## Lagrange multipliers

Consider $f^1, ..., f^k$ some $C^1$ functions from $\mathbf{R}^n$ to $\mathbf{R}$, where $k \leq n$. Assume the set of constraints is regular, and let
$C = \{x \in \mathbf{R}^n : f^1(x) = ... = f^k(x) = 0\}$.
consider the problem

$$(P) \min_{x \in C} f(x)$$

Then if $\bar{x} \in C$ is a solution of $(P)$ and $f$ is differentiable at $\bar{x}$, then there exists some reals $\lambda_1, ...\lambda_k$ such that:

$$\nabla f_{\bar{x}} = \sum_{i=1}^{k} \lambda_i \nabla f^i_{\bar{x}},$$

The coefficients $\lambda_i$ are called Lagrange multipliers.

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Sometimes, some author write the necessary first order conditions $\nabla\mathcal{L}(\bar{x}, \lambda_1, ..., \lambda_k) = 0$ where

$$\mathcal{L}(\bar{x}, \lambda_1, ..., \lambda_k) = f(\bar{x}) - \lambda_1.f^1(\bar{x}) - ... - \lambda_k.f^k(\bar{x})$$

is called the Lagrangian function.
You try to solve this sytem (with $n + k$ unknown) to find **candidates** to be solution of the optimization problem.

Example 1: Minimize $2x^2 + y^2$ under the constraint $x + y = 1$.

(P) $\max_{x_1^2+x_2^2+\ldots+x_n^2=1} x_1 x_2 x_3 \ldots x_n$

Geometric interpretation.

For a general reference on Lagrange Multipliers, see also Section 3.3. in Further MATHEMATICS FOR Economic Analysis.

- For constraints defined by inequalities and equalities, we can find similar lagrange multipliers (KKT theorem below), but the conditions are more complex.
- Again, the only difficulty is to be able to write the Normal cone, which, (again), requires Regularity conditions (see below).

## Intuition when we have inequalities through an example

Consider

$$C = \{(x, y) \in \mathbf{R}^2 : g(x, y) = x^2 + y^2 - 1 \leq 0\}.$$

Then for every $(\bar{x}, \bar{y}) \in C$,

$$N_C(\bar{x}, \bar{y}) = \{\mu \nabla g_{(\bar{x},\bar{y})}, \mu \geq 0\}$$

and

$$T_C(\bar{x}, \bar{y}) = \{h \in \mathbf{R}^2 : \nabla g_{(\bar{x},\bar{y})}.h \leq 0\}.$$

Difference with equality ?

## KKT (Karush, Kuhn and Tucker) Theorem

Consider $f^1, ..., f^k, g^1, ..., g^m$ some $C^1$ functions from $\mathbf{R}^n$ to $\mathbf{R}$, where $k \leq n$.
Assume the set of constraints satisfies **regularity (also called qualification constraints) constraints** we will see after.
Let $C = \{x \in \mathbf{R}^n : f^1(x) = ... = f^k(x) = 0, g^1(x) \leq 0, ..., ...g^m(x) \leq 0\}$.
consider the problem

$$(P) \min_{x \in C} f(x)$$

Then if $x^* \in C$ is a solution of $(P)$ and $f$ is differentiable at $x^*$, then there exists some reals $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_m$ such that:
$(i) \nabla f_{\bar{x}} + \sum_{i=1}^{k} \lambda_i \nabla f^i_{\bar{x}} + \sum_{j=1}^{m} \mu_j \nabla g^j_{\bar{x}} = 0,$
$(ii) \forall j = 1, ..., m, \mu_j \geq 0$ (Positivity of multiplicators associated to inequalities)
$(iii) \forall j = 1, ..., m, [\mu_j = 0$ or $g^j(\bar{x}) = 0]$. (Each inequality constraint is binded or the associated multiplicator is null)
$(iv) \forall i = 1, ..., k, f^i(\bar{x}) = 0$. (Equality constraints satisfied!)
$(v) \forall j = 1, ..., m, g^j(\bar{x}) \leq 0$. (Inequality constraints satisfied!)

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$$(P) \max_{x \in C} f(x)$$

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$(i) \nabla f_{\bar{x}} - \sum_{i=1}^k \lambda_i \nabla f_{\bar{x}}^i - \sum_{j=1}^m \mu_j \nabla g_{\bar{x}}^j = 0,$
$(ii) \forall j = 1, ..., m, \mu_j \geq 0$
$(iii) \forall j = 1, ..., m, \mu_j . g^j(\bar{x}) = 0.$
$(iv) \forall i = 1, ..., k, f^i(\bar{x}) = 0.$
$(v) \forall j = 1, ..., m, g^j(\bar{x}) \leq 0.$

Intuitively, regularity conditions are conditions on the constraints so that we have a nice formula for the normal cone, which allows to have the "simple" KKT condition.

**Condition 1** A first possible condition that is enough to get KKT theorem is **Slater's condition**

### Slater's condition

Consider $f^1, ..., f^k, g^1, ..., g^m$ some $C^1$ functions from $\mathbf{R}^n$ to $\mathbf{R}$, where $k \leq n$.

Slater's conditions are true if:

(i) All $g^j$ are convex.

(ii) All $f^i$ are affine.

(iii) There exists $\tilde{x}$ feasible point (i.e. it satisfies the constraints) such that for every $j$ such that $g^j$ is not affine, we have $g^j(\tilde{x}) < 0$.

A second possible condition for which KKT theorem is true are the following **regularity's conditions**

**Regularity (or qualification) conditions for system of equalities together with inequalitites**

The set of constraints defined by $f^1 = ... = f^k = 0, g^1 \leq 0, ..., g^m \leq 0$ is regular if (1) $k + m \leq n$, and (2) for every $\bar{x} \in \mathbf{R}^n$ such that $f^1(\bar{x}) = ... = f^k(\bar{x}) = 0, g^1(\bar{x}) \leq 0, ..., g^m(\bar{x}) \leq 0$,
the $n \times (k+m)$ matrix whose columns are $\nabla f^1_{\bar{x}}$, ...., $\nabla f^k_{\bar{x}}$, $\nabla g^1_{\bar{x}}$, ...., $\nabla g^m_{\bar{x}}$, has a rank $m + k$.

Example of use of KKT.

$$(P) \min_{x+y\leq 3, -2x+y\leq 2} x^2 - 4x + y^2 - 6y$$