Correspondences (set-valued mappings) B1

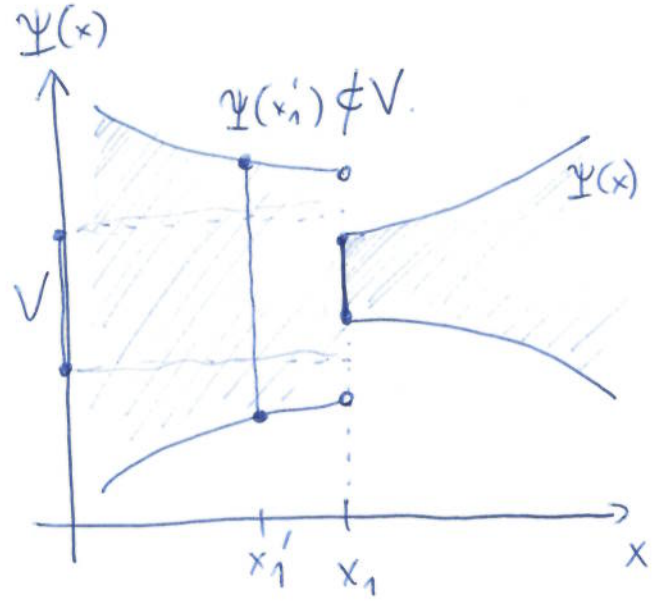
$\Psi: X \rightrightarrows Y$, with $\Psi(x) \subset Y$ – a set.

Def. Ψ is upper hemi-continuous (uhc) at $x \in X$ if for every open set $V \supset \Psi(x)$ there exists a neighborhood $\mathcal{U}$ of x such that $\forall x' \in \mathcal{U}$ $\Psi(x') \subseteq V$.

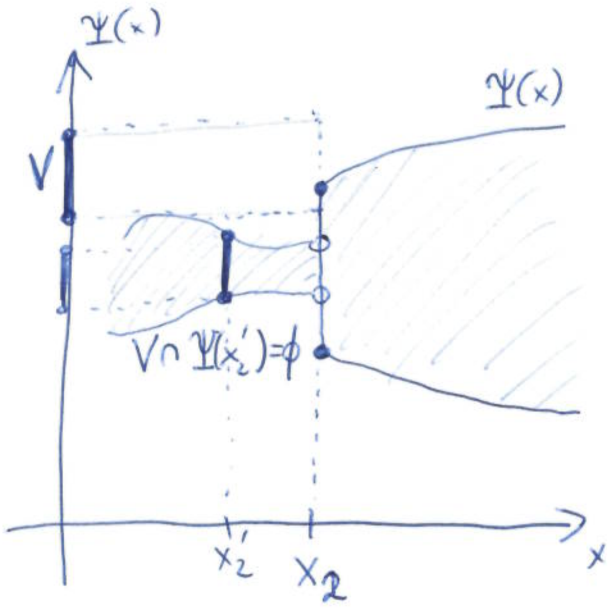
Def. Ψ is lower hemi-continuous (lhc) at $x \in X$ if for every open set $V \subset Y$, with $\Psi(x) \cap V \neq \phi$, there exists a neighborhood $\mathcal{U}$ of x such that $\Psi(x') \cap V \neq \phi$ for all $x' \in \mathcal{U}$.

Def. Ψ is continuous if it is both uhc and lhc.

Intuition:
- uhc — the set $\Psi(x)$ doesn't "explode" with x.
- lhc — the set $\Psi(x)$ doesn't "implode" with x.



Failure of uhc ("explosion")



Failure of lhc ("implosion")

$\boxed{B2}$

Ex. 1

f – continuous, $f : X \to Y$

$F : X \rightrightarrows Y$, defined as $F(x) = \{ f(x) \}$

SINGLETON SETS.

- Why F is <u>uhc</u>? Let V be an open set, $V \supset \{f(x)\}$, i.e, <u>$f(x) \in V$</u>. Then there exists $\mathcal{U}$, open, such that $x \in \mathcal{U} \Rightarrow f(x) \in V$.
- Why F is <u>lhc</u>? Let V be an open set, $\{f(x)\} \cap V \neq \emptyset$, i.e. <u>$f(x) \in V$</u>. The same result follows, directly from continuity of f.

Ex. 2

X, Y – metric spaces

$y = \sum_{i=1}^{n} \alpha_i f_i(x)$

$\sum \alpha_i = 1$

POLYTOPE

$f_i : X \to Y$ is continuous for all i

Define $F(x) := \operatorname{conv} \{ f_i(x),\ i = 1, \dots, n \}$ → CONVEX HULL OF n POINTS.

- Why F is <u>uhc</u>? Let V – open, $F(x) \subset V$.

By continuity of f_i, we have

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \underbrace{\forall i = 1, \dots, n}_{\text{POSSIBLE BECAUSE } n \text{ IS FINITE}} \ \big(d(x_1, x_2) < \delta \Rightarrow d(f_i(x_1), f_i(x_2)) < \varepsilon \big)$$

Hence there exists $\overset{x}{\mathcal{U}} \subset X$ such that $\forall x' \in \mathcal{U}$

$$d\Big(\underbrace{\sum_{i=1}^{n} \alpha_i f_i(x')}_{y'}, \underbrace{\sum_{i=1}^{n} \alpha_i f_i(x)}_{y} \Big) \leq d\big(f_j(x'), f_j(x) \big) < \varepsilon,$$

where $j = 1, \dots, n$ is the index for which $f_j(x') - f_j(x)$ is largest.

Hence $F(x') \subset V$.

• Why is F lhc? Let V - open, $F(x) \cap V \neq \phi$.

B3

→ Continuity is used again.

→ By continuity we have that there exists $\mathcal{U} \subset X$, with $x \in \mathcal{U}$, such that $F(x') \cap V \neq \phi$ for all $x' \in \mathcal{U}$.

→ Since $\forall i \; d(f_i(x'), f_i(x)) < \varepsilon$, it follows that

$$d\left(\sum_{i=1}^{n} \alpha_i f_i(x'), \sum_{i=1}^{n} \alpha_i f_i(x)\right) < \varepsilon, \text{ for all } \{\alpha_i\} \text{ summing to } 1.$$

Hence, the maximum distance between $y \in F(x)$ and $y' \in F(x')$ is less than ε (with the same $\{\alpha_i\}$)

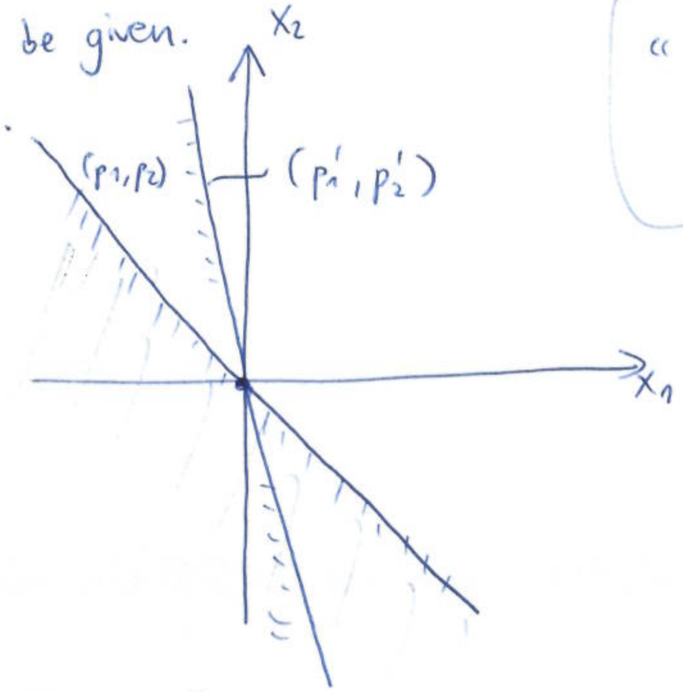
→ If $F(x) \cap V \neq \phi$ then (by openness of V), $F(x') \cap V \neq \phi$.

thus F is lhc. $\square$

Ex. 3.

Let $p_1 \geq 0$, $p_2 \geq 0$ be given.

"BUDGET SET"

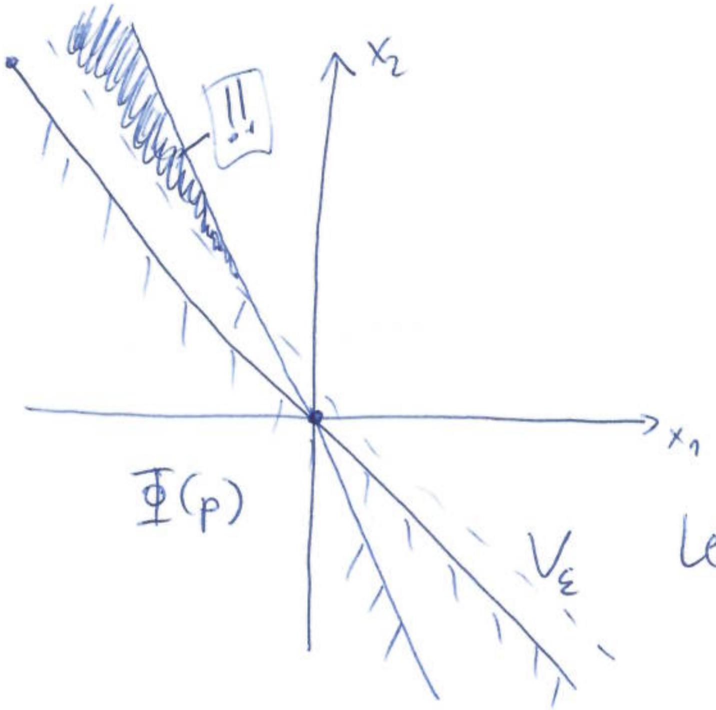


$\Phi : \mathbb{R}^2_+ \rightarrow\rightarrow \mathbb{R}^2$.

• Is the correspondence Φ uhc?

$$\Phi(p) = \{(x_1, x_2) \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq 0\}$$

B4

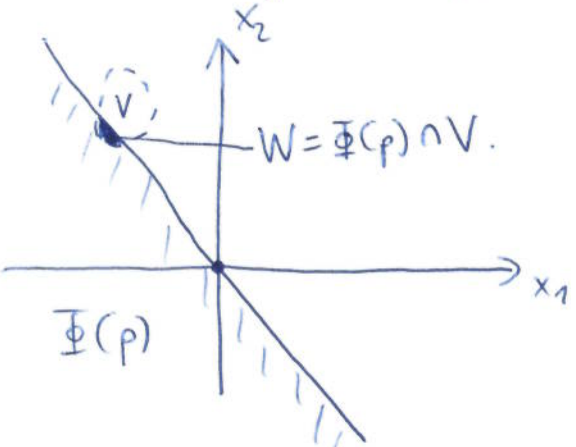


open set $\subset \mathbb{R}^2$

Let $V_\varepsilon = \{x \in \mathbb{R}^2 : p_1 \cdot x_1 + p_2 \cdot x_2 < \varepsilon\}$, with $\varepsilon > 0$.

- It is not true that for V_ε there exists a neighborhood $U \subset \mathbb{R}^2$ such that $\forall p' \in U \; \Phi(p) \subset V_\varepsilon$.
- In every neighborhood there exists $p' \in U$ with $\frac{p_1'}{p_2'} \neq \frac{p_1}{p_2}$ (different slope), and then $\Phi(p') \not\subset V_\varepsilon$.
- Hence Φ is not uhc.

- Is the correspondence lhc? YES! (BUDGET SETS ARE LHC CORRESPONDENCES.)



- Consider a neighborhood $U \subset \mathbb{R}^2$, $(p_1, p_2) \in U$.
- For all $p' \in U$, $d((p_1', p_2'), (p_1, p_2)) < \delta/2$.

  Hence $|p_1' - p_1| < \delta$, $|p_2' - p_2| < \delta$.
- For $(x_1, x_2) \subset \operatorname{int} W$, $\underbrace{p_1 x_1 + p_2 x_2}_{-\varepsilon} < 0$
- Then $|p_1' x_1 + p_2' x_2 - p_1 x_1 - p_2 x_2| =$

  $= |(p_1' - p_1) x_1 + (p_2' - p_2) x_2| \leq 2\delta |x_1 - x_2|$
- Finally, $p_1' x_1 + p_2' \cdot x_2 + \varepsilon \leq 2\delta |x_1 - x_2|$ and so $p_1' x_1 + p_2' x_2 \leq 0$ if $2\delta |x_1 - x_2| \leq \varepsilon \iff \boxed{\delta \leq \frac{\varepsilon}{2|x_1 - x_2|}}$
- Hence, $(x_1, x_2) \in \Phi(p')$, and thus $\Phi(p') \cap V \neq \emptyset \Rightarrow \Phi$ is lhc. $\square$

B5

Ex. 5

$\Phi : X \to\to Y$

X - compact

$\Phi(x)$ - compact for all $x \in X$

Φ - uhc

Show that $\Phi(X) = \bigcup_{x \in X} \Phi(x)$ is compact.

Remark. Not every union of compact sets is compact. Finite unions are, but infinite don't have to.

E.g. $A = \bigcup_{i=1}^{\infty} \underbrace{[0, i]}_{\text{COMPACT}} = \underbrace{[0, +\infty)}_{\text{NOT BOUNDED}}$

$B = \bigcup_{i=1}^{\infty} [1, (1+\frac{1}{i})^i] = [1, \overbrace{e)}^{\text{NOT CLOSED}}$

Proof. Let $\mathcal{O}$ be an open cover of $\Phi(X)$. We wish to find its $\boxed{\text{finite open subcover}}$. Note that $\forall x \in X$, $\mathcal{O}$ is also an open cover of $\Phi(x)$. Since $\Phi(x)$ is compact, there exists a finite collection of open sets $\{O_1(x), \dots, O_{m_x}(x)\} \subset \mathcal{O}$ such that $\Phi(x) \subset \bigcup_{i=1}^{m_x} O_i(x) := O(x)$. As Φ is uhc, $\Phi^{-1}(O(x)) = \{x' \in X : \Phi(x') \subset O(x)\}$ — the UPPER INVERSE IMAGE OF $O(x)$ — is open, for all $x \in X$.

Moreover, $\Phi(X) \subset \bigcup_{x \in X} \{O(x)\}$, so that $X \subset \bigcup_{x \in X} \{\Phi^{-1}(O(x))\}$.

It follows that $\{\Phi^{-1}(O(x))\}_{x \in X}$ is an open cover of X. By compactness of X, there exist a finite collection of points $\{x^1, \dots, x^m\} \subset X$ such that $\{\Phi^{-1}(O(x^i)) : i = 1, \dots, m\}$ covers X. But then $\{O(x^1), \dots, O(x^m)\}_{i=1,\dots,m}$ covers $\Phi(X)$. Therefore the collection $\{O_j(x^i) : i = 1, \dots, m;\ j = 1, \dots, m_{x^i}\}$ is a finite subset of $\mathcal{O}$ which covers $\Phi(X)$. Therefore $\Phi(X)$ is compact. $\square$

Theorem (Berge)

Let Θ, X be two metric spaces, $\Gamma: \Theta \rightrightarrows X$ be a compact-valued correspondence, and $\varphi: X \times \Theta \to \mathbb{R}$ be continuous.

Define:

$$\sigma(\theta) = \arg\max_{x \in \Gamma(\theta)} \varphi(x,\theta);$$

$$\varphi^*(\theta) = \max_{x \in \Gamma(\theta)} \varphi(x,\theta),$$

and assume that Γ – continuous (uhc & lhc) at $\theta \in \Theta$.

Then:
- $\sigma: \Theta \rightrightarrows X$ is compact-valued, uhc and closed at θ,
- $\varphi^*: \Theta \to \mathbb{R}$ is continuous at θ.

Application:

$\theta = [p; w]$

→ $X = \mathbb{R}^l$ – bundles of l commodities

→ $\Theta = \mathbb{R}^l_{++} \times \mathbb{R}_{++}$ – prices of l commodities + consumer's wealth

→ $\varphi(x,\theta)$ – utility obtained from consuming $x \in X$ commodities at prices $\theta \in \Theta$

[often $\varphi(x,\theta) \equiv u(x)$ $\forall \theta \in \Theta$]

CONTINUOUS

→ $\Gamma(\theta) = B(p,w) = \{x \in \mathbb{R}^l : \sum_{i=1}^{l} p_i x_i \le w\}$ – budget set

CONTINUOUS & COMPACT-VALUED

Then
- $\varphi^*(\theta) = \max_{x \in \Gamma(\theta)} \varphi(x,\theta)$ – INDIRECT UTILITY FCT $\Rightarrow$ CONTINUOUS
- $\sigma(\theta) = \arg\max_{x \in \Gamma(\theta)} \varphi(x,\theta)$ – DEMAND CORRESPONDENCE $\Rightarrow$ COMPACT-VALUED, UHC & CLOSED.

Application in Dynamic Programming.

<u>Theorem</u> (de la Fuente, p. 561).

Assume that the agent maximizes $\sum_{\tau=t}^{\infty} \beta^{\tau-t} F(x_\tau, u_\tau)$

s.t. $x_{\tau+1} = m(x_\tau, u_\tau)$ for all $\tau \geq t$, and $u_\tau \in \Gamma(x_\tau)$.

Assume that F - bounded and continuous, m - continuous, Γ - continuous (uhc & lhc), $\Gamma(x)$ - nonempty and compact for all x.

Then $T: C(X) \to C(X)$, with

$$T(v)(x) = \max_{u \in \Gamma(x)} \{ F(x,u) + \beta v(m(x,u)) \},$$

maps continuous bounded functions into continuous bounded functions.

Moreover, it is a <u>contraction</u>. Hence, there exists a unique $V \in C(X)$ - fixed point - which is the <u>value function</u> of the dynamic programming problem.

Moreover, the solution is a policy correspondence $g(x)$ -

- defined as $g(x) = \arg\max_{u \in \Gamma(x)} \{ F(x,u) + \beta V(m(x,u)) \}$

- which is nonempty, compact-valued and uhc.

$\to$ TO PROVE THIS, APPLY BERGE'S MAXIMUM THEOREM.

OTHER PARTS HAVE BEEN PROVEN.

A few more particular applications (corollaries)

I → assuming that F, m are concave, F - strictly concave, $\Gamma(x)$ - convex $\forall x \in X$, and Γ - convex in the sense that

$$\forall x_0, x_1 \in X \ \forall \lambda \in [0,1] \ u_0 \in \Gamma(x_0),\ u_1 \in \Gamma(x_1)$$
$$\Rightarrow (1-\lambda)u_0 + \lambda u_1 \in \Gamma((1-\lambda)x_0 + \lambda x_1),$$

- the value function V is strictly concave and strictly increasing.
- The policy "correspondence" g is a continuous function.

II → In the case of the Ramsey model as well as the fisheries/forest management model,

- the value function V is strictly concave and strictly increasing;
- the policy "correspondence" is a continuous function. It is also strictly increasing.