) = c$.

Bolzano's Theorem

- Let $f : A \to \mathbf{R}$ be a continuous function.
- Let $K \subset A$ be a connected set.
- Consider $a, b \in K$, such that $sign f(a) \neq sign f(b)$.
- There $\exists z \in K$ such that $f(z) = 0$.

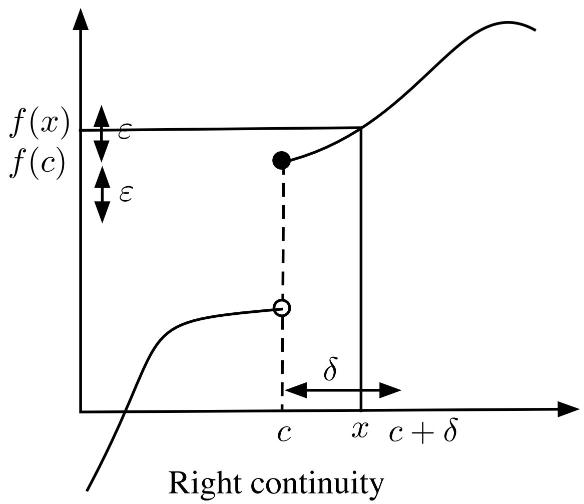
Intermediate Value and Bolzano Theorems - Illustration



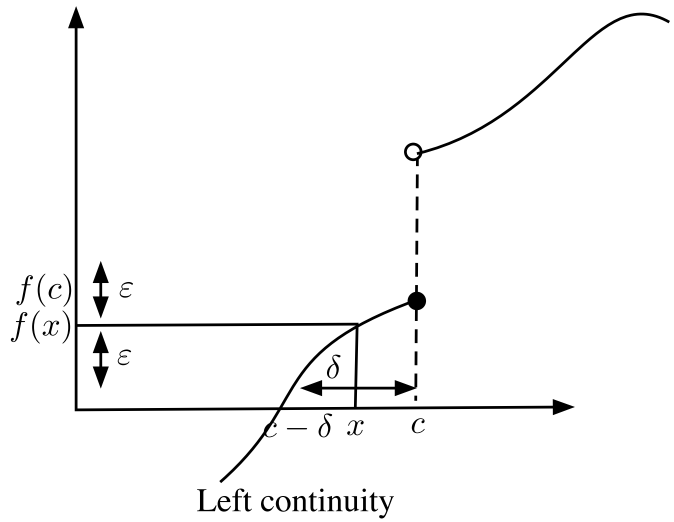
Weaker concepts of continuity

- directional continuity
  - right continuity - no jump when appraoching the limit from the right
  - left continuity - no jump when appraoching the limit from the left
- semi-continuity
  - upper semi-continuity: jumps if any, only go up
  - lower semi-continuity: jumps if any, only go down

Weaker concepts of continuity- Illustration

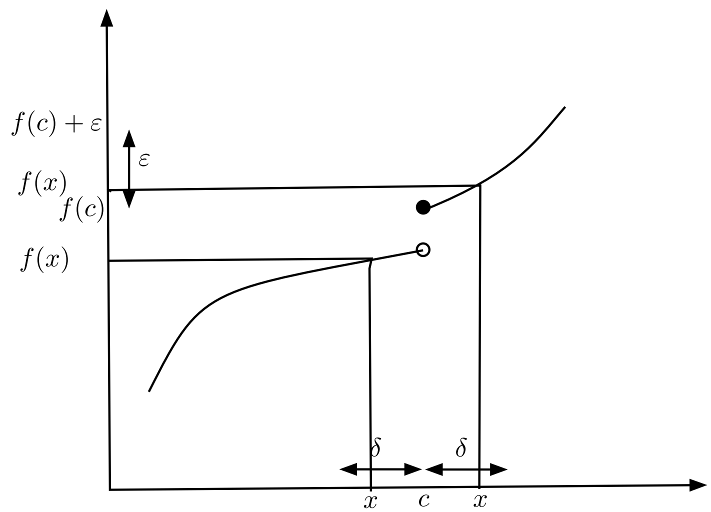


$\forall x \in (c, c+\delta),\ |(f(x) - f(c)| < \varepsilon$



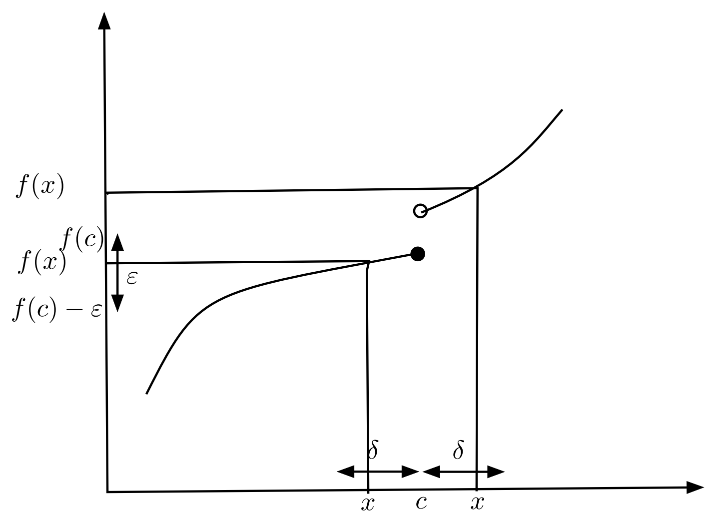
$\forall x \in (c-\delta, c),\ |(f(x) - f(c)| < \varepsilon$

Weaker concepts of continuity- Illustration



Upper semi-continuity

$\forall x, |x-c| < \delta, f(x) \leq f(c) + \varepsilon$



Lower semi-continuity

$\forall x, |x-c| < \delta, f(x) \geq f(c) - \varepsilon$