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Intergenerational Interactions in Human Capital Accumulation*

Lukasz Wozny and Jakub Growiec

Abstract

This paper considers an economy populated by a sequence of generations who decide over their consumption and investment in human capital of their immediate descendants. In such a framework, we first identify the impact of strategic interactions between consecutive generations on the time path of human capital accumulation. To this end, we characterize the decentralized Markov stationary Nash equilibrium (MSNE) and derive the sufficient conditions for its existence and uniqueness. We then provide sufficient conditions under which human capital accumulation is unambiguously (pointwise) lower in the "strategic" equilibrium than under the optimal dynastic policy, and discuss an example where this ordering does not hold. Secondly, we also run a numerical sensitivity analysis to assess the magnitude of discrepancies between the two analyzed cases and discuss the potential implications of overestimation of the human capital role if intergenerational interactions are not accounted for.

KEYWORDS: human capital, intergenerational interactions, Markov stationary equilibrium, stochastic transition, constructive approach

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1 Introduction

Human capital is nowadays widely acknowledged to be one of the most important factors determining the differences in wealth across nations as well as their growth potential. This variable is thus present in a wide range of micro- and macroeconomic theories, including those taking an explicitly intergenerational planning perspective. In such theories, various forms of altruism (Abel and Warshawsky 1987, Arrondel and Masson 2006, Bertola, Foellmi, and Zweimueller 2006) are proposed to deal with the empirically grounded intergenerational correlations and linkages in wealth, human capital, social status, and occupation choice. In particular, strategic interactions across generations are especially apparent in relation to schooling: on the one hand, a substantial fraction of investment in human capital of an individual is made by her parents, while on the other hand, the parents cannot fully anticipate what use will be eventually made of these personal assets (Becker and Tomes 1986, Galor and Tsiddon 1997, Haveman and Wolfe 1995, Lochner 2008, Loury 1981, Orazem and Tesfatsion 1997).[1]

If one assumes that within each generation, people derive their utility from – among other things – the utility of their children, then there logically follows an infinite-horizon planning problem: parents care for children who care for grandchildren who care for great-grandchildren, etc. A markedly different situation is encountered, however, if parents care for their children's consumption directly: it becomes then crucial if all consecutive generations can credibly commit to their future choices. If not, then the optimization problem becomes inherently strategic.

The direction of impact of such strategic interactions on the time path of human capital accumulation is not clear a priori. On the one hand, strategic interactions and the resulting lack of commitment may lower each generation's investment, as it is less productive then (under the assumption of Lipschitz continuous consumption policies) than in the Pareto optimal dynastic policy case. On the other hand, however, as noted by Bernheim and Ray (1987), with strategic interactions, the same result in terms of tomorrow's utility requires higher investment today. Lack of commitment thus lowers agents' utility derived from the next generation's consumption and (under the standard assumption of decreasing marginal utility) raises its marginal utility. As a result,

[1]The classic works within the human capital accumulation literature, such as Mincer (1958) or Ben-Porath (1967), focus primarily on the other component of investment in education which is individuals' own purposeful educational spending motivated by the expected increases in their future earnings. The Ben-Porath's model specification is however already flexible enough to allow for intergenerational transmission of human capital as well.

this second channel creates incentives to increase investment in the presence of commitment problems. Hence, in models decentralizing the Pareto-optimal allocation of the dynastic economy, the actual investment in human capital accumulation in the steady state (or at the balanced growth path) can also be potentially both under- and overestimated when strategic interactions are not taken into account.

Given this background, the contribution of the current paper to the literature is fourfold. First, we identify the impact of strategic interactions between consecutive generations on the time path of human capital accumulation in an economy populated by a sequence of generations allowed to decide over their consumption levels as well as over the levels of investment in human capital of their immediate descendants. We provide sufficient conditions under which human capital accumulation is unambiguously (pointwise) lower in the "strategic" equilibrium than under the optimal dynastic policy, and discuss an example where this ordering does not hold. We are able to obtain clear-cut results here by computing the Markov Stationary Nash equilibrium (MSNE) human capital investment policy at the aggregated level and benchmarking this time-consistent MSNE result against the optimal but time-inconsistent policy which neglects strategic interactions across generations.[2] This is achieved thanks to a novel constructive method, similar to the one offered by Balbus, Reffett, and Woźny (2011).

Secondly, we propose a decentralization of the MSNE allocation as a quasi-competitive recursive equilibrium. To the best of our knowledge, it is the first formal definition of prices for MSNE allocations in stochastic games. To obtain our result, we build on the original idea of Lane and Leininger (1986) on defining prices in deterministic economies with (consumption) externalities and add recent results of Magill and Quinzii (2009) on the so-called "probability approach" to general equilibrium modeling.

Thirdly, we provide a set of sufficient conditions guaranteeing that in the MSNE policy, human capital investment is unambiguously lower than under full commitment. In a simple example we also show that if our conditions are not satisfied, this inequality may not necessarily hold. We complement this result with an indication that the results obtained under full commitment generally do not lead to a (Markov stationary) Pareto optimal allocation. Hence, although strategic motives can promote human capital accumulation in comparison to the (Markov stationary) Pareto optimal allocation, it may be still

[2] In Appendix A, we also compare these two setups to a similar model, frequently used in the literature, i.e. the one of joy-of-giving altruism (used by, among numerous others, Abel and Warshawsky (1987), Artige, Camacho, and de la Croix (2004), Bruhin and Winkelmann (2009)).

insufficient as compared to the optimal dynastic allocation.

Fourthly, we work out a specific functional parametrization of the model, under which we obtain additional conditions for monotonicity of the (MSNE) equilibrium as well as the optimal policy. In a sensitivity analysis based on this parametrization, we also assess the magnitude of discrepancies between human capital policies in the two compared cases. Based on these results, we discuss the potential implications of overestimation of the human capital role if intergenerational interactions are not accounted for.

The current paper can be viewed as a methodological contribution to the discussion on intergenerational transfers, distribution of wealth and public policy in the class of OLG models – a discussion initiated by Barro (1974) and continued, among others, by Laitner (see Laitner (1979, 2002) and references within), Bernheim, Shleifer, and Summers (1985), or specifically in the context of human capital accumulation by Caucutt and Lochner (2011), Drazen (1978), Keane and Wolpin (2007). Our contribution constitutes a step towards proper assessment of the magnitude of impact of strategic interactions on the expected paths of human capital accumulation, often neglected in the quantitative macroeconomic literature (see papers streaming from Becker, Murphy, and Tamura 1990, Becker and Tomes 1979, Lucas 1988, among numerous others)

The remainder of the article is structured as follows. Section 2 discusses the related literature, both from the economic and the methodological/technical angle. In Section 3 we lay out our basic model with strategic interactions and present our principal theoretical results. In Section 4 we compare this model with a benchmark model where no strategic interactions are allowed, providing further analytical results. Section 5 provides an illustrative numerical example for our calculations of the preceding chapters. Section 6 discusses numerically the role of strategic interactions in shaping human capital investment decisions. Section 7 concludes.

2 Related literature

The vast majority of the macroeconomic literature on human capital formation is based on the assumption of full commitment across generations.[3] Based on

[3]More specifically, numerous articles also include children's human capital levels (or bequests) directly in the parents' utility function, rather than children's consumption. This is consistent with the argument that "[t]he model has similar implications if it is assumed that parents value the utility of their children, rather than their human capital or attainments" (Haveman and Wolfe 1995). See also Becker and Tomes (1979). For a discussion of a version

these premises, this literature has investigated the impact of endogeneously determined levels of human capital on economic growth (Gong, Greiner, and Semmler 2004, Lucas 1988), fertility (Becker, Murphy, and Tamura 1990) as well as income and wealth inequality (Bénabou 1996, Galor and Zeira 1993). Articles in this field have also addressed a range of research questions related to the efficiency of allocation of education funds and they have provided very detailed characterizations of the process of intergenerational human capital transmission (Aiyagari, Greenwood, and Seshadri 2002, Brown, Scholz, and Seshadri 2012, Caucutt and Lochner 2011, Cunha and Heckman 2007). The current paper argues that due to the omission of intergenerational interactions in the theoretical underpinnings of this literature, the incentives for parents to invest in their children's education might have been overestimated there (further discussion of this issue follows in Section 4).

Our setup, on the other hand, by allowing consecutive generations' utilities to be defined over their own and the successive generation's *consumption*, leads to strategic interactions and thus necessitates an application of the (Markov stationary)[4] Nash equilibrium concept rather than just an optimal planning solution. The framework is based on three broad theoretical considerations – altruistic preferences[5], hyperbolic discounting[6], and distributive justice[7], each providing a compelling motivation for studying economic settings

of our model with joy-of-giving altruism, please refer to Appendix A.

[4]Observe that many other interesting equilibria, featuring history-dependent punishment schemes, may also arise in the context of repeated games. Hence we owe the reader some justification for our exclusive use of the Markovian equilibrium concept. Firstly, our model studies an OLG-type economy populated by short-lived agents, each representing a separate generation. For this reason it would be problematic to interpret non-Markovian punishment schemes, i.e., the ones based on longer memory than one period behind. One should also mention that Markovian equilibria in our model are defined on a smaller state space than the one which includes all one-period histories. Indeed, in our model a player in period t does not know the action taken by the previous generation (in period $t-1$) and so cannot punish previous period deviations directly (this can be done only indirectly via the observed state). Secondly, one of the advantages of our paper lies with computational tractability of the proposed model which is obtained partly thanks to the employment of a Markovian/recursive equilibrium concept. See also Kubler and Polemarchakis (2004), Citanna and Siconolfi (2010) for a similar equilibrium concept.

[5]Leininger (1986), Bernheim and Ray (1987), Bernheim and Ray (1989), Amir (1996c) and Nowak (2006).

[6]Phelps and Pollak (1968), Peleg and Yaari (1973) or more recently Laibson (1997), Bernheim, Ray, and Yeltekin (1999) and Krusell and Smith (2003).

[7]The normative literature on distributional justice including, among others, works by Dasgupta (1974b), Dasgupta (1974a) and Lane and Mitra (1981), views the Nash equilibrium as a concept corresponding to the universalizability criterion of distributive justice discussed by Rawls (Dasgupta 1974b). In relation to the current paper, Lane and Mitra (1981) have

where consecutive generations (or current and future selves) play strategically in their consumption decisions. For empirical evidence supporting our baseline specification of preferences, the reader is referred to O'Donoghue and Rabin (1999a,b), and Angeletos, Laibson, Repetto, Tobacman, and Weinberg (2001), or Eisenhauer and Ventura (2006).

The commitment problem in intergenerational setups is also closely related to the issue of time (in)consistency of optimal plans and policy games, which has been studied in detail by economists ever since the works of Kydland and Prescott (1977) and Stokey (1991). Although Kydland, Prescott's or Stokey's pathbreaking contributions focused primarily on strategic interactions between the private economy and the government, whereas the current paper deals with strategic interactions between private agents only, the conceptual and numerical problems are the same for both approaches.

From the technical perspective, the point of departure of the current article is the problem where the parents optimally choose their consumption level as well as the level of investment in human capital of their children. This requires considering the possible options the children will face in the subsequent period – when they will themselves become independent utility maximizers. The parents would therefore like to embed their children's optimization problems in their own and thus become "leaders" of such an intergenerational strategic game. Unfortunately, this procedure cannot be carried out directly: since the children's optimization problem embeds the optimization problem of their own children, and so forth *ad infinitum*, we would end up with an infinite series of embedded games (or an infinite horizon game). Unfortunately, one cannot apply usual fixed-point arguments here because the strategic component of the embedded games creates a "vicious circle" (see Leininger (1986))[8] of strategy space which has obstructed the development of economic theories in this vein for many years (see e.g. Strotz (1955) and Phelps and Pollak (1968)).

Despite these problems, (Markov stationary) equilibrium existence results have been obtained by Bernheim and Ray (1983) and Leininger (1986) for a deterministic incarnation of the game, and by Amir (1996a,c) and Nowak (2006) for its stochastic version. These crucial technical developments are however based on topological arguments, existential rather than constructive in nature, and thus without additional results regarding the conditions for uniqueness of the analyzed equilibrium, their usefulness in applied work is

studied Pareto or modified Pareto optimality of a Nash equilibrium in a class of games of intergenerational altruism.

[8]Specifically, even very strong assumptions made on the strategy/policy of the subsequent generation cannot guarantee that the best response to that strategy would belong the the same strategy/policy space.

uncertain.

In a related series of papers on time-consistent taxation, Klein, Krusell, Quadrini and Ríos-Rull have proposed an intuitive numerical technique for equilibrium computation by value function iteration. Specifically, Klein and Ríos-Rull (2003) and Klein, Vincenzo, and Ríos-Rull (2005) analyzed the Markov stationary equilibrium in a growth model without (tax policy) commitment using techniques essentially based on numerical iteration of the value function under a linear-quadratic approximation. A potential application of this approach to our case is obstructed by two problems, though. Firstly, no controlled accuracy or error bounds have been provided for these approximations. Secondly and more importantly, the method is based on the assumption of differentiability of the policy function and the related strict concavity and twice differentiability of the (infinite horizon) value function which, perhaps apart from a few cases of specific functional forms representing preferences and technology, is very problematic (see e.g. the assumptions in Santos (1994), Montrucchio (1998) necessary for policy function differentiability). And although recently Klein, Krusell, and Ríos-Rull (2008) managed to solve the first of aforementioned problems by proposing a characterization of the time-consistent policy in terms of first order conditions (the so-called Generalized Euler Equation), the second argument, to our best knowledge, remains unsolved. Hence, as for our intergenerational human capital bequest economy, there are no results available yet on the uniqueness or differentiability of the Markov stationary equilibrium (see Kohlberg (1976) and Amir (1996c) for a discussion), and we cannot apply the methods proposed by Klein, Krusell, Quadrini and Ríos-Rull in our constructive study.

Certain problems are also encountered when one uses the methods for showing equilibrium existence in the class of dynamic games proposed by Abreu, Pearce, and Stacchetti (1990), henceforth APS. In this line of research, existence results come almost for free, but unfortunately almost no equilibrium characterization is available, not to mention uniqueness of the analyzed equilibria or computational possibilities (see Balbus, Reffett, and Woźny (2012a) for a detailed discussion).

Given the drawbacks of all discussed methods, in the current paper we propose a novel technical framework for studying our human capital bequest economy with strategic interactions. This framework not only allows us to obtain the equilibrium uniqueness result, but also features a constructive numerical algorithm for computing it. The algorithm guarantees uniform convergence, thanks to which we are able to solve the technical problem of computing error bounds. The technique itself is new, albeit linked to the one offered by Balbus, Reffett, and Woźny (2011). The main difference between our setup and theirs lies with the proof of the MSNE existence and uniqueness theorem. In this paper, the existence and uniqueness argument is based on an operator defined (implicitly within the first order conditions) on the space of inverse marginal utility functions; in theirs – it is defined on the space of value functions. For this reason our approach is closer in spirit[9] to the papers of Feng, Miao, Peralta-Alva, and Santos (2009), Phelan and Stacchetti (2001) where an APS operator maps the set of correspondences (of inverse marginal values/Lagrange multipliers) into itself. Although the set of assumptions allowing to represent solutions of the maximization problem using first order conditions is usually restrictive in dynamic economies with time-consistency problems[10], it is useful for computations in our particular setting. Specifically, as we are able to obtain uniform convergence of iterations of inverse marginal utilities to the equilibrium one, we obtain uniform error bounds for human capital investment policies directly (by just inverting marginal utilities, see also Santos 2000). To obtain such error bounds for policies using the value function technique applied in Balbus, Reffett, and Woźny (2011), one would have to impose additional Lipschitz continuity assumptions on the model primitives (especially on the stochastic transitions) – a feature which we would like to avoid in our model. Viewed from a different perspective, our method is a direct generalization of the monotone operators and inverse marginal utility method (Coleman 2000, Datta, Mirman, and Reffett 2002, Morand and Reffett 2003) applied to a class of dynamic games.

Our technique comes, however, at a cost as well. Specifically it is restrictive in terms of requiring a specific form of stochastic transition of the state variable – here, the human capital stock. Two main features of this transition are the following: (i) it is defined in terms of distributions over the next period's state space parameterized by the current period's investment and current state, and (ii) it "separates" the decision from distributions by requiring a certain functional form of the mixing functions. Specifically, it cannot be reduced to the deterministic case (see Assumption 2 for details). Such stochastic transition has already been widely used: by Amir (1997) in optimal growth theory; by Amir, Nowak, Curtat and coauthors[11] in the directly related context of dynamic games; by Horst (2005) in his study of "weak social interactions", as

[9]The critical difference between our approach and theirs is that our operator is suitable for the analysis of Markovian (short-memory) equilibria only, whereas their approach characterizes the set of all sequential equilibria. See also the discussion in footnote 4 and following Theorem 3.

[10]Especially to verify that the FOC is also sufficient. See our earlier discussion on the Klein, Krusell, and Ríos-Rull (2008) paper.

[11]See Amir (2002), Nowak (2003, 2007), Curtat (1996) and references therein.

well as (at a somewhat more general level) by Magill and Quinzii (2009) in the general equilibrium framework.

Economically, the assumed form of the stochastic intergenerational human capital transition function is also critical to obtain an unambiguous ordering of the MSNE (human capital) investment policy and the optimal (time-inconsistent) investment policy. It neutralizes one of the two effects of strategic interaction, discussed already in the introduction: through increased marginal utility.

Other characteristics of this transition function, e.g., the requirement a certain level of "mixing", have so far prevented scholars from proving existence of appropriate price systems decentralizing firms' decisions in such a framework. One of the key achievements of this paper is thus to overcome these difficulties and develop a way to decentralize the MSNE allocation. To this end, we build on Magill and Quinzii (2009), who have proposed a way to decentralize the optimal allocation in a (two-period) economy with technology being a probability distribution (over a finite number of states) rather than an Arrow-Debreu "state of nature" production function. We generalize the approach due to Magill and Quinzii, combine it with the result of Lane and Leininger (1986) generalized to a stochastic setting, and in this way we obtain a counterpart of the First Welfare Theorem and find a decentralization of a Markov (differentiable) stationary equilibrium allocation in the strategic case. To the best of our knowledge, this is the first proposal to decentralize inefficient equilibrium allocations of stochastic games[12] and hence the current paper should be seen as a first step towards bringing stochastic games/models with partial commitment closer to the current mainstream macroeconomic research. Several further conceptual and technical issues arise with such an equilibrium concept which we do not address directly here. Please refer to the paper by Balbus, Reffett, and Woźny (2012b) where some of these questions are answered in the context of a recursive equilibrium of an economy with endogenous risk.[13]

[12]On the other hand, decentralization of equilibria in deterministic, static games has been the subject of extensive research. Let us refer the reader to a few of them: Dubey (1982) Dubey and Shubik (1977), Gale (1992), Schmeidler (1980).

[13]See also the discussion by Karatzas, Shubik, and Sudderth (1997), Geanakoplos, Karatzas, Shubik, and Sudderth (2000) or Phelan and Stacchetti (2001).

3 The model

3.1 Setup of the model

Our model economy is populated by an infinite sequence of generations whose sizes are equal and normalized to unity. Each generation $t = 0, 1, 2, \ldots$ is characterized by the common utility function U, taking values $U(c_t, c_{t+1})$, where c_t is the total consumption of generation t. We assume U to be time-separable and take the form: $U(c_t, c_{t+1}) = u(c_t) + v(c_{t+1})$. The consumption set is $Y = [0, \bar{Y}]$ where $\bar{Y} \in \mathbb{R}_+$. Each household is endowed with the bequested human capital level and a unit of leisure time that can be split between working in two sectors: production and schooling (i.e., the "human capital investment" sector).[14]

The unique consumption good is produced by firms using technology f which requires two kinds of inputs: (i) time devoted to work l_1, and (ii) human capital h. Observe that we neglect all physical capital accumulation in our basic model.

Human capital is accumulated thanks to the schooling (investment) sector producing human capital using a stochastic technology[15] given by the transition function G parameterized by two inputs: (i) the current level of aggregated human capital H, and (ii) time devoted to human capital accumulation l_2. Let the set $\mathbf{H} = [0, \bar{H}]$, where $\bar{H} \in \mathbb{R}_+$, represents all possible levels of human capital.

[14]Our modeling approach can be interpreted in two ways: (i) straightforwardly – that each household lives for one period and derives utility from its own consumption, $u(c_t)$, and the consumption of its immediate successor, $v(c_{t+1})$; and (ii) with an OLG flavor – that each household lives for two periods but chooses the fraction of time devoted to the production of consumption goods and the fraction of time devoted to the accumulation of human capital of the subsequent generation in the first period only. Its consumption in the second period is chosen by the next generation, and thus is only indirectly influenced by the level of human capital left to the next generation. See also Galor and Zeira (1993).

[15]As noted earlier, the introduction of stochastic factors in human capital accumulation is motivated primarily by technical reasons. Such factors have sound economic motivation, though. Indeed, (i) heredity involves randomness: the unobservable skill levels are not inherited from one's parents deterministically; (ii) human capital is not homogenous: it is technology-specific and thus up-front investment in it might (but might not) be ineffective (Chari and Hopenhayn 1991), depending on the future pattern of technological progress; (iii) the motivation of children to learn is endogenous (Orazem and Tesfatsion 1997). All these factors taken together make it clear that treating investment in education as a lottery where future payoffs depend on stochastic factors is quite reasonable. Finally, it should be noted that we rule out all systematic human capital externalities from non-relatives here (Ben-Porath 1967, Rangazas 2000) and assume that children's human capital is created from parental human capital, education effort, and stochastic factors only.

The timing of the considered intergenerational game is straightforward: first the household (the "parents") decides upon its preferred time allocation and investment in human capital of the next generation, given its own human capital level and subject to the expected return of human capital accumulation of the consecutive generation (the "children"). Second, markets clear and random variables take their realizations, determining the human capital level of the next generation. Since we are looking for Markov stationary perfect equilibria, we also require that the perceived and actual human capital laws of motion coincide and that the optimal human capital accumulation policy is valid irrespective of time.

More specifically, we make the following assumptions:

Assumption 1 *Let:*

- $u, v : Y \to \mathbb{R}$ *be increasing, continuously differentiable, and satisfying* $\lim_{c\to 0} u'(c) = \lim_{c\to 0} v'(c) = \infty$*;* $(\forall c \in Y, c > 0) \quad u'(c) < \infty$ *and* $(\forall c \in Y, c > 0) \quad v'(c) < \infty$*. Moreover, let* u *and* v *be strictly concave and such that* $u(0) = v(0) = 0$*,*

- $f : \mathbf{H} \times [0, 1] \to Y$ *be strictly concave with respect to the second argument, twice continuously differentiable with finite partial derivatives, and satisfying* $(\forall l \in [0, 1]) \quad f(0, l) = 0$*,* $(\forall h \in \mathbf{H}) \ \lim_{l\to 0} f_2'(h, l) = \infty$*. Furthermore, assume that* $(\forall h \in (0, \bar{H}])\, f(h, \cdot)$ *and* $(\forall l \in (0, 1])\, f(\cdot, l)$ *are strictly increasing functions.*

The following assumption on the stochastic transition follows Amir (1996c) and Nowak (2006).

Assumption 2 (Technology) *The distribution* G *satisfies:*

- $\forall h \in \mathbf{H}, \quad G(0|h, 0) = 1$*,*

- $\forall h \in \mathbf{H}, l \in (0, 1], \quad G(\cdot|h, l) = (1 - g(h, l))\delta_0(\cdot) + g(h, l)\lambda(\cdot|h)$*, where*

- $g : \mathbf{H} \times [0, 1] \to [0, 1]$ *is strictly concave with respect to the second argument, twice continuously differentiable, and satisfies the condition:* $(\forall l \in (0, 1]), g(0, l) > 0$*,*

- $(\forall l \in (0, 1])\, g(\cdot, l)$ *and* $(\forall h \in (0, \bar{H}])\, g(h, \cdot)$ *are strictly increasing functions,*

- $(\forall h \in \mathbf{H}) \lim_{l\to 0} g_2'(h, l) = \infty$ *and* $(\forall h \in \mathbf{H}, l > 0), 0 < g_2'(h, l) < \infty$*,*

- $\lambda(\cdot|h)$ *is a family of Borel transition probabilities (possessing density) on* $(0, \bar{H}]$ *that is stochastically decreasing and continous with* h*, while* δ_0 *is a probability measure concentrated at zero.*

The crucial implications of this specification are as follows: with probability $1 - g(h, l_2)$, the next generation's human capital will be zero, indicating that the investment in it has been completely ineffective. The economic interpretation of this assumption can be twofold. First, it may capture human capital-dependent mortality: the next generation's zero human capital is then a synonym for not surviving until adult age. Such a setup is in good agreement with evidence: indeed, children of better educated parents face a generally lower risk of dying young. Second, this may also relate to the argument that skills are often technology-specific and that technology might change fast enough to make all previously acquired skills obsolete.[16]

With probability $g(h, l_2)$, conditional on survival and non-obsolescence of skills, human capital is however drawn from a distribution λ which does not depend on l_2. This relates to the stochastic heredity assumption, coupled with the random motivation of children to learn. We normalize $l_1 + l_2 = 1$ and require $h = H$ for consistency.

3.2 Equilibrium concepts

Let us now describe two ways of analyzing allocations in our economy: (i) the (decentralized) quasi-competitive recursive equilibrium allocation and (ii) the Markov stationary Nash equilibrium of the corresponding stochastic game. We start with the former one.

Formally, we adopt the following definition:

Definition 1 *A recursive quasi-competitive equilibrium of the economy under study is a list of functions* $(\Phi, \Pi, \tilde{c}, c^*, a^*, l_1^*, l_2^*, p^*, w^*, e^*)$ *satisfying the follow-*

[16] As demonstrated in Remark 1 further on in this paper, the assumed shape of the transition function $G(\cdot|h, l_2)$ is necessary for our main qualitative results to hold. Further technical justifications for this assumption can be found in Amir (1996c), Nowak (2006). Most importantly, though, our functional assumption on $G(\cdot|h, l_2)$ should be considered separately from the simplifying assumption that the utility of generation t at time $t+1$ depends on the next generation's consumption only (via $v(c_{t+1})$). The latter assumption is *not* crucial for our main results and can be relaxed to some degree (see Balbus, Reffett, and Woźny 2012c). It serves to separate the intergenerational altruism and strategic interactions channel from the intertemporal savings channel, both of which could potentially have an effect on equilibrium human capital formation. By switching off the second channel, we are able to obtain clear-cut results on the first one. This assumption also helps us avoid the need to specify intertemporal exchange markets for the consumption good in the decentralized allocation. We leave this question for further research.

ing conditions (for any given individual and aggregate human capital levels $h, H \in \mathbf{H}$):

- *taking prices $p^*(\cdot, H), w^*(H), e^*(H)$, profits $\pi^*(H)$ as given, the households' consumption policy $c^*(h, H)$ and schooling policy $a^*(\cdot, h, H)$ solve:*

$$\max_{c \geq 0, a(\cdot) \geq 0} u(c) + \int_{\mathbf{H}} v(\tilde{c}(a(y)))\Pi(y|H)dy,$$

under the budget constraint

$$w^*(H) + he^*(H) + \pi^*(H) \geq c + \int_{\mathbf{H}} a(y)p^*(y, H)dy,$$

- *taking prices $w^*(H), e^*(H)$ as given, the time devoted to work $l_1^*(H)$ and the human capital stock $h = H$ solve*

$$\max_{h, l_1 \geq 0} f(h, l_1) - w^*(H)l_1 - e^*(H)h,$$

- *taking prices $p^*(\cdot|H), w^*(H)$ and the perceived law of motion for human capital $\Pi(\cdot|H)$ as given, the time devoted to human capital accumulation $l_2^*(H)$ solves:*

$$\max_{l_2 \geq 0} \int_{\mathbf{H}} \Phi(y, H)dG(y|l_2, H) - w^*(H)l_2,$$

where $(\forall y \in H)\, \Phi_1'(y, H) = \frac{p^*(y,H)}{\Pi(y|H)}$ is the marginal revenue of investment good firms, and we set

$$\pi^*(H) = \int_{\mathbf{H}} \Phi(y, H)dG(y|l_2^*(H), H) - w^*(H)l_2^*(H),$$

- *markets clear: $f(H, l_1^*(H)) = c^*(H, H)$, $l_1^*(H) + l_2^*(H) = 1$, $(\forall y \in \mathbf{H})\, y = a^*(y, H, H)$,*

- *the perceived and actual law of motion for human capital coincide, i.e. $(\forall H, y \in \mathbf{H})\, \Pi(y|H) = \frac{\partial}{\partial y} G(y|l_2^*(H), H)$,*

- *the equilibrium consumption policy is $c^*(H, H) = \tilde{c}(H)$.*

Let us now explain and interpret our equilibrium definition. For this reason we shall now explicitly incorporate timing into our notation, although our definition of the recursive equilibrium as such is time-invariant.

First, all equilibrium conditions imposed on household behavior are standard apart from the inclusion of strategic interactions[17] between the current and future generation through the equilibrium consumption policy $\tilde{c}$ (hence the term "quasi-competitive"). Specifically for a given aggregate human capital level H_t, each household solves:

$$\max_{c_t, a_{t+1}} u(c_t) + \int_{\mathbf{H}} v(\tilde{c}_{t+1}(a_{t+1}(y_{t+1})))\Pi(y_{t+1}|H_t)dy_{t+1},$$

where $\tilde{c}_{t+1}$ is an increasing, Lipschitz continuous and Markov (consumption) policy of their immediate successor, and the perceived law of motion for human capital, $\Pi(y_{t+1}|H_t)$, is taken as given. The choice variable a_{t+1} reflects the next generation's human capital stock (as a function of the drawn state). The optimal equilibrium choice depends of course on H_t (via prices).

Furthermore, e_t, w_t stand for prices of human capital and leisure, respectively, whereas $p_t(y_{t+1}, \cdot)$ is the price of a state-contingent security purchased at time t, i.e. a commitment to receive (or deliver) a unit of human capital in the next period $(t+1)$, when state y_{t+1} is realized. Households take prices p_t, w_t, e_t, profits π_t and aggregate states H_t as given. The wealth accumulation process (budget constraint) is then given by:

$$w_t + h_t e_t + \pi_t \geq c_t + \int_{\mathbf{H}} a_{t+1}(y_{t+1})p_t(y_{t+1}, H_t)dy_{t+1}.$$

Note that in each period we normalize the prices of capital (and investment) goods to 1 and express prices for capital and (Arrow)-securities relative to consumption prices.

Second, the labor input is divided into two sectors but since in equilibrium, wages in both are the same, the household will be indifferent with regard to its division.

Third, the "investment sector firms" (that can be understood as private schools) present in the decentralized equilibrium use labor to increase the probability of the next generation reaching a high human capital state and obtain revenues $\Phi(y_{t+1}, H_t)$ that depend on today's aggregate human capital level and tomorrow's state. Hence, their objective is given by:

$$\max_{l_2 \geq 0} \int_{\mathbf{H}} \Phi(y_{t+1}, H_t)dG(y_{t+1}|l_{2,t}, H_t) - w_t^*(H_t)l_{2,t},$$

[17] Alternatively we can incorporate such interactions directly in prices, see Lane and Leininger (1986), but the allocations under both specifications would stay the same.

where $(\forall y_{t+1} \in \mathbf{H})\, \Phi_1'(y_{t+1}, H_t) = \frac{p^*(y_{t+1},H_t)}{\Pi(y_{t+1}|H_t)}$ is the marginal revenue of investment good firms. Observe that in equilibrium, these firms must only know the marginal Φ_1' given by the price-probability ratio. Introducing individual and aggregate human capital levels (h_t, H_t, respectively) separates price channels from strategic interactions.

Our definition of the (decentralized) quasi-competitive recursive equilibrium is inspired by Lane and Mitra (1981), Lane and Leininger (1986), and Magill and Quinzii (2009). There are two main differences between our decentralization and that of Magill and Quinzii (2009). Firstly, we apply it to an infinite horizon economy with a representative agent present in each period, whereas Magill and Quinzii (2009) study a two-period economy with multiple agents (in the context of a stakeholder equilibrium). Secondly and more importantly, we have an uncountable number of states, while they have a finite number of states only. Apart from specific technical difficulties arising in our context, an uncountable number of states allows us to restore the informative role of prices to coordinate the interest of firms and consumers to maximize social welfare. Loosely speaking, in our case prices are equal to Φ_1' and hence marginal utilities, whereas in the case of Magill and Quinzii (2009), firms need to calculate $\triangle_H \Phi$ and thus can only approximate consumers' marginal utilities.

Other conditions included in our equilibrium concept are standard and resemble the recursive competitive equilibrium concept (see Prescott and Mehra 1980).

We shall now drop the time subscripts and turn to the characterization of the interior, differentiable[18] equilibrium allocation by first order conditions:

$$\frac{v'(\tilde{c}(a^*(y,H,H)))\tilde{c}'(a^*(y,H,H))}{u'(c^*(h,H))} = \frac{p^*(y,H)}{\Pi(y,H)}, \tag{1}$$

$$\frac{d}{dl}\int_{\mathbf{H}} \Phi(y,H)dG(y|1-l_1^*(H),H) = w^* = f_2'(H,l_1^*(H)), \tag{2}$$

$$e^*(H) = f_1'(H,l_1^*(H)). \tag{3}$$

Using integration by parts, market clearing and condition (1), condition (2) becomes:

$$-\int_{\mathbf{H}} v'(\tilde{c}(y))\tilde{c}'(y)G_1'(y|1-l_1^*(H),H)dy = f_2'(H,l_1^*(H))u'(\tilde{c}(H)).$$

where $f(H,l_1^*(H)) = \tilde{c}(H)$. Carefully studying this equilibrium condition, one

[18] Although the current analysis requires a differentiable policy $\tilde{c}$, we think that it can also be generalized using subgradients methods. We leave this for further research.

observes that:

$$l_1^*(h) \in \arg\max_{\hat{l}} u(f(h,\hat{l})) + \int_{\mathbf{H}} v(f(y, l_1^*(y)))G(dy; h, 1-\hat{l}), \tag{4}$$

with $h = H$.

Let us pause the analysis of the decentralized equilibrium here and consider an (infinite horizon) stochastic game played by a sequence of generations that – as it will turn out – yields the same Markov stationary Nash equilibrium conditions as the one defined in (4).

From now and until the end of the paper, we will be using the notation $l = l_1$ and $1 - l = l_2$. Thus, l will capture the fraction of working time spent in the production sector.

We shall assume that the game is still described by the same technologies as before, but now each generation plays directly against the consecutive one. So we let the next generation follow a Markov strategy $l' \in L$ where $L = \{l : (0, \overline{H}] \to [0,1], l \in \mathcal{C}\}$[19]. Moreover, we shall let $\mathbf{0} \in L$ denote the constant zero function, and let $\mathbf{1} \in L$ denote a constant function whose values are always equal to 1. Under this notation, the maximization problem of the current generation reduces to:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_{\mathbf{H}} v(f(y, l'(y)))G(dy; h, 1-\hat{l}), \tag{5}$$

which gives solutions equivalent to (4).

Under Assumptions 1 and 2, the maximand of (5) (for a given $h \in (0, \bar{H}]$) is strictly concave and differentiable with respect to $\hat{l}$ on $(0,1)$. Furthermore, the unique optimal labor supply level l^* solves $\zeta(l^*, h, l') = 0$ whenever interior, where ζ is defined as:

$$\zeta(l,h,l') := u'(f(h,l))f_2'(h,l) - g_2'(h, 1-l)\int_{\mathbf{H}} v(f(y,l'(y)))\lambda(dy|h). \tag{6}$$

An interior Markov stationary Nash equilibrium (MSNE) of the economy with stochastic transition is then a function l which solves $\zeta(l(h), h, l) = 0$ for all $h \in (0, \bar{H}]$.

Given this result, we may summarize that the MSNE satisfying (4) can be equivalently interpreted as (i) a labor supply function l_1^* characterizing the recursive quasi-competitive equilibrium defined above, and (ii) a Markov stationary Nash equilibrium of a sequential intergenerational stochastic game. Moreover, observe that (ii) is equivalent to a (iii) time-consistent policy which is

[19] By $\mathcal{C}$ we denote the set of all continuous functions with the given domain and codomain.

equally well suited for any generation. All these three concepts are thus equivalent, at least for differentiable policies. Specifically having computed a differentiable MSNE policy one can deduce prices in the recursive quasi-competitive equilibrium from equations (1)–(3) (see Balbus, Reffett, and Woźny 2012b, on how to do that). From now on, for simplicity, we will study the (centralized) stochastic game allocation, but keep in mind the other two interpretations as well.

3.3 Characterization of the MSNE, existence and uniqueness

Before showing existence of the MSNE (and hence existence of the recursive quasi-competitive equilibrium allocation) we shall present some of the basic properties of this equilibrium, if it exists. They will be helpful in our further analysis.

Theorem 1 (Characteristics of MSNE) *Let Assumptions 1 and 2 hold and suppose that a MSNE l^* exists. If $f''_{12}(\cdot,\cdot) \leq 0$ and $g''_{12}(\cdot,\cdot) \geq 0$, then l^* is strictly decreasing on $(0, \bar{H})$ wherever interior.*

Proof. From Assumptions 1–2, a MSNE l^* must solve the following maximization problem:

$$\max_{\hat{l} \in [0,1]} u(f(h, \hat{l})) + g(h, 1 - \hat{l}) \int_{\mathbf{H}} v(f(y, l^*(y))), \tag{7}$$

which leads to a FOC given by:

$$u'(f(h, \hat{l}))f'_2(h, \hat{l}) - g'_2(h, 1 - \hat{l}) \int_{\mathbf{H}} v(f(y, l^*(y))). \tag{8}$$

If inequalities $f''_{12}(\cdot,\cdot) \leq 0$ and $g''_{12}(\cdot,\cdot) \geq 0$ hold together with strict concavity of u, then the marginal returns of the objective function, viewed as a function of h, are strictly increasing with $\hat{l}$. To see that inspect:

$$u''(f(h, \hat{l}))f'_1(h, \hat{l})f'_2(h, \hat{l}) + u'(f(h, \hat{l}))f''_{12}(h, \hat{l}) + \tag{9}$$

$$-g''_{12}(h, 1 - \hat{l}) \int_{\mathbf{H}} v(f(y, l^*(y))) < 0. \tag{10}$$

An application of the theorem[20] due to Amir (1996b) on strict monotone comparative statics completes the proof. ■

[20]See also Edlin and Shannon (1998) for a similar result.

The assertion follows from established theorems on strict monotone comparative statics (Amir 1996b) of optimal solutions to maximization problems featuring a supermodular function on a lattice. Please observe that the reverse to the second assertion need not hold. Generally, even if $f''_{12}(\cdot,\cdot) \geq 0$ and $g''_{12}(\cdot,\cdot) \leq 0$, the optimal labor supply policy l^* need not increase with h due to the strictly decreasing marginal utility.

The existence of a unique (and later, differentiable) MSNE is established in the following way. First, we rearrange the first order condition of maximization given by (5) as:

$$\xi_h(\hat{l}) := \frac{u'(f(h,\hat{l}))f'_2(h,\hat{l})}{g'_2(h,1-\hat{l})} = \int_{\mathbf{H}} v(f(y,l(y)))\lambda(dy|h), \tag{11}$$

where $h \in (0,\bar{H}]$. The function $\xi_h(0,1] \to \mathbb{R}_+$, with $\xi_h(1) = 0$, introduced just above, captures the marginal utility of consumption coupled with marginal labor productivities in both sectors. The next lemma establishes that the function ξ_h is continuously differentiable, strictly decreasing, and invertible with a continuously differentiable inverse.

Lemma 2 *Let Assumptions 1, 2 be satisfied. For all $h \in (0,\bar{H}]$, the function ξ_h is then strictly decreasing, and a diffeomorphism, i.e. invertible, continuously differentiable and with a continuously differentiable inverse. The inverse $\xi_h^{-1} : \mathbb{R}_+ \to (0,h]$ is also strictly decreasing.*

Proof. Let $h \in (0,\bar{H}]$ be given. Note that $g'_2 > 0$ for all arguments. Therefore, $\xi_h(l)$ is well defined and continuous at the point $l = 1$. Moreover, $\lim_{l\to 0}\xi_h(l) = \infty$. As u' and f'_2 are strictly decreasing, and g'_2 is strictly increasing in $l \in (0,1)$ we conclude that ξ_h is strictly decreasing on $(0,1]$. Therefore, ξ_h is invertible, and its inverse is strictly decreasing. It is straightforward to show (using strict monotonicity, strict concavity and continuous differentiability of u' and g'_2) that ξ_h is also continuously differentiable, with $\xi'_h \neq 0$. Finally, ξ_h is also proper because ξ_h is continuous and $\xi_h^{-1}(\mathbb{R}_+) \subset [0,1]$. Therefore, by the global implicit function theorem (e.g. Gordon (1973), Theorem, p. 674), we conclude that the function ξ_h^{-1} is continuously differentiable. ■

On $P = \{\bar{l} : (0,\bar{H}] \to [0,\infty), \bar{l}$ is Borel measurable$\}$, let us also define an operator B such that for any $h \in (0,\bar{H}]$, B satisfies:

$$B\bar{l}(h) = \int_{\mathbf{H}} v(f(y,\xi_h^{-1}(\bar{l}(y))))\lambda(dy|h). \tag{12}$$

The operator B is going to be central to the reasoning in the remainder of the paper: it will be used both in the proofs of our theoretical results and in their numerical implementation. Its importance stems from the fact that by definition, the fixed point of B, say $\bar{l}$, generates a policy l satisfying the FOC (11) with $\hat{l} = l(h) = \xi_h^{-1}(\bar{l}(h))$. This means that for any given "marginal utility" ξ of the succeeding generation, the operator B assigns the optimal "marginal utility" of the current generation (compare with Coleman 2000, Datta, Mirman, and Reffett 2002, Feng, Miao, Peralta-Alva, and Santos 2009, Phelan and Stacchetti 2001).

The next theorem gives the conditions under which B has a unique fixed point in P. This finding is equivalent to showing under which conditions the MSNE of the considered economy, l^*, exists and is unique. By E_x^f we denote the partial elasticity of a function f with respect to x: $E_x^f = \frac{\partial f(x)}{\partial x}\frac{x}{f(x)}$.

Theorem 3 (Existence and uniqueness) *Let Assumptions 1 and 2 be satisfied. Assume in addition that there exists an $r \in (0,1)$ such that for all $h \in \mathbf{H}$ the following holds:*

$$(\forall x > 0) \quad r \geq \left[-E^v_{f(h,\xi_h^{-1}(x))} E^{f,2}_{\xi_h^{-1}(x)} E_x^{\xi_h^{-1}(x)} \right]. \tag{13}$$

Then there exists a unique MSNE l^* of the economy under study.

Proof. The result follows by applying Theorem 7 (delegated to the Appendix; cf. Guo, Cho, and Zhu (2004)). We firstly show that B maps a cone P into itself and is decreasing. Secondly we show that under condition (13) operator B satisfies the geometric condition in Theorem 7.

Let $\bar{l} \in P, \bar{l} \neq 0$, $h \in (0, \bar{H}]$ and t such that $0 < t < 1$ be given. For a given r define a function $\phi_r : [0,1] \to \mathbb{R}_+$, $\phi_r(t) = t^r B(t\bar{l})(h)$. We will now show that there exists an $r, 0 < r < 1$, such that ϕ_r is increasing with t on $(0,1)$. By monotonicity and continuity of ϕ_r from the left at 1 we will conclude that there exists $r, 0 < r < 1$, for which the inequality $\phi_r(t) \leq \phi_r(1)$ is satisfied and so is $t^r B(t\bar{l}) \leq B\bar{l}$.

From the definition of $B(t\bar{l})$ we get: $\phi_r(t) = t^r \int_{\mathbf{H}} v(f(y, \xi_h^{-1}(t\bar{l}(y))))\lambda(dy)$. Note also that $\bar{l}(y)v'(f(y,\xi_h^{-1}(tp(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))$ is bounded for a given $\bar{l}$ and h. As a result, the function ϕ_r is continuously differentiable and

$$\begin{aligned} \phi_r'(t) = t^{r-1} \big[& r \textstyle\int_{\mathbf{H}} v(f(y,\xi_h^{-1}(t\bar{l}(y))))\lambda(dy) + \\ & + t \textstyle\int_{\mathbf{H}} \bar{l}(y)v'(f(y,\xi_h^{-1}(t\bar{l}(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))\lambda(dy) \big]. \end{aligned}$$

Denoting by E_x^f the elasticity of function f at point x in its domain observe that the second integral in the above expression can be reformulated to:

$$t\int_{\mathbf{H}} \bar{l}(y)v'(f(y,\xi_h^{-1}(t\bar{l}(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))\lambda(dy) =$$
$$\int_{\mathbf{H}} v(f(y,\xi_h^{-1}(t\bar{l}(y))))\left[\frac{v'(f(y,\xi_h^{-1}(t\bar{l}(y))))}{v(f(y,\xi_h^{-1}(t\bar{l}(y))))} f(y,\xi_h^{-1}(t\bar{l}(y)))\right]\cdot$$
$$\left[\frac{f_2'(y,\xi_x^{-1}(t\bar{l}(y)))}{f(y,\xi_h^{-1}(t\bar{l}(y)))}\xi_h^{-1}(t\bar{l}(y))\right]\left[\frac{(\xi_h^{-1})'(t\bar{l}(y))}{\xi_h^{-1}(t\bar{l}(y))}t\bar{l}(y)\right]\lambda(dy) =$$
$$\int_{\mathbf{H}} v(\xi_h^{-1}(t\bar{l}(y)))E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}(t\bar{l}(y))}E^{\xi_h^{-1}}_{t\bar{l}(y)}\lambda(dy).$$

Using the above reformulations and condition (13) we conclude that there exists an $r, 0 < r < 1$, such that $r \geq -E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}(t\bar{l}(y))}E^{\xi_h^{-1}}_{t\bar{l}(y)}$ holds for any $h \in (0,\bar{H}]$ and $t, 0 < t < 1$, and $\bar{l} \in P, \bar{l} \neq \mathbf{0}$. Adding non-negativity of v, we obtain:

$$\int_{\mathbf{H}} rv(f(y,\xi_x^{-1}(t\bar{l}(y))))\lambda(dy) \geq$$
$$-\int_{\mathbf{H}} v(f(y,\xi_x^{-1}(t\bar{l}(y))))E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}}(t\bar{l}(y))E^{\xi_h^{-1}}_{t\bar{l}(y)}\lambda(dy). \qquad (14)$$

It follows that $\phi_r'(t) \geq 0$ for $t \in (0,1)$ (since r does not depend on $\bar{l}$ or h). Hence, ϕ_r is increasing on $(0,1)$ for this r. Adding continuity of ϕ_r from the left at 1 we have: $t^r B(t\bar{l}) \leq B\bar{l}$ for any $t \in (0,1]$ and any $\bar{l} \in P$. We conclude therefore, that for all $h \in (0,\bar{H}]$ the inequality $t^r B(t\bar{l}) \leq B\bar{l}$ holds for any $t, \bar{l}$ as required by Theorem 7. ■

Theorem 3 provides the sufficient conditions for the existence and uniqueness of a fixed point of the operator B and thus a MSNE of the considered economy. Moreover, one can straightforwardly compute it using a Picard iterative procedure.

The mathematical intuition behind Theorem 3 is the following: since the fixed point operator B is decreasing, it may have multiple, unordered fixed points. The condition in Theorem 3 asserts, however, that this operator is "convex" (see Guo and Lakshmikantham (1988) for details) or – in other words – it is a "local contraction". This property is sufficient for existence of a unique fixed point of B. Economically, the condition (13) ("convexity" or "local contraction") could be interpreted in terms of partial elasticities: it requires that the product of elasticities of v, f and ξ_h^{-1} cannot exceed unity, i.e. that the percentage change in next-period utility v resulting from a one percent change in labor supply $\bar{l}$ cannot be "too high". Otherwise, it could be profitable to deviate from the given policy – the loss in instantaneous consumption sub-utility u would be more than compensated by the gain in next-period consumption sub-utility v – indicating that the given policy could not be an equilibrium any more. We leave the question of the number of equilibria when condition (13) is not satisfied for further work.

Let us comment on the alternative way of showing existence of an equilibrium in our model – using an APS-type procedure where incentive conditions are handled using the FOC. Such a procedure, applied in the space of correspondences (see Feng, Miao, Peralta-Alva, and Santos 2009, Phelan and Stacchetti 2001) or in the space of functions (see Balbus, Reffett, and Woźny 2012a, Doraszelski and Escobar 2012), could characterize the set of all sequential or (nonstationary) Markovian equilibria (specifically their inverse utilities) in our economy. Moreover, as our optimization problem is strictly concave, such a procedure would work for an uncountable number of states without the necessity to introduce sunspots or correlation devices that would convexify the set of inverse marginal values. Such methods, however, are not useful when one is interested in the *stationary* Markovian equilibrium[21]. Interestingly, we also observe that when our uniqueness condition is satisfied, we also obtain global stability of iterations of our operator B defined in the space of bounded, measurable functions. Such a condition, when applied to an APS-type procedure (in function spaces of nonstationary Markovian equilibrium inverse marginal utilities) would guarantee that APS iterations converge to a singleton set[22] where the only function is our unique MSNE (inverse marginal utility).

Having that, we can finally answer the question of existence of a smooth MSNE. This is especially important for obtaining prices via relation (1). Under a few additional conditions, the answer is affirmative.

Theorem 4 (Differentiable MSNE) *Let Assumptions 1 and 2 hold. If additionally $u''(\cdot) < 0, g''_{22}(\cdot) \leq 0, f''_{22}(\cdot) \leq 0$, and $h \mapsto \int v(f(y, l'(y))\lambda(dy|h)$ is $\mathcal{C}^1$ for any bounded and Borel measurable $l' \in L$, then there exists a differentiable MSNE.*

Proof. Observe that under our assumptions MSNE is interior and characterized my the first order condition $\zeta(l(h), h, l) = 0$ that is also sufficient. As $u''(\cdot) < 0, g''_{22}(\cdot) \leq 0, f''_{22}(\cdot) \leq 0$; $\zeta'_2(\cdot) < 0$ hence by implicit function theorem best response operator maps bounded, Borel measurable functions into differentiable ones. Hence there exists a MSNE in $\mathcal{C}^1$. ■

[21] To see that observe that our decreasing operator B may possess fixed edges (or exhibit cycles) x, y, s.t. $B(x) = y$ and $B(y) = x$, while not possessing a fixed point at all. In such case there is no stationary Markov equilibrum but there is an nonstationary Markovian equilibrium, where every odd generation uses x, while every even generation uses y.

[22] In such a case, operator B has only one fixed edge, being its fixed point.

4 Human capital dynamics with and without strategic interactions

In the current section, we shall compare the time-consistent Markov stationary policy l^* (or the corresponding decentralized allocation), discussed in the previous sections, to the outcomes obtained within a similar setup which does not allow for strategic interactions across generations.

To this end, following Bernheim and Ray (1987), we will benchmark our results under strategic interactions to the optimal policies of a planner taking care of all generations simultaneously. In order to attain comparability of utilities across different periods, we must assume that $v(\cdot) = \delta u(\cdot)$ where $\delta \in (0, 1)$ is the discount factor.

There are at least two justifications for the argument that such comparisons are really assessing the role of strategic interactions, without convoluting it with anything else. First, we shall be formally comparing the outcomes of two economies with the same technology and the same within-period preferences, the only difference being that the first one is governed by two-period bequest motives and the other – by an omniscient planner whose social welfare function is $\sum_{t=0}^{\infty} \delta^t u(c_t)$. Observe that a similar optimization problem can be obtained when we reformulate the model such that individuals do not derive utility directly from their successors' consumption, but from their *utility*. Hence, generations' choices can be embedded in the first generation's optimization problem, ultimately yielding a "dynastic" model with infinite-horizon planning where each generation $t > 0$ maximizes $\sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau)$. To see it formally (from $t > 0$), consider an economy populated by a sequence of generations each represented by a single household with preferences $U(c_t, V_{t+1})$ over its consumption c_t and its immediate descendants' utility V_{t+1}. Since all generations' utility functions are the same, their choices can be embedded in the first generation's optimization problem. The solution to this maximization problem corresponds to a stationary solution of an infinite-horizon dynastic model with stochastic transition in human capital levels: $\max_{\{c_\tau\}} \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau)$, where $\delta \in (0, 1)$ is a discount factor.

The second justification is that the same benchmark will be used to compare the allocation obtained in the bequest economy under strategic interactions, and an allocation chosen by the planner taking care of all generations:

$$\max_{\{c_t\}} \left\{ u(c_0) + \sum_{t=0}^{\infty} \delta^t u(c_t) + \sum_{t=1}^{\infty} \delta^t v(c_t) \right\} = 2 \max_{\{c_t\}} \left\{ u(c_0) + \sum_{t=1}^{\infty} \delta^t u(c_t) \right\},$$

where the first term $u(c_0)$ indicates that the planner takes also care of an initial (old) "−1" generation.

Let us finally comment that now we are *not* aiming at comparing allocations obtained under strategic interactions to the Pareto optimal (or modified Pareto optimal) allocation for *the same* economy. We refer the reader to the works of Lane and Leininger (1986) or Bernheim and Ray (1987) concerning conditions under which an allocation governed by strategic interactions is Pareto (or modified Pareto) optimal.

Keeping this in mind, we consider the first order condition for the Markov stationary optimal policy of the planner:

$$u'(f(h,l(h)))f_2'(h,l(h)) = \delta g_2'(h,1-l(h)) \int_{\mathbf{H}} V(y)\lambda(dy|h), \tag{15}$$

where $V(h)$ is the Bellman's value function defined as

$$V(h) = \max_{\hat{l}\in[0,1]} \left\{ u(f(h,\hat{l})) + \delta \int_{\mathbf{H}} V(y)G(dy;h,1-\hat{l}) \right\}. \tag{16}$$

Standard arguments of dynamic programming (see e.g. Stokey, Lucas, and Prescott (1989)) guarantee that under our assumptions the functional equation (16) has a unique solution V and that the solution corresponds to a function l which solves $V(h) = u(f(h,l(h))) + \delta \int_{\mathbf{H}} V(y)G(dy,h,1-l(h))$. The first order condition (15) guarantees that the marginal utility of consumption of the current generation, acquired thanks to an extra unit of time devoted to work, is exactly equal to the expected marginal cost in terms of utility lost by the next generation because of having marginally less human capital.

Since the optimal setup rules out all strategic aspects of the decision process, the optimal Markov policy for the dynastic optimization economy is (generally) not a MSNE of an economy with strategic interactions,[23] and it is also not a Pareto optimal allocation.

It turns out, however, that equilibrium policies of our basic model with strategic interactions and the optimal policy abstracting from such interactions can be directly compared:

[23] A related class of models frequently encountered in the human capital accumulation literature uses the framework of joy-of-giving altruism. In such models, generations do not derive their utility directly from their successors' consumption, but are instead interested in providing them with the *means* allowing for consumption. In the context of human capital accumulation it means that their utility function is $u(c_t) + v(h_{t+1})$. Hence, the decisions made by the next generation do not matter for the utility of the current generation. Unfortunately, although widely used in the literature, the "joy-of-giving" altruism utility function and hence the whole model is not directly comparable to the ones studied in this paper. Hence, we only briefly discuss the implications of joy-of-giving altruism models in the context of our argument in the Appendix.

Theorem 5 (On comparing equilibria) *Let l_{MSNE} be a MSNE of an economy with strategic interactions with $v(\cdot) = \delta u(\cdot)$, and l_R be the optimal stationary policy of a dynastic economy with utility u. Then $l_{MSNE}(h) > l_R(h)$ for all $h \in (0, \bar{H}]$.*

Proof. Consider two families of functions parameterized by $h \in (0, \bar{H}]$, denoted as S_h,$Z_h : [0,1] \to \mathbb{R}_+$, such that for a given $h \in (0, \bar{H}]$,

$$S_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dy|h)$$

and

$$Z_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_{\mathbf{H}} V(y)\lambda(dy|h),$$

where V is the value function corresponding to the Bellman equation (16).

We would like to show that for any given h, $S'_h(l) > Z'_h(l)$ in their whole domain. To this end, first note that for $h > 0$:

$$u(f(h, l_{MSNE}(h))) \leq \max_{l\in[0,1]} u(f(h,l)) <$$
$$< \max_{l\in[0,1]}\{u(f(h,l)) + \delta g(h, 1-l) \int_{\mathbf{H}} V(y)\lambda(dy|h)\} = V(h). \tag{17}$$

From the above reasoning, it immediately follows that

$$\int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dy|h) < \int_{\mathbf{H}} V(y)\lambda(dy|h) \tag{18}$$

and hence:

$$S'_h(l) = u'(f(h,l))f'_2(h,l) - \delta g'_2(h, 1-l) \int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dy|h) >$$
$$u'(f(h,l))f'_2(h,l) - \delta g'_2(h, 1-l) \int_{\mathbf{H}} V(y)\lambda(dy|h) = Z'_h(l), \tag{19}$$

which completes the first part of the proof.

Now let us impose another function $T : \{1,2\} \times [0,1] \to \mathbb{R}_+$ on top of that, such that $T(1,l) = Z(l)$ and $T(2,l) = S(l)$. From inequality (19) we have that $T'_2(2,l) > T'_2(1,l)$, and thus T has increasing marginal returns with $i = 1, 2$. For $i = 1, 2$, the function $T(i, \cdot)$ defined on the lattice $[0, 1]$ is thus supermodular. Hence, by the theorem due to Amir (1996b) we obtain that $(\forall h \in (0, \overline{H}])\, l_{MSNE}(h) = \arg\max_{l\in[0,1]} T(2,l) > \arg\max_{l\in[0,1]} T(1,l) = l_R(h)$. ■

Theorem 5 asserts that equilibrium human capital investment is *unambiguously lower* in an economy with strategic interactions than in an economy following the optimal policy. The rationale is that with strategic interactions, utility acquired from second period consumption is conditional on the strategy chosen by the subsequent generation, whereas in the optimal policy model it is certain. If such a strategy is Lipschitz, then certainly some part of parents' investment will be wasted.

As mentioned in the Introduction, Bernheim and Ray (1987) have identified, however, another force at work here: since in the strategic model, each generation views the investment made by their children, $(1-l')$, as pure waste, it must invest more to obtain the same effect. As the marginal utility is decreasing, this channel makes the investment decision more productive when strategic interactions are present. However, under our assumptions on the transition technology, the latter channel is shut down.

To see formally how the assumption on transition G closes this second channel observe in the proof of Theorem 5 that Assumption 2 allows us to replace the increasing difference between investment and the marginal utility from the next generation's consumption with the increasing difference between investment and the value (i.e., utility itself). As a result, under our Assumption 2, we obtain monotonicity of the optimal investment choice with continuation value and thus (since the non-strategic model has a longer planning horizon) a direct comparison of both analyzed policies.

Conversely, upon relaxing the conditions imposed on the transition G in Assumption 2, the ordering of both policies ceases to be unambiguous. This can be seen in the following remark.

Remark 1 *Let Assumption 1 be satisfied and modify Assumption 2 letting*

$$G(\cdot|h, 1-l) = (1 - g(h, 1-l))\lambda_0(\cdot) + g(h, 1-l)\lambda(\cdot|h),$$

with $\lambda(\cdot|x)$ stochastically dominating $\lambda_0(\cdot)$ for any $x \in I$. We leave the rest of the Assumption 2 unaltered. Hence the only difference lies in replacing the Dirac's delta distribution δ_0 with λ_0. The first order condition of an optimal (unique and interior) investment $\hat{l}$ requires equating

$$u'(f(h,\hat{l}))f_2'(h,\hat{l}) - \\ \delta g_2'(h, 1-\hat{l}) \left[\int_{\mathbf{H}} u(f(y,l(y)))\lambda(dy|h) - \int_{\mathbf{H}} u(f(y,l(y)))\lambda_0(dy) \right]$$

to zero. Observing this we can only conclude that the objective function has increasing differences between investment and the marginal value (the term *in the square brackets). As a result, the above described effects of the second force come into play and obtaining the conclusions of Theorem 5 nor MSNE uniqueness cannot be expected.*[24]

Finally let us mention that, generally, one also should not expect the optimal (Markov) dynastic policy to lead to a Pareto dominating allocation of our bequest economy.

Remark 2 *Consider a modification of our setup where each subsequent generation can fully commit to some stationary policy l_{PO} (or equivalently, that the current generation is allowed to choose the policy for the subsequent one). In such a case we obtain ($\forall h \in \mathbf{H}$):*

$$u(f(h, l_{PO}(h))) + g(h, 1 - l_{PO}(h))\delta \int_{\mathbf{H}} u(f(y, l_{PO}(y)))\lambda(dx|h) :=$$
$$\max_{l \in L} \left\{ u(f(h, l(h))) + g(h, 1 - l(h))\delta \int_{\mathbf{H}} u(f(y, l(y)))\lambda(dx|h) \right\} \geq$$
$$u(f(h, l_{MSNE}(h))) + g(h, 1 - l_{MSNE}(h))\delta \int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dx|h) \geq$$
$$u(f(h, l_{PO}(h))) + g(h, 1 - l_{PO}(h))\delta \int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dx|h),$$

where the last inequality follows from the definition of MSNE. But this immediately implies that ($\forall h \in \mathbf{H}$):

$$\int_{\mathbf{H}} u(f(y, l_{PO}(y)))\lambda(dx|h) \geq \int_{\mathbf{H}} u(f(y, l_{MSNE}(y)))\lambda(dx|h)$$

which gives: $l_{PO} \not\leq l_{MSNE}$. Hence, the (Markov stationary) Pareto optimal policy cannot be (pointwise) lower than the MSNE. But as it has been already shown in Theorem 5, $l_R \leq l_{MSNE}$, with strict inequality for interior arguments. Hence, l_R shall not be expected to yield a (Markov stationary) Pareto optimal allocation in our economy.

This comparison sheds new light on the impact of strategic interactions on human capital accumulation policies. On the one hand, strategic motives can promote human capital accumulation in comparison to the (Markov stationary) Pareto optimal allocation. But on the other hand, it is still insufficient as compared to the optimal dynastic allocation.

[24] In a different setup this problem was also noticed by Curtat (1996) who only managed to show monotonicity of the choice variable with the marginal value and not the value itself.

Let us finally discuss how the above observations relate to some well known results from the literature preoccupied with the macroeconomic effects of human capital accumulation. First, the literature following the seminal works of Lucas (1988) as well as Becker, Murphy, and Tamura (1990), analyzing optimal (social planner's) human capital accumulation paths identifies, depending on the assumed functional specification, a strong impact of human capital investment either on long-run growth rates of key macroeconomic variables in the model (output, capital per worker, etc.), or on their levels in the steady state. Once strategic interactions are accounted for and our stochastic technology assumption is accepted, however, then following Theorem 5, the impact of human capital investment on the development paths is unambiguously diminished. As is clear from Remark 1, this finding does not necessarily have to carry forward to different technology specifications, though.

Second, similar forces are present in the context of the OLG economies without strategic interactions (see Becker and Tomes (1979) or more recently Aiyagari, Greenwood, and Seshadri (2002)). The comparison is more subtle here, though. Still, when the marginal utility effect is canceled using our technology assumption, the identified role of human capital accumulation is higher than under strategic interactions. On the other hand, strategic interactions might promote human capital accumulation as compared to the (Markov stationary) Pareto-optimal solution. Thus, in particular, when strategic interactions are present then policies focused on welfare maximization may actually decrease incentives for human capital accumulation instead of increasing them.

5 Computation of the MSNE

The objective of the current section is to compute numerically the equilibrium policy l^* for an economy with strategic interactions and to analyze the equilibrium dynamics of human capital accumulation given certain functional assumptions on u, v, f and G. To facilitate economic interpretation, we will concentrate on the case of iso-elastic utility and Cobb-Douglas production functions. We will then benchmark these numerical results against the corresponding one obtained within the non-strategic (dynastic) model discussed in the previous section. Our workhorse example which will be used in all our subsequent numerical exercises is the following.

Example 1 *Let* $U(c_1, c_2) = c_1^{\gamma_1} + \delta c_2^{\gamma_2}$, $f(h,l) = h^{\alpha_1} l^{\beta_1}$ *and* $g(h, 1-l) = \frac{1}{H^{\alpha_2}} h^{\alpha_2}(1-l)^{\beta_2}$, *with* $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in (0,1)$ *and* $\delta \in (0,1]$. *The function*

ξ_h is then given by:

$$\xi_h(l) = \frac{\beta_1\gamma_1}{\beta_2}\bar{H}^{\alpha_2}h^{\alpha_1\gamma_1-\alpha_2}\frac{l^{\beta_1\gamma_1-1}}{(1-l)^{\beta_2-1}}. \tag{20}$$

We assume that $1 > \beta_1(\gamma_1+\gamma_2)$, so that there exists a unique MSNE in L. To simplify further computations, we assume additionally that $\beta_2 = \beta_1\gamma_1$.

The last equality assumption has been made for the sole purpose of analytical tractability: it is only when $\beta_2 = \beta_1\gamma_1$ that the ξ_h mapping is invertible in a closed form. Relaxing it increases the computational burden significantly but does not overturn any of our results. If $\beta_2 = \beta_1\gamma_1$, we obtain:

$$\xi_h^{-1}(\bar{l}) = \frac{\bar{l}^{\frac{1}{\beta_2-1}}h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}}\bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}^{\frac{1}{\beta_2-1}}h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}}\bar{H}^{\frac{\alpha_2}{1-\beta_2}}}. \tag{21}$$

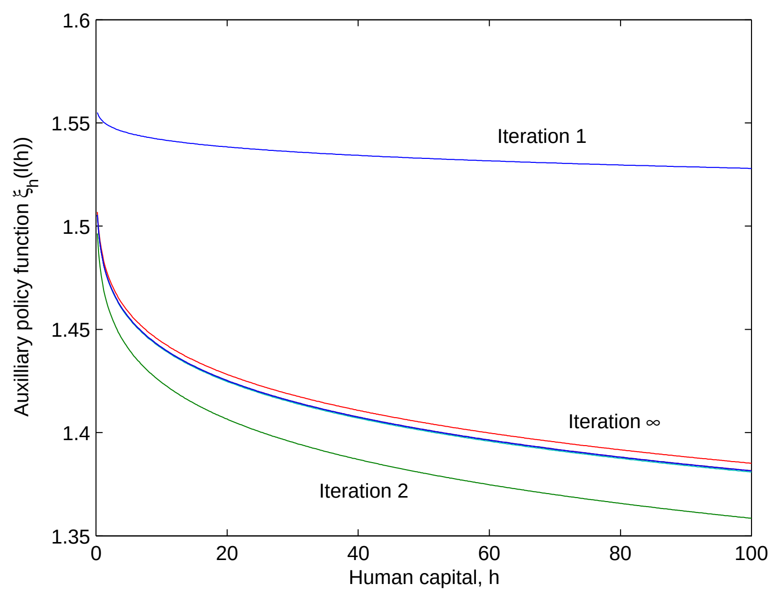

Figure 1: Convergence to the fixed point of operator B. The fixed point is the auxilliary policy function $\bar{l}(\cdot) = \xi_h(l(\cdot))$. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma_1 = .6; \gamma_2 = .5; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

Assuming furthermore that the distribution λ is uniform on $\mathbf{H}$, the MSNE policy can be found as $l^*(y) = \xi_h^{-1}(\bar{l}(y))$ where $\bar{l}$ is found as the fixed point of the operator B given by

$$B\bar{l}(h) = \frac{\delta}{\bar{H}} \int_0^{\bar{H}} y^{\alpha_1\gamma_2} \left(\frac{\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}} \right)^{\beta_1\gamma_2} dy. \tag{22}$$

As stated in Theorem 3, repeated iteration of B guarantees convergence to the MSNE (see Figure 1).[25]

5.1 Monotonicity

Based on the parametric assumptions spelled out in Example 1, we obtain the following additional result.

Theorem 6 *The MSNE policy l^* is monotone. It is everywhere decreasing iff $\alpha_1\gamma_1 < \alpha_2$, everywhere increasing iff $\alpha_1\gamma_1 > \alpha_2$, and constant iff $\alpha_1\gamma_1 = \alpha_2$.*

Proof. In equilibrium, $\bar{l}(h) = \xi_h(l(h))$ can by defined as the right-hand side of (22).

We will now differentiate $l(h) = \xi_h^{-1}(\bar{l}(h))$ with respect to h. Observe that it is justified since ξ_h^{-1} is differentiable while from equations (21) and (22) we also have that functions η_z (where, for given $z \in [0,\infty)$, $\eta_z(h) := \xi_h^{-1}(z)$) and $\bar{l}$ are differentiable with respect to h on $(0,\bar{H})$. It is obtained that:

$$\begin{aligned} \frac{dl(h)}{dh} &= \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial \bar{l}(h)} \frac{\partial \bar{l}(h)}{\partial h} + \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial h} = \\ &= \frac{1}{(1+\Xi(h))^2} \left(\frac{l(h)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_1}-1}}{1-\beta_2} \right) (\alpha_1\gamma_1 - \alpha_2) \times \\ &\times \left(1 - \frac{\frac{\beta_1\gamma_2}{1-\beta_2} \frac{\delta}{\bar{H}} \int_0^{\bar{H}} y^{\alpha_1\gamma_2} \left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2} \frac{1}{1+\Xi(y)} dy}{\frac{\delta}{\bar{H}} \int_0^{\bar{H}} y^{\alpha_1\gamma_2} \left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2} dy} \right), \end{aligned} \tag{23}$$

with $\Xi(y) \equiv \bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}$. Since $\beta_1\gamma_1 = \beta_2$, and by assumption, $1 > \beta_1(\gamma_1+\gamma_2)$, it follows that $\frac{\beta_1\gamma_2}{1-\beta_2} < 1$ and thus the ratio of two integrals in the last parenthesis is smaller than one, we find the expression in the last

[25] To calculate the equilibrium policies of any of the three models numerically, we have used the discretization method discussed by Judd (1998). Matlab codes used to compute the numerical results quoted throughout the paper as well as to produce Table 1 are available from the authors upon request.

parenthesis to be positive. In conclusion, $\frac{dl(h)}{dh} > 0$ and thus $l(h)$ is increasing in its domain iff $\alpha_1\gamma_1 > \alpha_2$, $\frac{dl(h)}{dh} < 0$ and thus $l(h)$ is decreasing in its domain iff $\alpha_1\gamma_1 < \alpha_2$, and $l(h)$ is constant iff $\alpha_1\gamma_1 = \alpha_2$. ■

Having specified the three cases in which the optimal labor supply policy is increasing, decreasing, or constant in the human capital endowment, let us discuss the empirical plausibility of each of the cases. The results are somewhat reassuring here. Namely, the case where $\alpha_2 > \alpha_1\gamma_1$, guaranteed to hold e.g. if $\alpha_1 \approx \alpha_2$ (i.e. if the shares of human capital in production of the consumption good and of human capital, respectively, are approximately equal), turns out to be significantly more plausible empirically than any of the other cases.[26] This case, implying that labor supply decreases (and human capital accumulation increases) with the stock of human capital, is thus going to be our benchmark case.

5.2 Dynamics

The dynamic properties of the economy are as follows. If all generations play the MSNE strategy, then in the limit as $t \to \infty$, average human capital tends to $\bar{h}$ solving the implicit equation:

$$\bar{h} = 2^{\frac{1}{\alpha_2 - 1}} \bar{H}(1 - l(\bar{h}))^{\frac{\beta_2}{1-\alpha_2}}. \tag{24}$$

This result has been confirmed numerically.[27]

The distribution of human capital will also evolve over time as consecutive generations will invest different fractions of time to work and education. By definition, however, the stationary distribution of human capital over $\mathbf{H}$ will have a constant density $\frac{1}{\bar{H}} g(\bar{h}, 1-l(\bar{h})) = \frac{1}{\bar{H}^{\alpha_2+1}} \bar{h}^{\alpha_2}(1-l(\bar{h}))^{\beta_2}$ and a probability mass $1 - g(\bar{h}, 1 - l(\bar{h})) = 1 - \frac{1}{\bar{H}^{\alpha_2}} \bar{h}^{\alpha_2}(1 - l(\bar{h}))^{\beta_2}$ concentrated at zero.

5.3 Role of the transition distribution λ

The MSNE policy $l^*(h)$ depends on the underlying transition distribution λ but this impact turns out to be rather modest. As a robustness check of our earlier numerical results, we have substituted the uniform distribution λ with two alternatives:

[26] Becker and Tomes (1986), Lochner (2008), among numerous others, discuss the empirical evidence that the educational effort and children's school attainments are unambiguously positively related to the parental human capital level.

[27] The results are available from the authors upon request.

- a triangular distribution with density

$$\varphi(h) = \begin{cases} \frac{4}{\bar{H}^2} h, & h \in (0, \frac{\bar{H}}{2}), \\ \frac{4}{\bar{H}} - \frac{4}{\bar{H}^2} h, & h \in (\frac{\bar{H}}{2}, \bar{H}); \end{cases} \tag{25}$$

- a one-point distribution[28] with all probability mass concentrated in $\bar{H}/2$: $P(h = \bar{H}/2) = 1$.

As we have confirmed numerically, the greatest labor supply is obtained when the distribution is uniform, and the least labor is supplied when the probability mass is concentrated at the mean human capital level. The policy for the triangular distribution falls in between these two extreme cases (uniform and one-point). The interpretation of this result is straightforward: the more risk remains that human capital of the successive generation would be low despite substantial investment, the less willing the decision maker would be to invest in human capital. Since individuals are risk-averse in this model, additional risk lowers education effort and increases labor supply which guarantees a certain payoff.

6 Numerical assessment of the role of strategic interactions

Let us now compare the equilibrium dynamics obtained in the numerical example presented above to the ones generated by the optimal-policy, dynastic model of Section 3.

Example 2 *Let $u(c) = c^\gamma$, $f(h,l) = h^{\alpha_3} l^{\beta_3}$, $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_4}} h^{\alpha_4}(1-l)^{\beta_4}$. Let the decision maker born at t maximize $u(c_t) + \delta u(c_{t+1})$. From (15), we obtain the first order condition for the optimal policy function $l(h)$. It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_3\gamma}}{(1-l)^{1-\beta_4}} = \frac{\bar{H}^{\alpha_4}}{\delta I} h^{\alpha_3\gamma - \alpha_4}, \tag{26}$$

where $I \equiv \int_{\mathbf{H}} V(y)\lambda(dy|h)$ is a predetermined constant.

[28]Note that even when λ is one-point, there remains a probability that the next generation's human capital will be zero. Hence, the assumptions and interpretations of the economy with strategic interactions studied in Section 3 are still satisfied.

Using the implicit function theorem, it can again be easily shown that $l(h)$ is everywhere decreasing whenever $\alpha_4 > \alpha_3\gamma$ and everywhere increasing whenever $\alpha_4 < \alpha_3\gamma$. In the special case where $\alpha_3\gamma = \alpha_4$, (26) implies that $l(h)$ is constant, independent of h. This finding parallels Theorem 6 precisely: there are absolutely no qualitative differences in the optimal policy behavior between the strategic and the non-strategic model. *Quantitative* differences are substantial, though, as we shall see shortly.

Moreover, just like in the strategic case, the first order condition (26) can be solved for $l^*(h)$ explicitly in the special case $\beta_3\gamma = \beta_4$. In such case,

$$l^*(h) = \frac{\left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}{1 + \left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}. \tag{27}$$

What remains to be derived is the constant $I = \int_{\mathbf{H}} V(y)\lambda(dy|h)$. It can be found as an implicit solution of the following equation:

$$I = \frac{\int_{\mathbf{H}} y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma}\lambda(dy|h)}{1 - \delta\int_{\mathbf{H}} \left(\frac{y}{\bar{H}}\right)^{\alpha_4}(1-l^*(y))^{\beta_4}\lambda(dy|h)}, \tag{28}$$

with l^* defined as in (27) and thus containing I. The approximate solution to this equation can be easily computed numerically. Please note that knowing I, we can also obtain an explicit formula for the value function:

$$\begin{aligned} V(h) &= h^{\alpha_3\gamma} l^*(h)^{\beta_3\gamma} + \\ &+ \left(\frac{\delta\int_{\mathbf{H}} y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma}\lambda(dy|h)}{1 - \delta\int_{\mathbf{H}} \left(\frac{y}{\bar{H}}\right)^{\alpha_4}(1-l^*(y))^{\beta_4}\lambda(dy|h)}\right)\left(\frac{h}{\bar{H}}\right)^{\alpha_4}(1-l^*(h))^{\beta_4}. \end{aligned} \tag{29}$$

The direct computation of I would not have been possible if not for the introduction of stochastic transition in human capital levels. Thanks to that step, the infinite series expansion of $V(h)$ can be computed as a simple geometric series which has a closed-form sum. It also enables us to use the law of iterated expectations to convert an n-tuple integral into a product of n simple integrals.

We are now in the position to compare the equilibrium labor supply policy function derived from the model with strategic intergenerational interactions with the alternative non-strategic scenario. To attain direct comparability of both setups, we must assure $\gamma = \gamma_1 = \gamma_2$ – in the dynastic model, the shape parameters of utility functions u and v must be equal. We shall also fix our other parameters at equal levels, $\alpha_1 = \alpha_3, \beta_1 = \beta_3, \alpha_2 = \alpha_4, \beta_2 = \beta_4$.

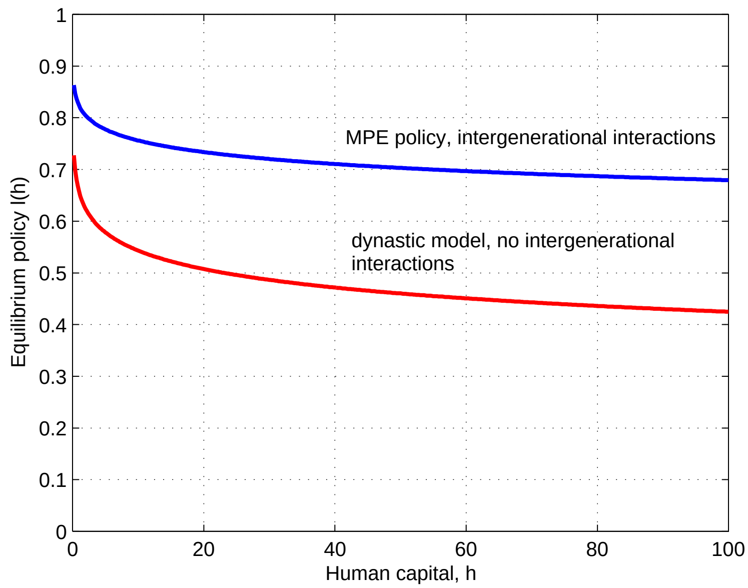

Figure 2: The difference between equilibrium policy functions l^* in the time-consistent policy and the optimal but time-inconsistent policy. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma = .6; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

The results are apparent in Figure 2. Significantly more labor is supplied (and thus, less human capital is accumulated) in the case of the MSNE policy in our baseline model with strategic interactions than in the optimal policy model which does not include such interactions. This directly confirms Theorem 5, providing a quantitative edge to that result.

Furthermore, even though there is a marked difference in the levels of human capital investment between the models, the *shapes* of the policy functions are remarkably similar. With iso-elastic utility and Cobb-Douglas production functions, and under our benchmark parametrization, labor supply functions $l^*(h)$ always decrease with h, indicating that human capital and education effort are positively related, in line with empirical observations (e.g. Becker and Tomes (1986)).

The uniform ordering of labor supply functions obtained from the models under consideration (the policy curves such as the ones depicted in Figure 2 never intersect) offers an intuitive and convincing explanation. In simple words: the more directly does child's human capital enter parent's utility function, the more willing will she be to invest in it.

In order to obtain a rough approximation of the magnitude of difference between equilibrium policies in the two considered models, we have carried out a numerical sensitivity analysis exercise: we have manipulated the parameters of the models under study and compared the resultant equilibrium policy functions $l^*(h)$. For each parameter configuration, we calculated two measures of distance between the functions. Since by Theorem 5, we know that $l_{MSNE} > l_R$ (where $MSNE$ stands for the Markov stationary equilibrium of our baseline strategic model and R denotes the model featuring dynastic optimization), our proposed distance measures have been defined as follows:

1. The area between l_{MSNE} and l_R: $D_1 = \int_{\mathbf{H}} (l_{MSNE}(h) - l_R(h))dh > 0$.

2. The minimum distance between l_{MSNE} and l_R:
   $D_2 = \inf_{h \in \mathbf{H}} |l_{MSNE}(h) - l_R(h)| > 0$.

One crucial finding which facilitates the subsequent analysis and justifies the above definitions is that the policy functions never intersect. Hence, the first measure captures the average overestimation of the human capital accumulation policy function in the non-strategic case, while the second measure – its minimum overestimation.

For simplicity of computations, we have maintained the assumption $\beta_2 = \beta_1\gamma_1$; for comparability of our results, we have also retained the condition $\gamma_1 = \gamma_2$. This limits the scope of this sensitivity analysis exercise markedly, but our intention was not to search through the whole parameter space anyway. Even under these restrictions, we find both important departures from the baseline parametrization illustrated in Figure 2 and potentially large distances between the two policy functions.

First of all, the numerical exercise confirms that equilibrium policy functions l^* from different models indeed never intersect ($D_2 > 0$). Furthermore, the numerical results on the ordering of policy functions obtained from the strategic model and from the optimal policy ($l_{MSNE} > l_R$) are obviously consistent with implications of Theorem 5. The distance between these two policy functions can vary considerably, though: under some parametrizations (such as the baseline parametrization), it is large, while under others, in particular those involving radically low δ's, it may even be close to zero.

The results of our sensitivity analysis exercise have been summarized in Table 1. The baseline parametrization is: $\alpha_1 = 0.3; \beta_1 = 0.7; \alpha_2 = 0.3; \gamma = 0.6; \beta_2 = \beta_1\gamma_1 = 0.42; \bar{H} = 100; \delta = 0.9$, just like in the previous section. Unless indicated otherwise, these parameter choices are maintained throughout the table.

Table 1: Sensitivity analysis results.

Case	D_1	D_2
Close to Baseline		
Baseline	23.7462	0.1353
$\beta_1 = 0.5$	25.9257	0.1884
$\alpha_1 = 0.6$	24.2728	0.2336
$\alpha_1 = \alpha_2 = 0.6$	13.0828	0.0215
$\alpha_2 = 0.6$	13.2903	0.0043
$\beta_1 = 0.6; \gamma = 0.8$	22.3790	0.1617
$l_{MSNE} \approx l_R$: low δ		
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.6$	4.0759	0.0044
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.3$	0.4628	0.0004
$\delta = 0.6$	7.7896	0.0296
$\beta_1 = 0.6.\gamma = 0.8; \delta = 0.6$	6.4581	0.0361
$\delta = 0.3$	0.9392	0.0026
$\beta_1 = 0.6; \gamma = 0.8; \delta = 0.3$	0.5958	0.0027

Source: own computations.

7 Conclusion

The purpose of the current paper has been to accomplish the two principal tasks: (i) to show how a Markov stationary equilibrium (MSNE) policy function can be computed in a decentralized model with fully-specified intergenerational interactions in human capital accumulation, within an otherwise standard discrete-time framework; (ii) to compare the outcomes of the strategic model with a benchmark model which neglects intergenerational interactions.

To this end, we have first defined an appropriate price system decentralizing the MSNE policy. To the best of our knowledge, we are the first to provide a formal definition of prices for MSNE allocations in stochastic games. Using this definition, we have computed the prices of Arrow securities insuring future human capital levels, thus demonstrating how our approach allows one to price the relevant commitment devices.

Our second contribution to the literature has been to prove analytically that, when compared to a model with dynastic optimization, under our assumptions, the strategic model predicts *unambiguously lower* equilibrium investment in human capital accumulation. On the other hand, as we have also shown, strategic motives can nevertheless promote human capital accumulation in comparison to the (Markov stationary) Pareto optimal allocation; but this is still insufficient as compared to the optimal dynastic allocation.

Third, we have put forward a constructive algorithm for computing MSNE policies in models of intergenerational altruism such as the one discussed here. This can be viewed as a further significant step towards modeling strategic linkages across generations.

Fourth, using this novel technique, we have characterized the conditions under which the MSNE policy exists and is unique, proven its monotonicity, and also presented a workhorse example for which most calculations could be done analytically, and for which the numerical convergence of our iterative procedure to the MSNE is quick and easy. Consequently, applying this numerical procedure in a series of sensitivity analysis exercises, we have assessed the magnitude of the possible overestimation of the role of human capital resulting from strategic interactions, both in terms of the expected steady-state (long-run) human capital level and the minimum distance between both scenarios along the equilibrium path.

Now, it is important to mention that our results are sensitive to the assumptions we make. Once our assumptions on transition probability are not satisfied not only equilibrium uniqueness and related computation method may fail, but also equilibrium comparison result will not hold in general.

The research presented in the current paper can be extended in various directions, but we feel that the foremost thing that needs to be done is a generalization of our constructive algorithm for computing MSNE policies into higher dimensions and onto multi-period economies. This is enforced by the fact that most economic models featuring intergenerational altruism are set up with multiple choice/state variables and multi-period life time economy.

We feel that this step is still necessary in order to bring models with strategic interactions in human capital accumulation to the level of sophistication which is now common with models lacking such strategic interactions.

A Appendix: A model of joy-of-giving altruism

Let us now proceed to one different example of a model which could be compared against our benchmark model with intergenerational interactions in human capital accumulation: a model with joy-of-giving altruism.

A model with joy-of-giving altruism (and, to guarantee direct comparability, with a stochastic transition in human capital levels) can be generally specified as:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_{\mathbf{H}} v(y)G(dy;h,1-\hat{l}). \tag{30}$$

The crucial difference between this model and the main model of the current paper consists in the fact that here, parents' utility depends directly on their children's *human capital* and not on their *consumption* ($v(h_{t+1})$ instead of $v(c_{t+1})$).

Concentrating on Markovian policies, the first order condition for optimal labor supply $l(h)$ is given by:

$$u'(f(h,l(h)))f_2'(h,l(h)) = g_2'(h,1-l(h)) \int_H v(y)\lambda(dy|h), \tag{31}$$

guaranteeing that the marginal utility of consumption acquired thanks to an extra unit of time devoted to work is exactly equal to the expected marginal cost in terms of lost human capital of the next generation.

Example 3 *Let* $u(c) = c^{\gamma_5}$, $v(h') = (h')^{\gamma_6}$, $f(h,l) = h^{\alpha_5} l^{\beta_5}$, $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_6}} h^{\alpha_6}(1-l)^{\beta_6}$. *From* (31), *we obtain the first order condition for the optimal policy* $l(h)$. *It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_5\gamma_5}}{(1-l)^{1-\beta_6}} = \frac{\beta_5\gamma_5}{\delta\beta_6}(1+\gamma_6)\bar{H}^{\alpha_6-\gamma_6} h^{\alpha_5\gamma_5-\alpha_6}. \tag{32}$$

Using the implicit function theorem, it is straightforward to show that $l(h)$ is everywhere decreasing whenever $\alpha_6 > \alpha_5\gamma_5$ and everywhere increasing whenever $\alpha_6 < \alpha_5\gamma_5$. In the special case where $\alpha_5\gamma_5 = \alpha_6$, (32) implies that $l(h)$ is constant, independent of h. This finding is crucial here because it is an exact analogue to Theorem 6 and an equivalent theorem which holds for the dynastic model: whenever the MSNE labor supply policy of the model with strategic interactions is decreasing/increasing, it is also decreasing/increasing in the model with "joy-of-giving" altruism.

Just like in Example 1, the above equation (32) can be solved for $l^*(h)$ explicitly in the special case $\beta_5\gamma_5 = \beta_6$. In such case,

$$l^*(h) = \frac{\left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}{1+\left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}. \tag{33}$$

For the highest available level of comparability, one has to impose $\gamma_6 = \beta_1\gamma_2$ in order to equalize the elasticities of h' in both utility functions. The functions themselves remain different, though.

B Appendix: an auxiliary theorem

Definition 2 *Let E be a real Banach space and $P \subseteq E$ be a nonempty, closed, convex set. Then:*

- *P is called a cone if it satisfies two conditions: (i) $x \in P, \epsilon > 0 \Rightarrow \epsilon x \in P$ and (ii) $x \in P, -x \in P \Rightarrow x = \theta$, where θ is a zero element of P,*
- *suppose P is a cone in E and $P^\circ \neq \emptyset$, where P° denotes the set of interior points of P, we say that P is a solid cone,*
- *every cone P in E defines an order relation $\leq$ in E as follows:*

$$x \leq y \text{ if } y - x \in P,$$

- *a cone P is said to be normal if there exists a constant $N > 0$ such that:*

$$(\forall x, y \in P) \quad \underline{\theta} \leq x \leq y \Rightarrow \|x\| \leq N\|y\|.$$

Theorem 7 (Guo, Cho, and Zhu (2004)) *Let P be a normal solid cone in a real Banach space with partial ordering $\leq$ and $B : P \to P$ be a decreasing operator (i.e. if $l_1 < l_2 \in P$ then $Bl_2 \leq Bl_1$) satisfying:*

$$(\exists r, 0 < r < 1)(\forall l \in P^\circ), (\forall t, 0 < t < 1) \quad t^r B(tl) \leq Bl, \tag{34}$$

then B has a unique fixed point in P° and the following holds:

$$(\forall l_0 \in P^\circ) \quad \lim_{n\to\infty} \|l_n - l^*\| \to 0, \tag{35}$$

where $(\forall n \geq 1)\, l_n = B(l_{n-1})$.

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