Optimization. A first course of mathematics for economists

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I.1.- Topology

Metric Spaces

Definitions

- Let E be a set over which a notion of "distance" between any two elements can be applied.
- Distance between x and y, $(x, y) \in E$ is a function d,

$$d : E \times E \to R,$$

satisfying the following properties:

$$\forall (x, y) \in E,\ d(x, y) \geq 0$$

$$\forall (x, y) \in E,\ d(x, y) = 0 \Leftrightarrow x = y$$

$$\forall (x, y) \in E,\ d(x, y) = d(y, x) \text{ (symmetry)}$$

$$\forall (x, y, z) \in E,\ d(x, y) \leq d(x, z) + d(y, z) \text{ (triangle inequality).}$$

- A pair (E, d) is called a metric space.

On the notion of distance

Lemma: In a metric space (E, d)
$\forall x, y, z, t \in E, |d(x, y) - d(z, t)| \leq d(x, z) + d(y, t).$
In particular,
$\forall x, y, z \in E, |d(x, z) - d(y, z)| \leq d(x, y).$

Distance between a point and a set

- Let (E, d) be a metric space. Let $x_0 \in E$ and $A \subset E$.
- Denote by $\{d(x_0, x)\}_{x \in A}$ the set of real numbers defined by the distances from x_0 to each element of A. This set has a lower bound of zero. Thus, it admits an infimum not smaller than zero.
- The distance from x_0 to the set A is the real number $d(x_0, A) = \inf\{d(x_0, x)\}_{x \in A}$.

Remark: infimum vs. minimum

- Infimum ($\inf$): greatest lower bound (GLB)
- If GLB belongs to the set $\rightarrow \inf = \min$
- example: Let $A = \{2, 3, 4\}$. Then,
  - $\inf\{2, 3, 4\} = 2$
  - Note 1 is also a lower bound but it is not the GLB.
  - $2 = \min\{2, 3, 4\}$
- If GLB $\notin$ set:
- example: Let $A = \{x \in \mathbf{R} | 0 < x < 1\}$. Then,
  - $\inf\{0 < x < 1\} = 0$.
  - $\min\{0 < x < 1\} = \nexists$.
- Parallel argument for $\sup$ vs. $\max$

On the notion of distance (2)

Distance between two sets

- Let (E, d) be a metric space. Let $A, B \subset E$, $A, B \neq \emptyset$.
- Denote by $\{d(x, y)\}_{x \in A, y \in B}$ the set of real numbers defined by the distances between a point of A and a point of B. This set has a lower bound of zero. Thus, it admits an infimum not smaller than zero.
- The distance between sets A and B is the real number $d(A, B) = \inf\{d(x, y)\}_{x \in A, y \in B}$

Euclidean Spaces

Definition

- Particular case of a metric space where $E = \mathbf{R}^n$

Properties

- Let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$; let $\alpha \in \mathbf{R}$.
- Define the following vector operations ($i = 1, \ldots, n$)

$$x + y = (x_1 + y_1, \ldots, x_n + y_n) \in \mathbf{R}^n \qquad \text{(addition)}$$

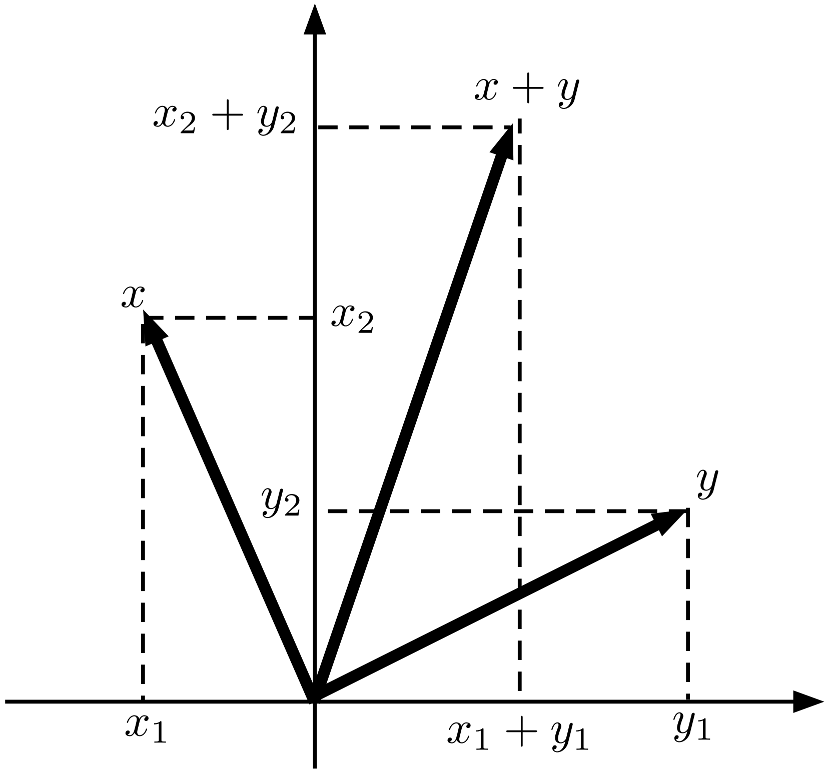
$$\alpha x = (\alpha x_1, \ldots, \alpha x_n) \in \mathbf{R}^n \qquad \text{(scalar product)}$$

$$\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} \in \mathbf{R} \qquad \text{(euclidean norm)}$$

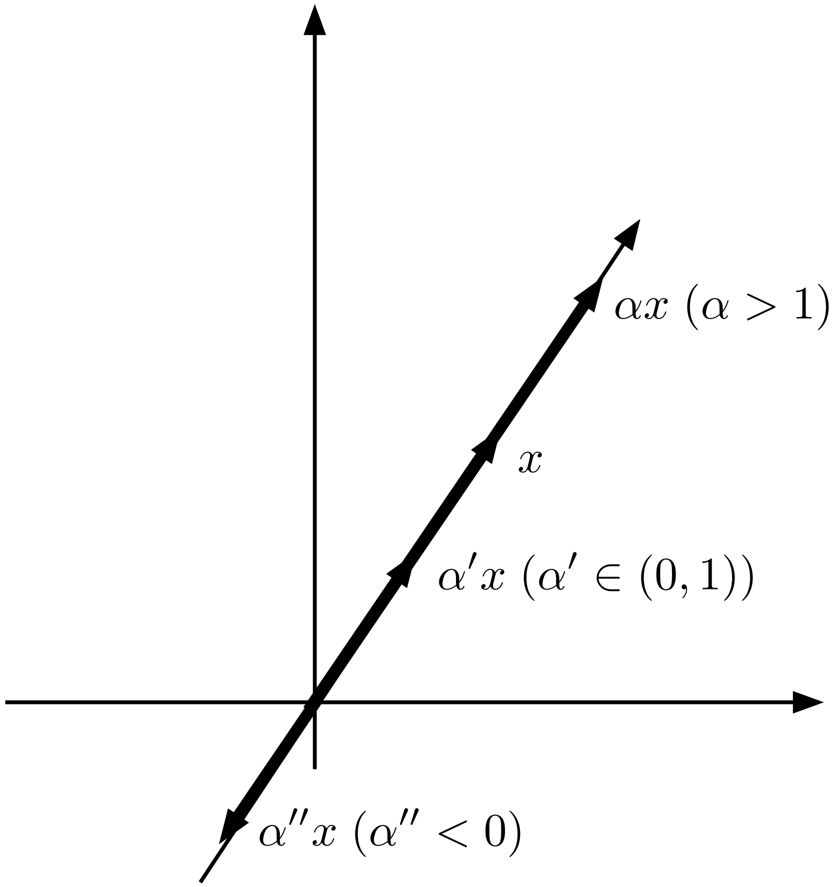
$$< x, y > = \sum_{i=1}^{n} x_i y_i \in \mathbf{R} \qquad \text{(inner [dot] product)}$$

$$< x, y > = \|x\| \|y\| \cos(\theta_x - \theta_y) \in \mathbf{R} \qquad \text{(inner product)}$$

Vector operations - Illustration

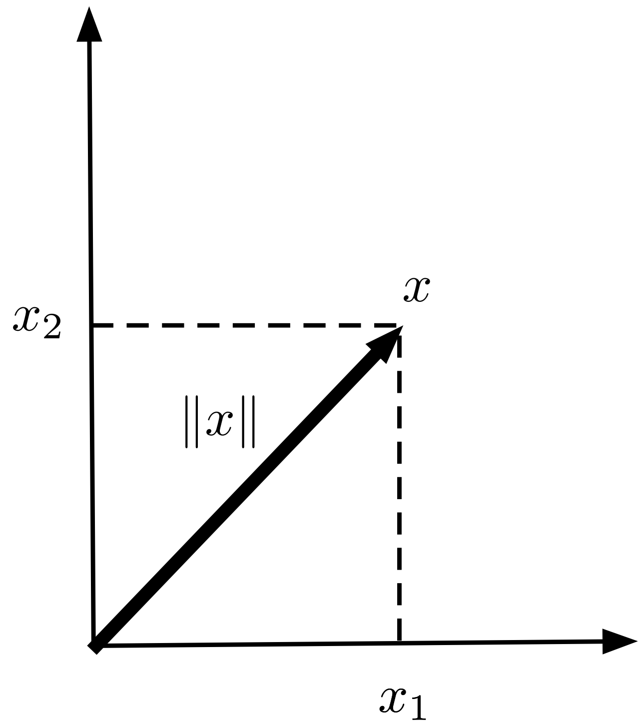


Vector addition

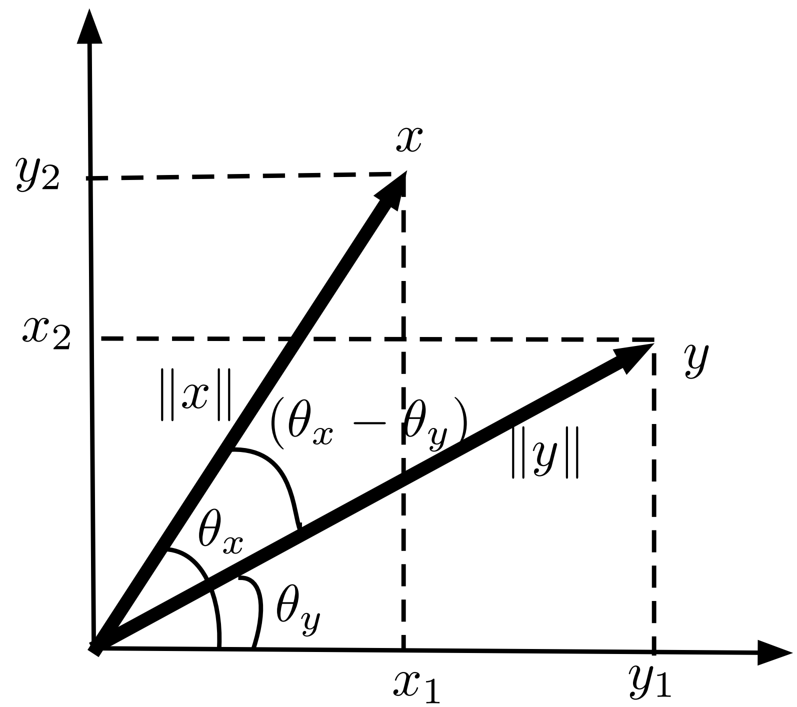


Scalar product

Norm and inner product - Illustration



Norm



Inner product

Euclidean Spaces (2)

The euclidean norm - properties

$$\forall x \in \mathbf{R}^n, \|x\| \geq 0, \quad \text{and } = 0 \Leftrightarrow x = 0,$$

$$\forall x \in \mathbf{R}^n, \forall \alpha \in \mathbf{R}, \|\alpha x\| = |\alpha| \|x\|,$$

$$\forall x, y \in \mathbf{R}^n, \|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)}.$$

Triangle inequality - proof

$$\|x + y\|^2 = < x + y, x + y >$$

$$= < x, x > +2 < x, y > + < y, y >$$

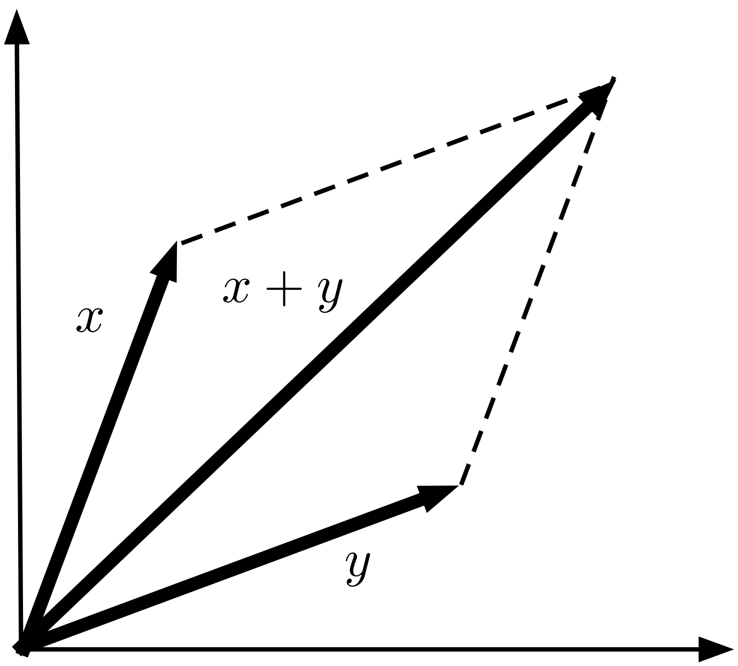
$$\leq \|x\|^2 + 2| < x, y > | + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \text{[apply Cauchy-Schwartz ineq]}$$

$$= (\|x\| + \|y\|)^2$$

Triangle inequality - Illustration

$$\|x+y\| \leq \|x\| + \|y\|$$



- if not satisfied x, y and $x+y$ cannot draw a triangle

Triangle inequality - example

$$x = (1,2) \to \|x\| = \sqrt{5}$$

$$y = (2,1) \to \|y\| = \sqrt{5}$$

$$\|x\| + \|y\| = 2\sqrt{5} \approx 4.47$$

$$x + y = (3,3) \to \|x+y\| = \sqrt{18} = 3\sqrt{2} \approx 4.24$$

and $4.24 \approx \|x+y\| < \|x\| + \|y\| \approx 4.47$

Exercise: Show when $\|x+y\| = \|x\| + \|y\|$

Euclidean Spaces (3)

Euclidean distance - definition

$\forall x, y \in \mathbf{R}^n, d(x,y) = \|x-y\| = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{\frac{1}{2}}$

Euclidean distance - properties

$$\forall (x,y) \in E,\ d(x,y) \geq 0$$

$$\forall (x,y) \in E,\ d(x,y) = 0 \Leftrightarrow x = y$$

$\forall (x,y) \in E,\ d(x,y) = d(y,x)$ (symmetry)

$\forall (x,y,z) \in E,\ d(x,y) \leq d(x,z) + d(y,z)$ (triangle inequality).

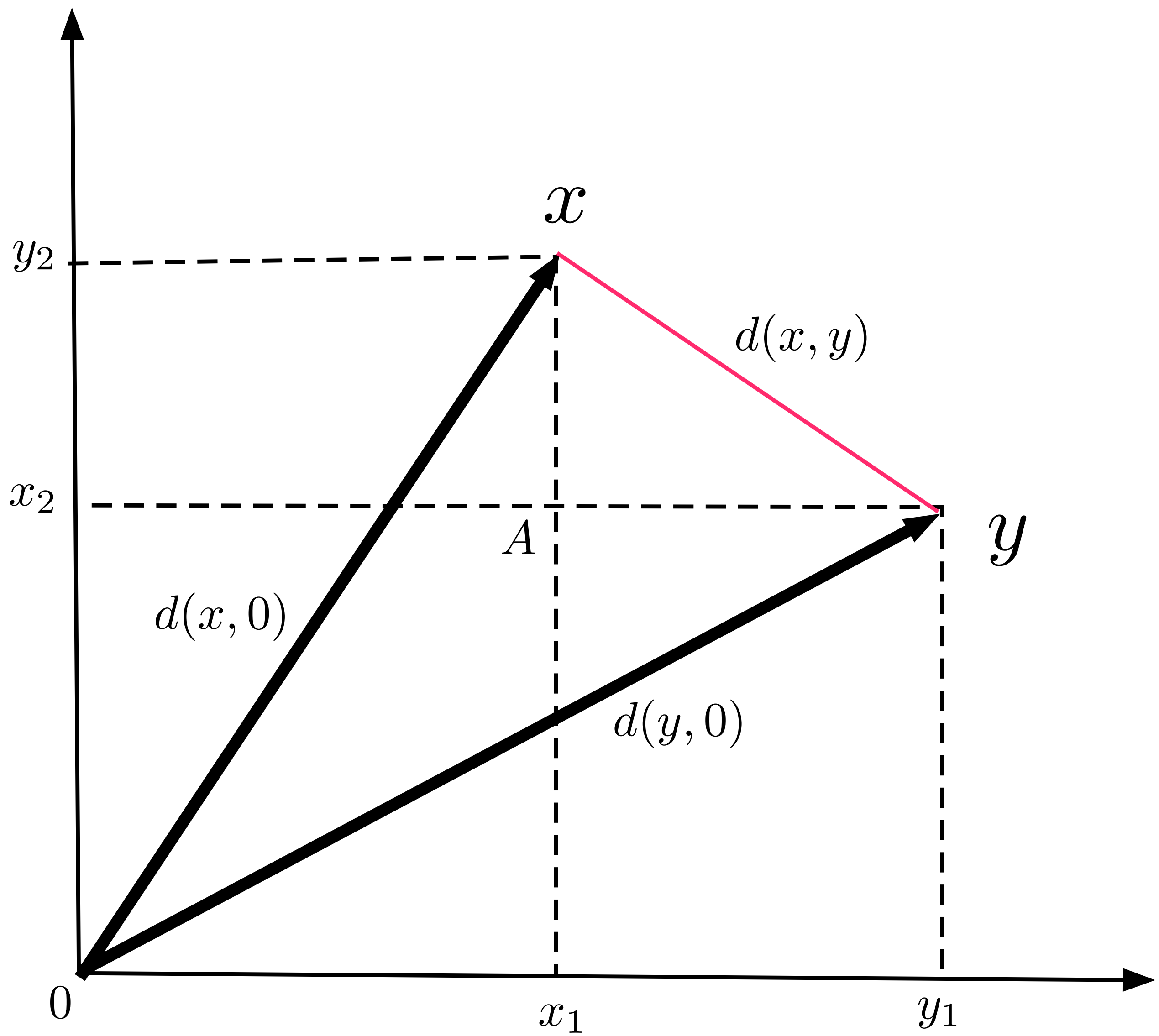
Triangle inequality - proof

$$\begin{aligned} d(x,y) &= \|x-y\| = \|(x-z)+(z-y)\| \\ &\leq \|x-z\| + \|z-y\| = d(x,z) + d(y,z) \end{aligned}$$

Euclidean distance - remark

- Application of Pythagoras' theorem
- Illustration in $\mathbf{R}^2$
  - Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$
  - Consider the right-angled triangle Axy
  - The distance between vectors x and y, $d(x, y)$ is its hypotenuse.
  - Applying Pythagoras' theorem
    $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- Similarly, the length of vector x (its norm) is the hypotenuse of the right-angled triangle $0xx_1$. Hence,
  $d(x, 0) = \|x\| = \sqrt{x_1^2 + x_2^2}$
- Mutatis mutandis, $d(y, 0) = \|y\| = \sqrt{y_1^2 + y_2^2}$

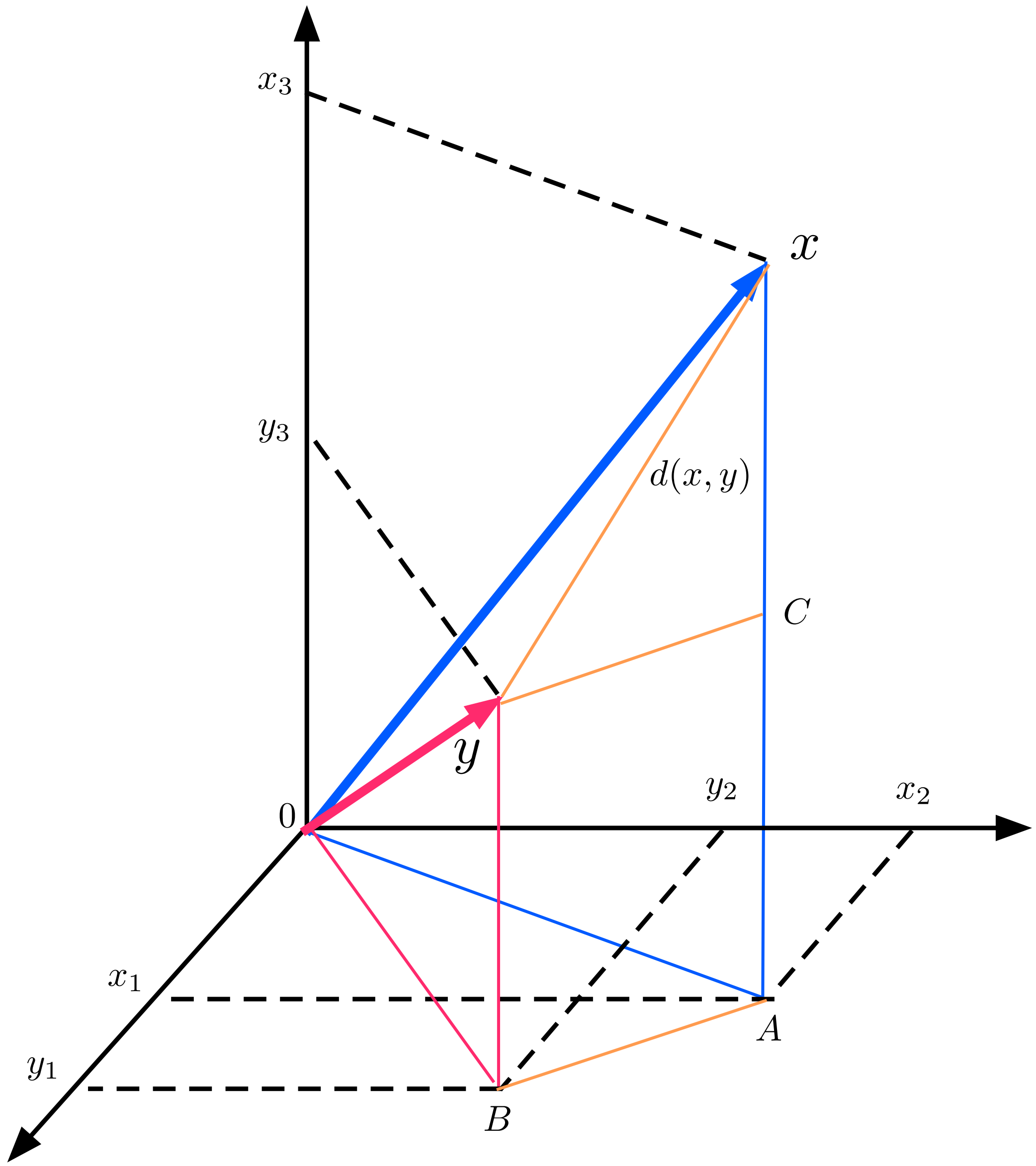
Euclidean distance in $\mathbb{R}^2$ - Illustration



Euclidean distance - remark (2)

- Illustration in $\mathbf{R}^3$
  - Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$
  - Consider the right triangle yxC
  - The distance between vectors x and y, $d(x, y)$ is its hypotenuse.
  - Applying Pythagoras' theorem
    $d(x, y) = \sqrt{\overline{yC}^2 + \overline{xC}^2}$
  - where
    $\overline{yC}^2 = \overline{AB}^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$ and
    $\overline{xC}^2 = (x_3 - y_3)^2$
  - Therefore, $d(x, y) = \sqrt{x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$

Euclidean distance in $\mathbb{R}^3$ - Illustration



The inner product

Geometric and algebraic definitions

- Consider vector $x \in \mathbf{R}^n$.

$$\cos\theta_x = \frac{x_1}{\|x\|} \rightarrow x_1 = \|x\| \cos\theta_x$$

$$\sin\theta_x = \frac{x_2}{\|x\|} \rightarrow x_2 = \|x\| \sin\theta_x$$

- Consider vector $y \in \mathbf{R}^n$.

$$\cos\theta_y = \frac{y_1}{\|y\|} \rightarrow y_1 = \|y\| \cos\theta_y$$

$$\sin\theta_y = \frac{y_2}{\|y\|} \rightarrow y_2 = \|y\| \sin\theta_y$$

The inner product (2)

Geometric and algebraic definitions (cont'd)

$$
\begin{aligned}
< x, y > &= \|x\|\|y\| \cos(\theta_x - \theta_y) \\
&= \|x\|\|y\|(\cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y) \\
&= \|x\| \cos\theta_x \|y\| \cos\theta_y + \|x\| \sin\theta_x \|y\| \sin\theta_y \\
&= x_1 y_1 + x_2 y_2 = \sum_i x_i y_i
\end{aligned}
$$

Remarks

- If x and y are orthogonal, $\cos(\theta_x - \theta_y) = 0$ and $< x, y >= 0$.
- If x and y are parallel, $\cos(\theta_x - \theta_y) = \pm 1$ and $< x, y >= \pm\|x\|\|y\|$

The inner product - properties

$\forall x, y_1, y_2 \in \mathbf{R}^n, < x, y_1 + y_2 >=< x, y_1 > + < x, y_2 >,$

$\forall x, y \in \mathbf{R}^n, \forall \alpha \in R, < x, \alpha y >= \alpha < x, y >,$

$\forall x, y \in \mathbf{R}^n, < x, y >=< y, x >,$

$\forall x, y \in \mathbf{R}^n, x$ and y orthogonal $\Leftrightarrow < x, y >= 0,$

$\forall x \in \mathbf{R}^n, < x, x > \geq 0$ and $< x, x >= 0 \Leftrightarrow x = 0,$

$\forall x, y, z \in \mathbf{R}^n,$ If $< x, y >=< x, z >, x \neq 0,$ then $< x, y - z >= 0 \Rightarrow$ x orthogonal $(y - z)$. Thus, it allows for $(y - z) \neq 0$ and thus $y \neq z$.

$\forall x, y \in \mathbf{R}^n, | < x, y > | \leq \|x\| \|y\|$ (Cauchy-Schwartz inequality).

If vectors x and y are linearly dependent, then equality

Cauchy-Schwartz inequality - proof

- Recall $| < x, y > | \leq \|x\| \|y\|$
- and rewrite it as $| < x, y > |^2 \leq < x, x > \cdot < y, y >$ or $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$
- Consider the following quadratic polynomial in $z \in \mathbf{R}$:
$$\|zx + y\|^2 = (x_1 z + y_1)^2 + \cdots + (x_n z + y_n)^2 = z^2 \sum(x_i^2) + 2z \sum(x_i y_i) + \sum(y_i^2)$$
- It is non-negative (as it is the sum of non-negative terms). Also, it has at most one real root in z if the discriminant is non-positive, ie, if
$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 - \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq 0$$
- and this is Cauchy-Schwartz inequality.
- Remark: Equality if x and y linearly dependent.

Open sets

Preliminary defs

- Let $x \in \mathbf{R}^n$ and $r > 0$. An open ball of radius r centered at x is the set: $B(x,r) = \{y \in \mathbf{R}^n | d(x,y) < r\}$
- Let $x \in \mathbf{R}^n$ and $r > 0$. An closed ball of radius r centered at x is the set: $\overline{B}(x,r) = \{y \in \mathbf{R}^n | d(x,y) \leq r\}$
- Let $A \subset \mathbf{R}^n$. We say that $x \in A$ is an interior point of A, if $\exists r > 0$ such that, $B(x,r) \subset A$.
- Let $A \subset \mathbf{R}^n$. We define the interior of set A as $int(A) = \{x \in A | x \text{ is an interior point of} A\}$
- Trivially, $int(A) \subset A$
- Let $A \subset \mathbf{R}^n$. Let $x \in \mathbf{R}^n$. We say that x is an accumulation point of A if $(B(x,r) \setminus \{x\}) \cap A \neq \emptyset$.

Open sets (2)

Definitions

- Let $A \subset \mathbf{R}^n$. We say that A is open if $\forall x \in A, \exists r > 0$ such that $B(x,r) \subset A$.
  Remark: In general r depends of x.
- Let $A \subset \mathbf{R}^n$. We say that A is open if $A = int(A)$, i.e. if all points are interior.

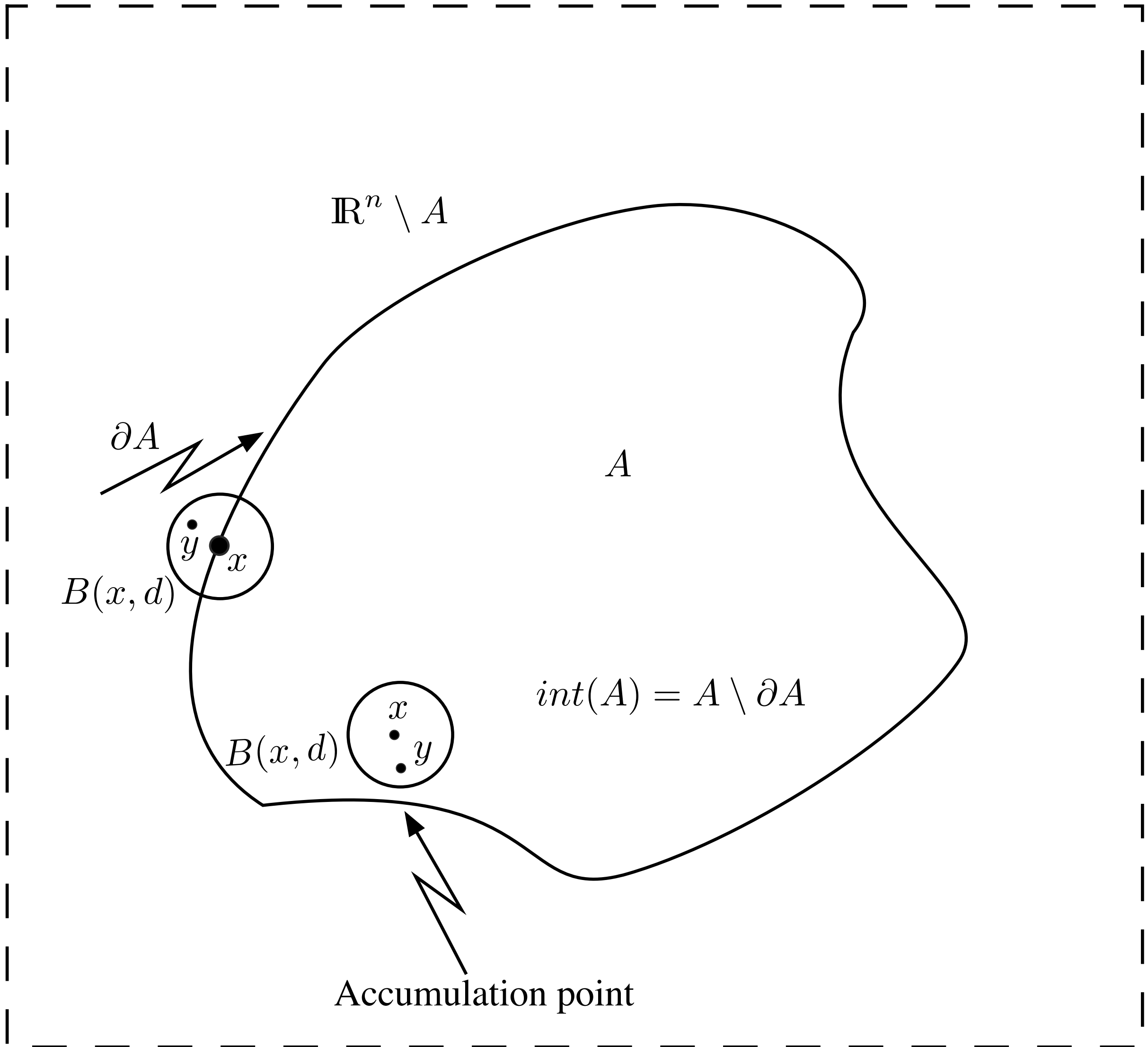
Two theorems

- Theorem 1: Any open ball is an open set.
- Theorem 2:
  - The union of an *arbitrary* number of open sets is an open set.
  - The intersection of a *finite* number of open sets is an open set.

Closed sets

- Let $A \subset \mathbf{R}^n$. We say that A is closed if its complement, $\mathbf{R}^n \setminus A$ is open.
- Let $A \subset \mathbf{R}^n$. We say that $x \in A$ belongs to the frontier of A, ∂A, if we can find a ball $B(x, d)$, with d arbitrarily small, such that $\exists y \in B(x, d)$ and $y \notin A$.
- Let $A \subset \mathbf{R}^n$. We say that A is closed if $\forall x \in \partial A \Rightarrow x \in A$.
- Let $A \subset \mathbf{R}^n$. We define the interior of set A as the set of points that belong to A but do not belong to ∂A
- Let $A \subset \mathbf{R}^n$. We say that $x \in \mathbf{R}^n$ is an accumulation point of A if $\forall d > 0,\ \exists y \in A$ with $y \neq x$ such that $y \in B(x, d)$.
- (Thm) Let $A \subset \mathbf{R}^n$. We say that A is closed if it contains all its acumulation points

Closed sets - Illustration



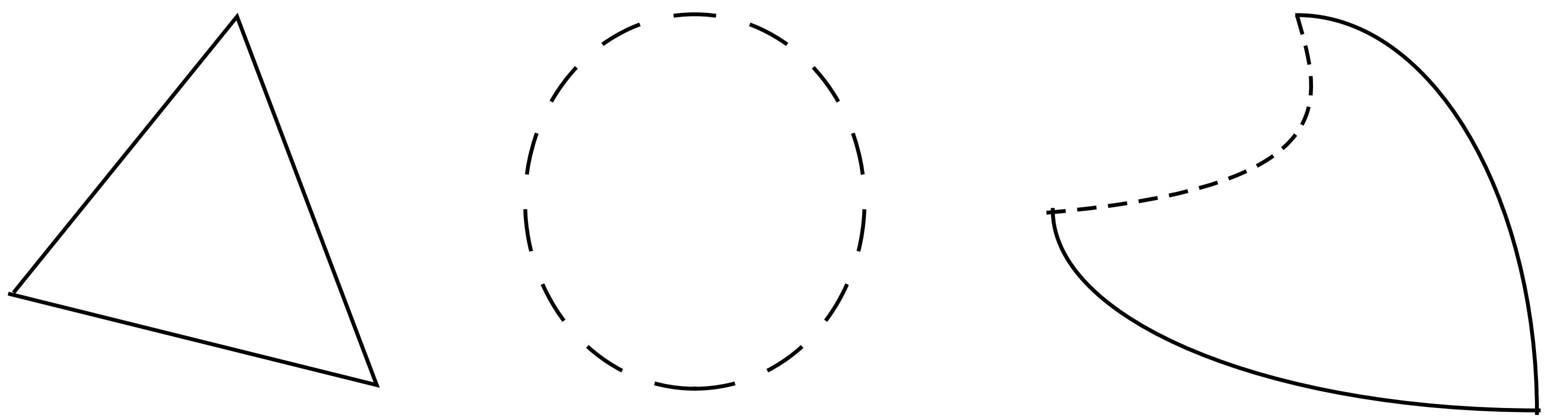
Compact sets

Bounded set

- Let $A \subset \mathbf{R}^n$. We say that A is bounded iff $\exists M \geq 0$ allowing to define $B(0, M)$ such that $A \subset B(0, M)$.

Compact set

- Let $A \subset \mathbf{R}^n$. We say that A is compact if it is closed and bounded.



Compact and non-compact sets

Convex sets

Intuition

- Let $A \subset \mathbf{R}^n$. We say that A is convex if $\forall (x, y) \in A$ any point c in the segment linking x and y also belongs to A.

Preliminary definitions

- Consider a finite number of points $x_i \in \mathbf{R}^n,\ i = 1, 2, \ldots, s$. A linear combination is a point of the form $\sum_{i=1}^{s} \alpha_i x_i$
- Consider a finite number of points $x_i \in \mathbf{R}^n,\ i = 1, 2, \ldots, s$. An affine combination is a point of the form $\sum_{i=1}^{s} \alpha_i x_i,\ \sum_{i=1}^{s} \alpha_i = 1$
- Consider a finite number of points $x_i \in \mathbf{R}^n,\ i = 1, 2, \ldots, s$. A convex combination is a point of the form $\sum_{i=1}^{s} \alpha_i x_i,\ \sum_{i=1}^{s} \alpha_i = 1, \alpha_i \geq 0$

Convex sets (2)

Definitions

- Let $A \subset \mathbf{R}^n$. We say that A is convex if given any two points (x, y) of the set, any convex combination of these two points, $[x, y]$, is also in the set.
- Let $A \subset \mathbf{R}^n$. We say that A is convex if $(x, y) \in A$ implies $[x, y] \subset A$.
- Let $A \subset \mathbf{R}^n$. We say that A is strictly convex if $(x, y) \in A$ implies $[x, y] \subset int(A), \alpha > 0$.
- Let $A \subset \mathbf{R}^n$. The set of all convex combinations of points in A constitute de convex hull of A. Smallest convex set containing A.
- The unit simplex of $\mathbf{R}^n$ is a convex and compact set defined by $S^{n-1} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n_+ | \sum_i^n \lambda_i = 1\}$
- A simplex is the convex hull of a finite set of points called the vertices of the simplex. Smallest convex set containing the given vertices

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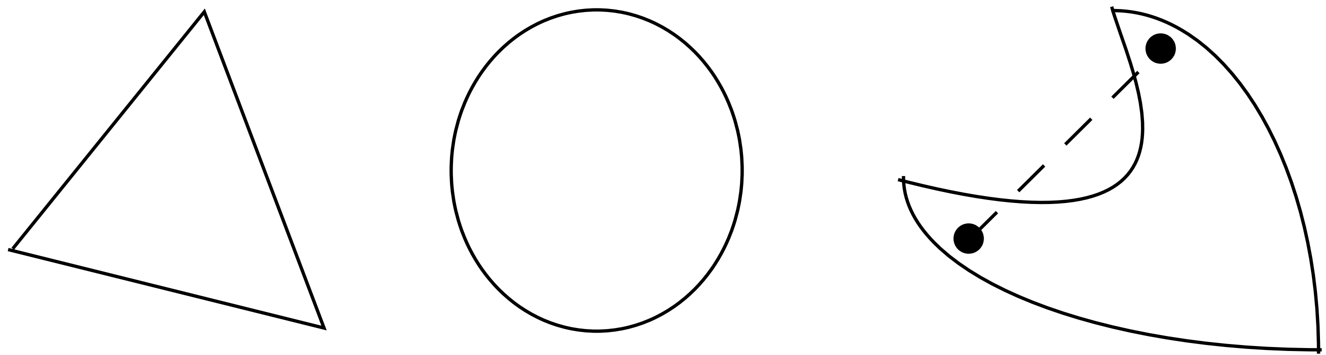
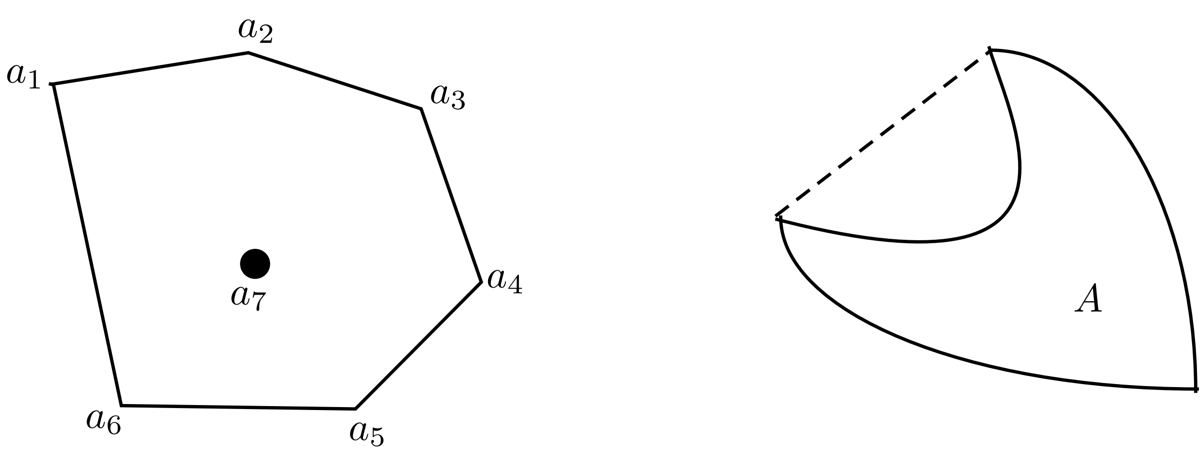
Convex sets - Illustration



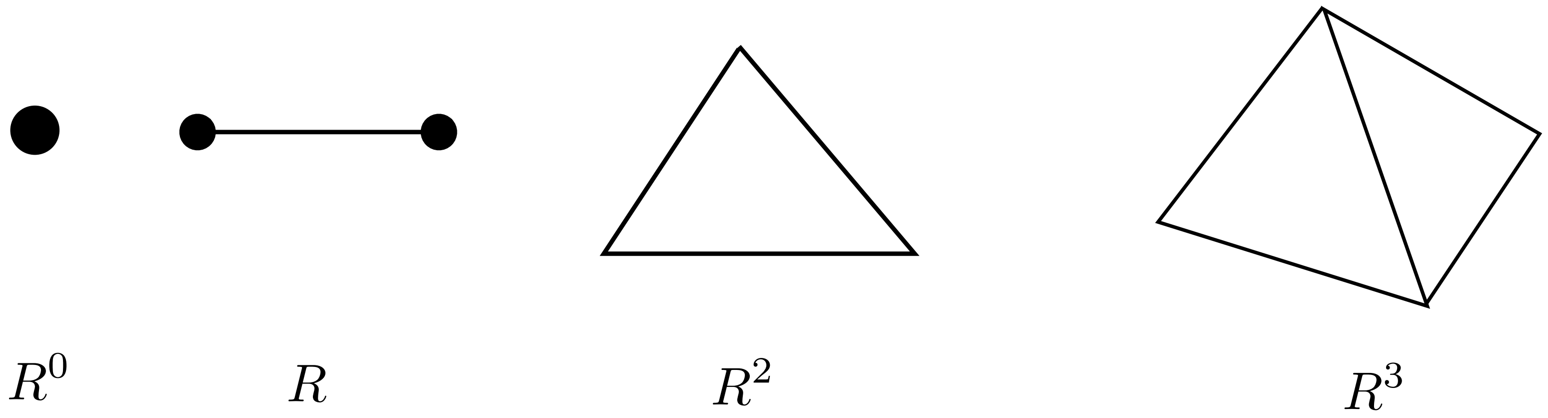
Convex, strictly convex and non-convex sets



$$A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$$

Convex hull of A

Simplex - Illustration



Connected and disconnected sets

Intuition

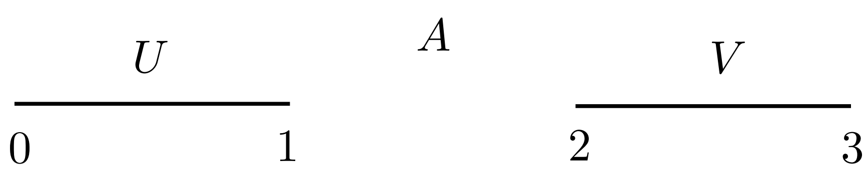
- A set A is called disconnected if it can be separated into two open, disjoint sets in such a way that neither set is empty and both sets combined give the original set A

Definitions

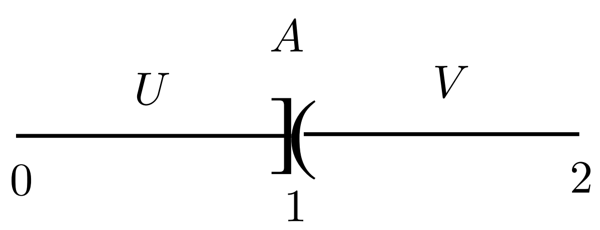
- An open set A is called disconnected if there are two open, non-empty sets U and V such that:
  $U \cap V = \emptyset$ and
  $U \cup V = A$
- A set A (not necessarily open) is called disconnected if there are two non-empty open sets U and V such that
  $(U \cap A) \neq \emptyset$ and $(V \cap A) \neq \emptyset$
  $U \cap V \cap A = \emptyset$
  $U \cup V \supseteq A$
- If A is not disconnected it is called connected.

Connected and disconnected sets - Illustration

- Example 1: $[0, 1]$ does not contain any limit points of $[2, 3]$, and vice versa.
- Example 2: $[0, 2]$ can be written as $[0, 1] \cup (1, 2]$, but 1 is a limit point of $(1, 2]$.



A is disconnected



A is connected

Hyperplanes

Definitions

- Let $A \subset \mathbf{R}^n$. Let $p \in \mathbf{R}^n$, and $\beta \in \mathbf{R}$. A hyperplane is the set of points $H = \{x \in A | \sum_{i=1}^n p_i x_i = \beta\} \subset \mathbf{R}^{n-1}$
  Remark: For any two points $(x, y) \in H,\ px = py = \beta$ so that $p(x - y) = 0$ i.e. p is orthogonal to the hyperplane.
- Let $A \subset \mathbf{R}^n$ be a convex set. Let $p \in \mathbf{R}^n$, and $\beta \in \mathbf{R}$. A hyperplane $H = \{x \in A | \sum_{i=1}^n p_i x_i = \beta\}$ is a supporting hyperplane of A if,
  - A belongs to either one of the two closed semi-spaces $\sum_{i=1}^n p_i x_i \leq \beta$ or $\sum_{i=1}^n p_i x_i \geq \beta$, and
  - the hyperplane has a common point with A.
  - Remark: If a is the intersection point, we refer to the support hyperplane of A at a.

Hyperplanes (2)

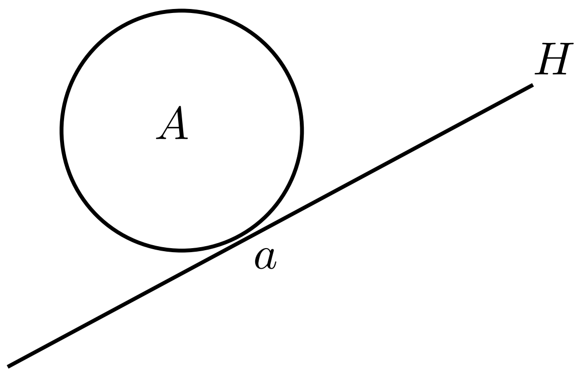
Definitions (cont'd)

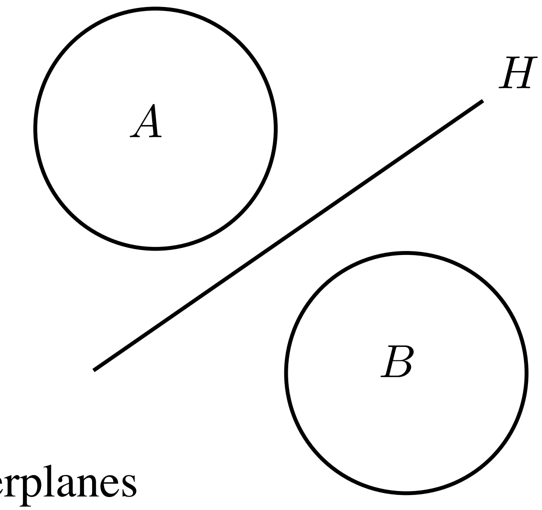
- Let $A, B \subset \mathbf{R}^n$ be nonempty convex disjoint sets i.e., $A \cap B = \emptyset$. A separating hyperplane for A and B is a hyperplane that has A on one side of it and B on the other.

Minkowski separation theorems

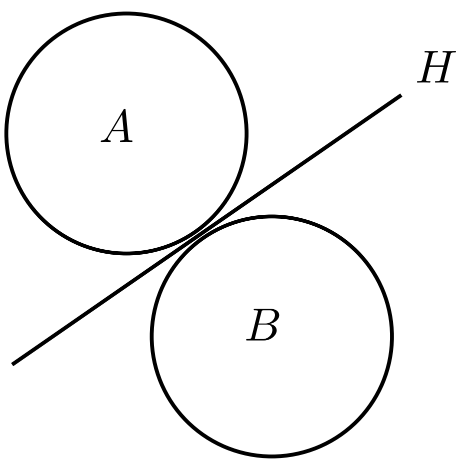
- Theorem 1
  Let $A \subset \mathbf{R}^n$ be a convex set. Then we can construct a hyperplane H passing through a point a that is separating for A if $a \notin int(A)$.
- Theorem 2
  Let $A, B \subset \mathbf{R}^n$ be two non-empty convex sets such that $int(A) \bigcap int(B) = \emptyset$. Then we can construct a hyperplane H separating both sets, i.e. $\exists \mathbf{p} \in \mathbf{R}^n$ and $\beta \in \mathbf{R}$ such that $\forall \mathbf{x} \in A, \mathbf{px} \leq \beta$ and $\forall \mathbf{x} \in B, \mathbf{px} \geq \beta$.

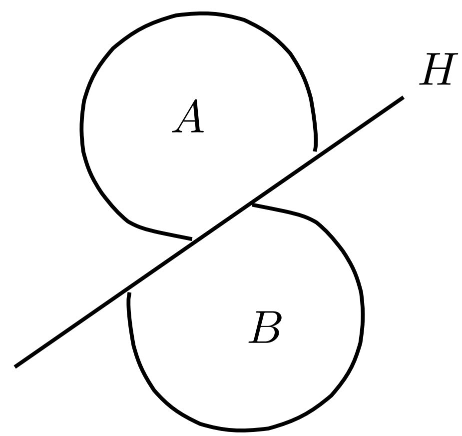
Hyperplanes - Illustration





Supporting and separating hyperplanes





Fixed point theorems

Theorem 1 (Brower)

- Let $A \subset \mathbf{R}^n$ be a convex, compact and non-empty set.
- Let $f : A \to A$ a continuous function associating a point $x \in A$ to a point $f(x) \in A$.
- Then, f has a fixed point $\widehat{x}$ so that $\widehat{x} = f(\widehat{x})$.

Intuition

- Let $g(x) = f(x) - x$ maps $[a, b]$ on itself.
- Thus, $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$
- If any of them holds with equality the fixed point is one of the end points of the interval.
- Otherwise the intermediate value theorem implies the existence of an interior zero of $g(x)$, i.e. a fixed point of $f(x)$.

Fixed point theorems (2)

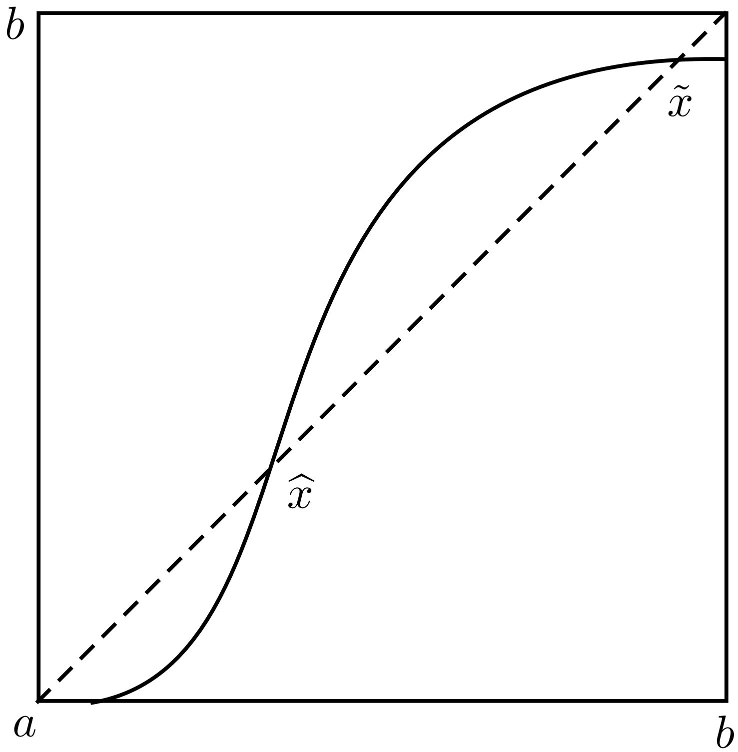
Theorem 2 (Tarsky)

- Let f be a non-decreasing function mapping the n-dimensional cube $[0,1] \times [0,1]$ into itself.
- Then, f has a fixed point $\widehat{x}$ so that $\widehat{x} = f(\widehat{x})$.

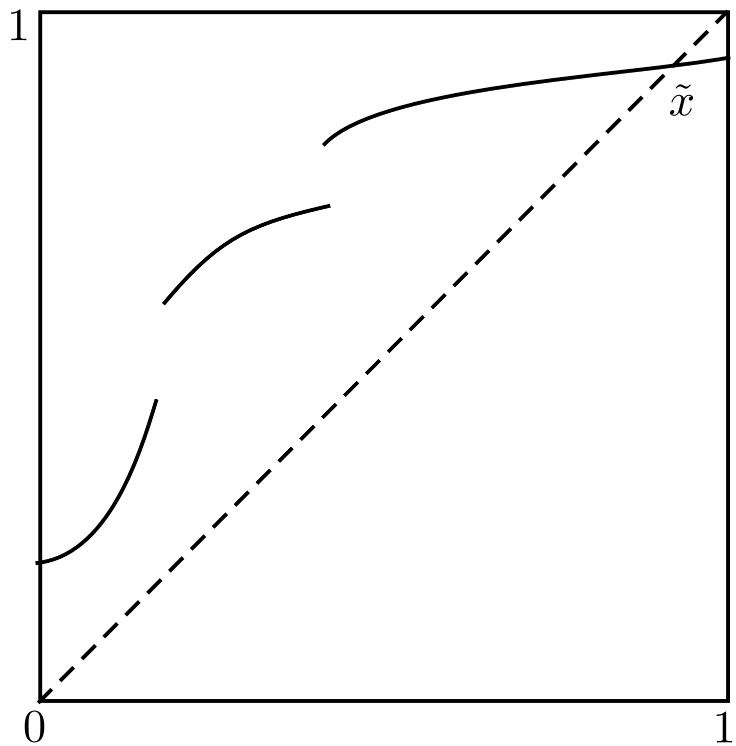
Intuition

- If $f(0) = 0$ and/or $f(1) = 1$ We have a fixed point.
- If $f(0) > 0$, Then f starts above the $45°$-line. Since it can only jump upwards at points of discontinuity, it cannot cross the diagonal at those points.
- If $f(1) < 1$ the graph of f must cross the diagonal at some point.

Fixed points - Illustration



Brower's fixed point



Tarsky's fixed point

Other fixed point theorems

- Border, K.M., 1990, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press.