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A new class of production functions and an argument against purely labor-augmenting technical change

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This paper derives the macro-level production function from idea-based microfoundations. Labor-augmenting and capital-augmenting developments are assumed to be Pareto-distributed and mutually dependent. Using the Clayton copula family to capture this dependence, a new "Clayton–Pareto" class of production functions is derived that nests both the Cobb–Douglas and the constant elasticity of substitution. In the most general case, technical change is not purely labor-augmenting over the long run, but it augments both capital and labor. Under certain parametrizations, the derived elasticity of substitution between capital and labor exceeds unity and, therefore, gives rise to long-run endogenous growth.

Key words production function, Pareto distribution, Clayton copula, technology frontier, microfoundation

JEL classification E23, O30, O40

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1 Introduction

This paper accepts the view that the production function, commonly assumed by macroeconomists to be a primitive, is in fact only a reduced form which should be derived from microfoundations. It is also acknowledged that such an economy-wide production function has to be viewed as an assembly of a multiplicity of production techniques, particular methods of producing the final good.

By means of a substantial generalization of Jones's (2005) setup, we derive from idea-based microfoundations a new class of production functions, baptized herein *Clayton–Pareto functions*. It is indeed a large class: it nests both Cobb–Douglas and constant elasticity of substitution (CES) functions.[1] We think of this as a particularly desirable property

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[1] The parameters of our microfounded production function are interrelated. Consequently, it makes sense to indicate that "the Clayton–Pareto family nests both Cobb–Douglas and the CES": saying that it nests the CES only

because both Cobb–Douglas and CES functions are argued to be plausible forms for the real-world economy-wide production function: an in-depth discussion and an empirical investigation of this issue can be found in Duffy and Papageorgiou (2000). When one carries out econometric tests to check whether it is the Cobb–Douglas or the CES function that is more useful for explaining observed data, why couldn't one check if there exists an even more general class of functions that provides more degrees of freedom and thus provides an even better fit? The Clayton–Pareto class could be useful in this regard.

To our knowledge, both the derivation of a class of production functions that nests the Cobb–Douglas and the CES functions and the provision of an idea-based microfoundation for the CES production function are novel to the published literature.

Taking the results in Jones (2005) as our benchmark, we show that when his assumptions are modified, even very slightly, the "Cobb–Douglas global production function and purely labor-augmenting technical change" result is overturned. We also build a formal link between Jones (2005) and Caselli and Coleman (2006): we present an idea-based explanation for (a generalization of) the shape of the technology frontier Caselli and Coleman postulate. Therefore, in a sense, we bring their model to a common denominator with Jones's. These two models have been empirically justified on diverse bases,[2] but they are characterized by a profound mathematical unity.

As a by-product of our analytical method, we also enrich the literature by providing alternative proofs for Jones's results and finding explicit solutions for the firms' technology choices wherever it is possible.

Unfortunately, we are unable to provide closed-form formulae for all Clayton–Pareto functions. We carry out a detailed study of analytically solvable special cases instead.

To discuss the long-run implications of the new class of production functions, we embed our microfounded framework in the standard neoclassical (Solow 1956) growth model. We find that the results derived by Jones (2005) and Acemoglu (2003), that technical change is purely labor-augmenting in the long run, do not hold in general, but do hold in certain important cases, among which independence of the capital-augmenting and labor-augmenting developments (the case studied by Jones) is probably the most prominent. In general, *technical change augments both capital and labor,* even in the long run. One simple intuition for this result is that if productivities of production factors are correlated, then it is impossible to achieve higher and higher productivity of labor without altering the productivity of capital as well. We shall study this issue in greater detail in Section 4. In Section 4 we shall also discuss the possibility of endogenous growth that arises if the derived production function exhibits elasticity of substitution greater than one in the long run.[3]

It is beyond all doubt that the problems discussed within the present paper are important for growth theory. Indeed, we are considering such fundamental questions as: "What is

(which itself nests the Cobb–Douglas as a limiting case) would not reflect the result properly. At the intersection of conditions guaranteeing the CES, and the Cobb–Douglas result, we would obtain a Cobb–Douglas function with exponents (0.5, 0.5) only.

[2] The idea that countries with different factor endowments use different production technologies has been studied by Atkinson and Stiglitz (1969) and Basu and Weil (1998), among others.

[3] Such a possibility has already been noticed in the seminal work of Solow (1956). There exists substantial literature that deals with these issues, including de La Grandville (1989), Jones and Manuelli (1990), Yuhn (1991), Klump and de La Grandville (2000) and Palivos and Karagiannis (2004).

the true shape of the economy-wide production function?", "What direction of technical change should we expect in the long run?", and "What drives the elasticity of substitution between production factors?". We manage to obtain new insights here because we derive what most economists assume.[4]

The remainder of the article is structured as follows. In Section 2, we generalize the setup put forward by Jones (2005) by allowing for dependence between capital-augmenting and labor-augmenting developments and we derive the class of Clayton–Pareto functions. In Section 3, we discuss the cases of particular interest and describe the determinants of the elasticity of substitution. In Section 4, we determine the long-run direction of technical change and check the conditions for endogenous growth. Section 5 concludes.

2 The microfounded framework

Our microfounded framework includes a representative profit-maximizing firm. In order to produce, the firm picks a single production technique (henceforth a local production function (LPF)) from the range of available techniques. Intuitively, one could associate each LPF with a "recipe," a list of instructions to follow. In line with this intuition, LPF should be rigid and not allow for much substitutability between the factors of production.[5]

The second assumption is that factors can be utilized, according to a given LPF, with certain efficiency levels only. Following Jones (2005), we identify the notion of an *idea* with these efficiency levels (unit factor productivities). Because we consider two factors of production only: capital K and labor L, an idea is consequently a pair (a, b), where b and a are unit productivities of capital and labor, respectively.[6] Technical change is then identified with the sequential arrival of new, better and better, ideas. The global production function (GPF) is the convex hull of LPF; that is, such an assembly of LPFs that for each K and L, productivities b and a are chosen optimally by the firm whose choice is constrained within the set of available ideas. The *technology frontier* is a subset of this set which contains only the ideas that could possibly be used by the profit-maximizing firm.

Following Jones (2005), we assume that each dimension of an idea, be it a or b, is randomly drawn from a Pareto distribution.[7] However, when a new idea arrives, it is already a pair (a, b) chosen from some joint bivariate distribution. The individual Pareto distributions of factor productivities serve only as marginal distributions here.

[4] Earlier literature that deals with closely related issues includes Houthakker (1955–56), Uzawa (1961), Kortum (1997) and Acemoglu (2003). Moreover, endogenous technology choice has common features with the assignment problem with heterogenous workers and tasks, analyzed by Sattinger (1975), and vintage capital theory where technological improvements are embodied in consecutive vintages of machines. This strand of literature dates back to Solow (1960), as well as Solow, Tobin, von Weizsäcker, and Yaari (1966).

[5] An extreme but probably the most intuitive example is the Leontief LPF (fixed coefficients), where the factors have to be used up in fixed proportions. More flexible functional forms are plausible as well, as we shall see shortly.

[6] We reverse the order of a and b in order to stick to Jones's (2005) notation where possible.

[7] Empirical evidence for the prevalence of Pareto distributions in scientific productivity dates back to Lotka (1926). Theoretical literature linking Pareto distributions to exponential growth includes Kortum (1997) and Gabaix (1999).

As opposed to Jones, we allow for dependence between the marginal idea distributions. On the positive side, this poses a new problem: we do not have any empirical evidence on the pattern of dependence. On the normative side, however, the Clayton family of copulas[8] can be used to show that marginal Pareto distributions imply neither a Cobb–Douglas global production function, nor purely labor-augmenting technical change in the long run. On the contrary, they produce a wide variety of possibilities, the empirically relevant CES function being among them.

Our derivation procedure of the GPF from the LPFs and the technology frontier can be decomposed into three steps:

1 Derive the technology frontier from the joint distribution of ideas. This is done in Proposition 2.
2 Find the optimal unit factor productivities a^* and b^* (i.e. pick the optimal LPF) given the available technology level and the associated technology frontier as well as stocks of capital K and labor L. This task is accomplished in Proposition 3.
3 Build the convex hull of LPFs; that is, insert the optimal pair of technologies (a^*, b^*) to the LPF, separately for each pair of endowments (K, L). The convex hull of LPFs is the GPF, as in Proposition 4.

2.1 Step 1. The technology frontier

It is assumed that when new ideas arrive, their levels are stochastic and drawn from Pareto distributions.

Assumption 1 *Unit factor productivities $\tilde{a}$ and $\tilde{b}$ are Pareto-distributed:*

$$P(\tilde{a} > a) = \left(\frac{\gamma_a}{a}\right)^{\alpha}, \qquad a \geq \gamma_a > 0, \qquad \alpha > 0, \tag{1}$$

$$P(\tilde{b} > b) = \left(\frac{\gamma_b}{b}\right)^{\beta}, \qquad b \geq \gamma_b > 0, \qquad \beta > 0. \tag{2}$$

We allow $\tilde{a}$ and $\tilde{b}$ to be mutually dependent. A multiplicity of joint (bivariate) ideas distributions, with (1) and (2) as marginal distributions, is produced using the Clayton family of copulas.

Assumption 2 *Dependence between the unit factor productivities $\tilde{a}$ and $\tilde{b}$ is represented by the Clayton copula, specified in* (3).

All members of the Clayton family of copulas are characterized by the following formula (Nelsen, 1999):

$$C(u, v) = \max\{0, (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}\}, \tag{3}$$

[8] There exist many families of copulas different to the one we have chosen. The Clayton family belongs to the Archimedean class (as the Frank and the Gumbel family do). Another widely recognized class of copulas is the elliptic class. See Nelsen (1999) for an introduction to the copula theory.

where u and v are random variables, uniformly distributed over the unit interval, and $\delta \geq -1$ captures the degree and sign of dependence between the marginal idea distributions: if $\delta < 0$, they are negatively correlated; if $\delta > 0$, they are positively correlated. $\delta = 0$ denotes independence and calls for a replacement of (3) with $C(u, v) = uv$. We consider δ to be *the* crucial parameter here, because it is exactly Jones's (2005) assumption of $\delta = 0$ that we relax and whose relaxation yields such interesting results.

The random variables u and v should be replaced with suitable cumulative distribution functions (CDF). For the ease of exposition, we relegate this point to Appendix A, which contains the proof of the following proposition.

Proposition 1 *Distribution of the two-dimensional random variable* $(\tilde{a}, \tilde{b})$ *is given by*

$$P(\tilde{a} > a, \tilde{b} > b) = \max\left\{0, \left(\left(\frac{\gamma_a}{a}\right)^{-\alpha\delta} + \left(\frac{\gamma_b}{b}\right)^{-\beta\delta} - 1\right)^{-\frac{1}{\delta}}\right\}, \tag{4}$$

if $\delta \in [-1, +\infty)/\{0\}$*; or*

$$P(\tilde{a} > a, \tilde{b} > b) = \left(\frac{\gamma_a}{a}\right)^{\alpha}\left(\frac{\gamma_b}{b}\right)^{\beta}, \tag{5}$$

if $\delta = 0$ *(the marginal distributions are independent).*

PROOF: See Appendix A. □

The probability (4) stands a chance of being zero only if $\delta < 0$. We also note that imposing $\delta = 0$ (independence) leads to (5), which is equation (20) in Jones (2005).

We shall now define formally one of the most important concepts of the present paper: the technology frontier. Application of this definition will make the correspondence between the stochastic arrival of ideas and the deterministic GPF clearer.

Definition 1 *The technology frontier is a curve in the* (a, b) *space, such that the probability* $P(\tilde{a} > a, \tilde{b} > b)$ *is constant along this curve.*

This seemingly simple definition helps us move away from the tedious probabilistic considerations back to the deterministic world. It is consistent with the approach of Caselli and Coleman (2006), and section II of Jones (2005). However, its motivation turns out to be not so simple after all. In quest of such, we build a little model of research where each invention needs to be understood and connected with the existing stock of knowledge by a given percentage of researchers before it comes into industrial use. An outline of this model has been relegated to Appendix D so that the main line of reasoning is not obstructed by this digression.

Using Definition 1 and (4)–(5), we can write the formula for the technology frontier.

Proposition 2 *Assume* $\delta \in [-1, +\infty)\setminus\{0\}$*. Then, the technology frontier* $H(a, b)$ *is given by*

$$H(a, b) = \gamma a^{\alpha\delta} + b^{\beta\delta} = N, \tag{6}$$

where $\gamma = \gamma_b^{\beta\delta}/\gamma_a^{\alpha\delta} > 0$, *and* $N = \gamma_b^{\beta\delta}[P(\tilde{a} > a, \tilde{b} > b)^{-\delta} + 1] > 0$. *See that* $N \in [\gamma_b^{\beta\delta}, 2\gamma_b^{\beta\delta}]$ *if* $\delta < 0$, *and* $N \geq 2\,\gamma_b^{\beta\delta}$ *if* $\delta > 0$.

If $\delta = 0$, *then the technology frontier becomes*

$$H(a, b) = a^{\alpha} b^{\beta} = N, \tag{7}$$

where $N \equiv \frac{\gamma_a^{\alpha}\gamma_b^{\beta}}{P(\tilde{a} > a, \tilde{b} > b)} \geq \gamma_a^{\alpha}\gamma_b^{\beta}$.

PROOF: The proof is simple and requires only algebraic manipulations. □

Equation (7) above is equation (8) in Jones. Please note the following consequence of Proposition 2: if there is negative dependence between the unit factor productivities ($\delta < 0$), then perspectives for technological progress are limited: N is bounded. In such a case, we have a strong "fishing-out" effect: with $N = \gamma_b^{\beta\delta}$, the probability $P(\tilde{a} > a, \tilde{b} > b)$ becomes zero and further progress is impossible. Intuitively, because of the negative dependence, all further labor-(capital-)augmenting innovations would have to be unambiguously capital-(labor-)impairing.

However, if there is positive dependence between individual factor productivities ($\delta > 0$), then R&D can go on forever. We will need these results for our considerations on the long-run direction of technical change in Section 4.

2.2 Step 2. Endogenous technology choice

We shall now make an important assumption that assures analytical tractability of our model. Namely, we assume the LPF to be CES.

Assumption 3 *The LPF* $\tilde{Y}$ *is given by*

$$\tilde{Y}(K, L; a, b) = \tilde{A}(\psi(bK)^{\theta} + (1-\psi)(aL)^{\theta})^{\frac{1}{\theta}}, \tag{8}$$

where $\tilde{A} > 0$, $\theta \in (-\infty, 1]\backslash\{0\}$ *and* $\psi \in (0, 1)$, *or*

$$\tilde{Y}(K, L; a, b) = \tilde{A}(bK)^{\psi}(aL)^{1-\psi}, \tag{9}$$

where $\tilde{A} > 0$ *and* $\psi \in (0, 1)$, *for the limiting case* $\theta = 0$.

This assumption is taken from Caselli and Coleman (2006). At this point, we do not impose $\theta < 0$ on top of it, as they do, but doing so would be natural for anyone who sticks to Jones's "recipe" interpretation of the LPFs. We allow for more flexibility here, keeping in mind the interesting limiting case of Leontief LPF ($\theta = -\infty$).[9]

In the following analysis, we shall also assume that $\alpha\delta > \theta$ and $\beta\delta > \theta$, so that the curvature of the LPF is always greater than the curvature of the technology frontier. This would

[9] Assuming $\theta < 0$ is equivalent to saying that the elasticity of substitution of the LPF, $\sigma = 1/(1-\theta)$, lies below unity. The "recipe" understanding of an LPF leads to $\theta = -\infty$, so that the LPF are Leontief. See Jones (2005) for an intuitive clarification of his understanding of an LPF. However, if the LPF denotes a country's production function as in Caselli and Coleman (2006), one would expect $\theta \approx 0$ rather than $\theta \approx -\infty$.

guarantee an interior solution to the optimization problem presented below. These two inequality constraints have been derived from the second-order conditions in Appendix C.

The competitive firm faces a continuum of LPF given by (8) or (9), and indexed by a and b along the available technology frontier (6) or (7). It chooses one of them to maximize its profit. It also optimally chooses its demand for capital and labor.

The problem of the competitive firm can be decomposed into two separate phases: first, choosing the optimal technology (a, b) for given endowments K, L and prices r, w; and then choosing the demand for both factors of production, taking their prices as given. Because we are only interested in finding the shape of the GPF and not in the general equilibrium of the economy, we shall skip the second phase.[10]

The first phase of the competitive firm's optimization problem in the typical case $\delta \neq 0, \theta \neq 0$ can be written as

$$\max_{a,b} \left\{ \tilde{A}(\psi(bK)^{\theta} + (1-\psi)(aL)^{\theta})^{\frac{1}{\theta}} \right\} \text{ s.t. } \gamma a^{\alpha\delta} + b^{\beta\delta} = N. \tag{10}$$

First-order conditions imply that[11]

$$\frac{b^{\beta\delta-\theta}}{a^{\alpha\delta-\theta}} = \frac{\gamma\psi}{1-\psi}\frac{\alpha}{\beta}k^{\theta}, \tag{11}$$

where we denoted $k \equiv K/L$ for convenience. Solving this equation for b yields

$$b = ck^{\frac{\theta}{\beta\delta-\theta}} a^{\frac{\alpha\delta-\theta}{\beta\delta-\theta}}, \tag{12}$$

which, plugged into (6), gives

$$\Phi(a, k; N) = \gamma a^{\alpha\delta} + c^{\beta\delta} k^{\frac{\theta\beta\delta}{\beta\delta-\theta}} a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}} - N = 0, \tag{13}$$

where $c \equiv (\frac{\gamma\psi}{1-\psi}\frac{\alpha}{\beta})^{\frac{1}{\beta\delta-\theta}}$.

Proposition 3 *Let $\alpha\delta > \theta$ and $\beta\delta > \theta$. Then, the optimization problem* (10) *allows a unique positive solution* $(a^*(k; N), b^*(k; N))$.

PROOF: See Appendix B. □

Equation (13) suffices to infer existence and uniqueness of the solution; unfortunately, it contains a sum of two arbitrary powers of a and, therefore, unless some particular equality constraint holds, no explicit formula for a as a function of capital per worker k and technology level N can be obtained from it. If $\delta = 0$ (the Jones case) or $\theta = 0$ (Cobb–Douglas LPF), then a Cobb–Douglas GPF is obtained. The most interesting case is nevertheless the one with $\alpha = \beta$ (equal Pareto exponents); it yields the CES result.

Comparing our results to those of Jones, we see that here the optimal technology levels a^* and b^* depend on θ, the curvature parameter of the LPF. Hence, Jones's (2005, p. 528) claim, that

[10] Caselli and Coleman (2006) close the model by assuming that the produced good is the numeraire and in the whole economy, stocks of available capital K and labor L are fixed. Jones (2005) skips the second phase.

[11] Exact derivations have been relegated to Appendix C.

> [o]f course, the intuition regarding the global production function suggests that it is determined by the distribution of ideas, not by the shape of the local production function

does not hold in general. Equation (13) clearly suggests that in the typical Clayton–Pareto case, the shape of the LPF *matters* for the shape of the GPF.

2.3 Step 3. Global production function

We shall now find the convex hull of LPFs, with $a^* \equiv a^*(k;\, N)$ and $b^* \equiv b^*(k;\, N)$ defined as above.

The most straightforward (and underestimated) way to do this is to insert a^* and b^* directly into the LPF formula (8). Another method is to characterize it in terms of partial elasticities, following equation (7) in Jones:

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\eta_b}{\eta_a}, \tag{14}$$

where $\varepsilon_K = (\partial Y/\partial K)(K/Y)$ is the partial elasticity of the GPF with respect to capital; and η_a, η_b are partial elasticities of the technology frontier H with respect to a and b, respectively. Deriving (14) requires use of the Envelope Theorem and the degree-one homogeneity of the GPF.

Although both methods give the same results for the *shape* of the GPF, they are not equivalent. The first method is probably computationally more demanding, but it does not incur a loss of information due to differentiation as the second one does. Therefore, all parameters of the GPF, including its intercept term, can be recovered only thanks to the first method.

Summing up, we obtain the following characterization of a GPF which belongs to the Clayton–Pareto family:

Proposition 4 *Given Assumptions 1, 2 and 3, and the firms' optimizing behavior summarized in* (10), *the GPF is given by*

$$Y(K,\, L;\, N) \equiv \tilde{A}(\psi(b^*K)^\theta + (1-\psi)(a^*L)^\theta)^{\frac{1}{\theta}}, \tag{15}$$

where a^* and b^* satisfy (11), (12) *and* (13). *This implies that*

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\psi}{1-\psi}\left(\frac{b^*K}{a^*L}\right)^\theta = \frac{\beta}{\gamma\alpha}\frac{(b^*)^{\beta\delta}}{(a^*)^{\alpha\delta}}. \tag{16}$$

PROOF: Already given in text. □

Proposition 4 implicitly defines all functions that belong to the Clayton–Pareto class. A few of them have been depicted in Figure 1.

The most important advantage of the Clayton–Pareto class of functions is its generality. We have shown that from a specific setup that is based on Pareto distributions of factor

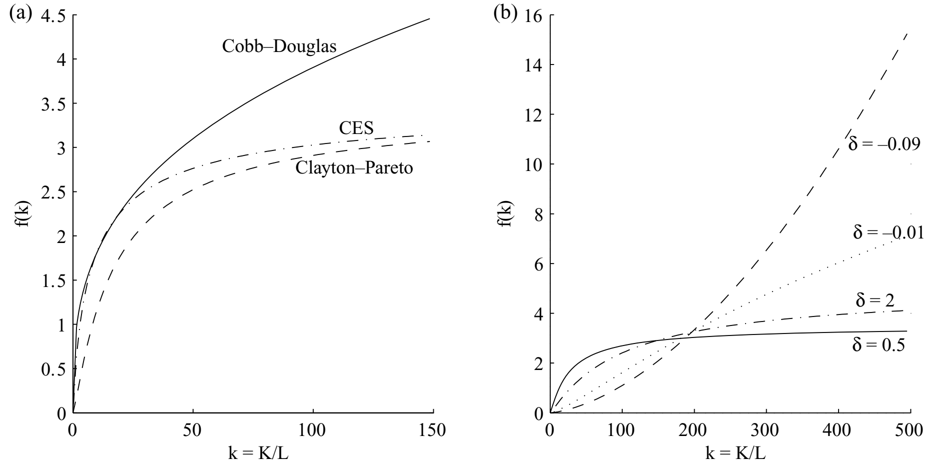

Figure 1 (a) A representative Clayton–Pareto production function compared to the Cobb–Douglas and the constant elasticity of substitution (CES) function. (b) Impact of the degree of dependence δ on the resultant Clayton–Pareto production functions. Assumed parameter values (unless indicated otherwise): $\tilde{A}=1$, $\gamma_a=1$, $\gamma_b=0.2$, $\theta=-1$, $\alpha=5$, $\beta=2.5$, $\psi=0.5$, $\delta=0.5$, $P(\tilde{a}>a,\ \tilde{b}>b)=0.1$. For the CES case, we equalized $\alpha=\beta=5$; for the Cobb–Douglas case, $\delta=0$.

productivities, a much wider class of functions can be derived that had been previously acknowledged.

Unfortunately, not all Clayton–Pareto functions are really *production* functions: some of them are not concave and, therefore, violate the diminishing marginal utility requirement. Such a situation is most likely to emerge if $\alpha\delta \approx \theta$; that is, if the second-order conditions are satisfied with a very narrow margin. In Figure 1, we present such a case for $\delta=-0.09$. See also that for $\delta=-0.01$, the Clayton–Pareto function is concave, albeit its implied marginal product of capital decreases *very* slowly. As we will see in the next section, in the CES case the concavity condition can be put in a simple analytical form, $\alpha\delta-\theta-\alpha\delta\theta>0$.

3 Special cases

Let us now dwell on some special cases that provide explicit solutions. They all have meaningful economic interpretations. We find that the Clayton–Pareto family nests the Cobb–Douglas function and the CES function.

3.1 Independent Pareto distributions

Independence of the marginal idea distributions (i.e. $\underline{\delta=0}$) yields a particularly nice and interpretable result: the benchmark result due to Jones (2005). It links *independent* Pareto idea distributions with Cobb–Douglas production functions.

With $\delta = 0$, the technology frontier is given by (5), which implies that the firm's optimality condition equivalent to (13) can be solved explicitly as

$$\begin{cases} a^*(k; N) & = N^{\frac{1}{\alpha+\beta}} \tilde{c}^{\frac{\beta}{\alpha+\beta}} k^{\frac{\beta}{\alpha+\beta}} \\ b^*(k; N) & = N^{\frac{1}{\alpha+\beta}} \tilde{c}^{-\frac{\alpha}{\alpha+\beta}} k^{-\frac{\alpha}{\alpha+\beta}}, \end{cases} \tag{17}$$

where $\tilde{c} = (\frac{\alpha}{\beta} \frac{\psi}{1-\psi})^{1/\theta}$ and $\theta < 0$. As Jones determines, the GPF comes to satisfy:

$$\frac{\varepsilon_K}{1 - \varepsilon_K} = \frac{\beta}{\alpha}, \tag{18}$$

so it needs to be Cobb–Douglas with constant returns to scale, and exponents proportional to α and β; that is, it needs to be

$$Y(K, L; N) = A(N) K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}}. \tag{19}$$

Further calculations yield that the factor-neutral productivity term $A(N)$ is given by

$$A(N) = \tilde{A} N^{\frac{1}{\alpha+\beta}} \left[\psi^{\frac{\beta}{\alpha+\beta}} (1-\psi)^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \right) \right]^{\frac{1}{\theta}}.$$

Moreover, if we take $\theta \to -\infty$, so that the LPF are approximately Leontief, then we obtain $A(N) \to \tilde{A} N^{\frac{1}{\alpha+\beta}}$. This precisely corresponds to equation (28) in Jones.

In the case of independent Pareto distributions, we could have missed the second step (finding a^* and b^*), and yet we would have arrived at the same result. This is because in (18), the GPF is required to exhibit *constant* partial elasticities with respect to its arguments, which is a defining property of Cobb–Douglas functions. So here, shapes of the LPFs have no impact on the shape of the GPF.[12]

3.2 Cobb–Douglas local production functions

It turns out that if local production functions are Cobb–Douglas (i.e. $\theta = 0$), then the GPF is Cobb–Douglas as well, irrespective of the shape of the technology frontier.[13] This is a fairly confusing result, taking into account the assertion by Jones that we cited in Section 2: namely, once we assume that the LPF are given by (9), we obtain that the shape of the technology frontier takes no part in determining the shape of the GPF. To see this

[12] However, we have to maintain the assumption that local production functions exhibit constant returns to scale. Moreover, to remain in the interior solution, we still have to assume that the elasticity of substitution of the LPF is everywhere lower than unity. Finally, shapes of the LPFs (described by the parameters θ and ψ) do have an impact on $A(N)$.

[13] Cobb–Douglas LPF are not plausible if one maintains a "recipe" understanding of an LPF, because they are characterized by unitary elasticity of substitution between factors. However, for the Caselli and Coleman understanding of an LPF as a country-wide production function, they are perfectly plausible.

implication, note that in the last step of our procedure, we obtain

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\psi}{1-\psi}, \tag{20}$$

implying a GPF of a Cobb–Douglas form:

$$Y(K, L; N) = A(N)K^{\psi}L^{1-\psi}, \tag{21}$$

irrespective of the shape of the technology frontier, provided that the two marginal distributions of unit productivities are positively dependent ($\delta > 0$), which is necessary for an interior solution.

Further calculations yield $A(N) = \tilde{A}\bar{c}^{\frac{\psi}{\beta\delta}}(\frac{N}{\gamma+\bar{c}})^{\frac{\psi}{\beta\delta}+\frac{1-\psi}{\alpha\delta}}$, where $\bar{c} = \frac{\gamma\psi}{1-\psi}\frac{\alpha}{\beta}$. This means that technology choice is irrelevant for the shape of GPF, but it is not irrelevant for the level of $A(N)$. To facilitate comparisons, we shall point out here that in the optimum, firms choose technologies a^* and b^* according to:

$$\begin{cases} a^*(k; N) = \left(\frac{N}{\gamma+\bar{c}}\right)^{\frac{1}{\alpha\delta}}, \\ b^*(k; N) = \left(\frac{N\bar{c}}{\gamma+\bar{c}}\right)^{\frac{1}{\beta\delta}}. \end{cases} \tag{22}$$

Note that both expressions are independent of k.

3.3 The constant elasticity of substitution production function

Let us now assume that the shape parameters of both Pareto idea distributions are equal (i.e. $\alpha = \beta$). In this important knife-edge case, an explicit formula for the GPF is obtained. This function exhibits a CES, being a function of α, δ and θ. The CES result is preserved for Leontief LPFs as well.

With $\alpha = \beta$, the technology frontier (6) becomes:

$$H(a, b) = \gamma a^{\alpha\delta} + b^{\alpha\delta} = N, \tag{23}$$

which is apparently Caselli and Coleman's equation (5). By writing it down as a special case of our model, we provide a "Clayton–Pareto" microfoundation for it. Please note that this result is obtained already in Step 1 of our derivation procedure: the link between the setups of Jones as well as Caselli and Coleman is valid regardless of the shape of the LPFs, which enters the analysis at a later stage.

The first-order condition (11) for the representative firm's optimization problem (10) can be now solved explicitly as:

$$\begin{cases} a^*(k; N) = \left(\frac{N}{\gamma + c^{\alpha\delta}k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}\right)^{\frac{1}{\alpha\delta}}, \\ b^*(k; N) = \left(\frac{Nc^{\alpha\delta}k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}{\gamma + c^{\alpha\delta}k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}\right)^{\frac{1}{\alpha\delta}}. \end{cases} \tag{24}$$

Therefore, the GPF satisfies the equation:

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{c^{\alpha\delta}}{\gamma} k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}. \tag{25}$$

Consequently, it takes the CES form (see Arrow, Chenery, Minhas, and Solow 1961):[14]

$$Y(K, L; N) = A(N)(\zeta K^{\xi} + (1-\zeta)L^{\xi})^{1/\xi}, \tag{26}$$

where

$$\xi = \frac{\alpha\delta\theta}{\alpha\delta-\theta}, \tag{27}$$

$$\zeta = \frac{c^{\alpha\delta}}{c^{\alpha\delta}+\gamma}, \tag{28}$$

$$A(N) = \tilde{A} N^{\frac{1}{\alpha\delta}} \left((1-\psi)^{\frac{\alpha\delta}{\alpha\delta-\theta}} \gamma^{-\frac{\theta}{\alpha\delta-\theta}} + \psi^{\frac{\alpha\delta}{\alpha\delta-\theta}} \right)^{\frac{\alpha\delta-\theta}{\alpha\delta\theta}}, \tag{29}$$

and $c = (\frac{\gamma\psi}{1-\psi})^{\frac{1}{\alpha\delta-\theta}}$. The assumption $\alpha\delta > \theta$ guarantees an interior solution to the competitive firm's optimization problem.

Let us also note that if one takes $\theta = -\infty$ (i.e. assumes that the LPF are Leontief, as in the reference case discussed by Jones (2005)), then the above result simplifies greatly while still implying a CES GPF: $\xi = -\alpha\delta$, $A(N) = \tilde{A} N^{\frac{1}{\alpha\delta}} (\frac{1}{\gamma+1})$, $\zeta = \frac{1}{\gamma+1}$, and $c = 1$.

The exponent ξ of the Pareto-microfounded CES function, derived in (27), implies that the elasticity of substitution σ of the GPF is equal to

$$\sigma = \frac{1}{1-\xi} = \frac{\alpha\delta-\theta}{\alpha\delta-\theta-\alpha\delta\theta}. \tag{30}$$

For Leontief LPF we obtain $\xi = -\alpha\delta$ and, accordingly, $\sigma = \frac{1}{1+\alpha\delta}$. This clear-cut result clarifies that we have obtained the CES GPF here not because we have assumed CES LPFs, but because we have assumed Pareto idea distributions with equal exponents, dependent according to the Clayton copula. Indeed, for Leontief LPFs that have zero elasticity of substitution, the CES result is preserved; if any of the latter assumptions are relaxed, the CES result is *not* preserved.

One important remark is due here. To assign a serious economic interpretation of the above formulae (i.e. to assure that the elasticity of substitution σ is positive so that the function is concave and the marginal products of production factors are decreasing), we need to assume $\alpha\delta - \theta - \alpha\delta\theta > 0$.

[14] To prove this, note that $\frac{\varepsilon_K}{(1-\varepsilon_K)} = \frac{K/L}{MRS}$, where *MRS* is the marginal rate of substitution between capital and labor. Straightforward algebraic manipulations yield that the elasticity of substitution $\sigma = \frac{\partial(K/L)}{\partial MRS}\frac{MRS}{K/L} = \frac{1}{1-\xi}$, where $\xi = \frac{\alpha\delta\theta}{\alpha\delta-\theta}$. Now use the theorem due to Arrow, Chenery, Minhas, and Solow (1961) to obtain (26). To obtain the coefficients A and ζ, more algebra is necessary. They have been computed by equalizing $\tilde{Y}(K, L; a^*, b^*) = Y(K, L; N)$, and then gradually simplifying the resultant expression.

Having this assumption in mind, we find that the elasticity of substitution σ of the GPF and the degree of dependence between the marginal Pareto distributions δ are inversely related. Moreover, we also find that for the smallest possible values of δ, σ explodes to infinity, and for $\delta \to \infty$, it approaches the (low) elasticity of substitution of the LPF:

$$\lim_{\delta \to \left(\frac{\theta}{\alpha(1-\theta)}\right)_+} \frac{\alpha\delta - \theta}{\alpha\delta - \theta - \alpha\delta\theta} = +\infty \quad \text{and} \quad \lim_{\delta \to +\infty} \frac{\alpha\delta - \theta}{\alpha\delta - \theta - \alpha\delta\theta} = \frac{1}{1-\theta}. \tag{31}$$

The inverse relation between δ and σ becomes the most apparent once θ is evaluated at $-\infty$. Indeed, in the case of Leontief LPFs, the relation between δ and ξ is linear, the elasticity of substitution σ exceeds unity if and only if $\delta < 0$, and it approaches zero as $\delta \to +\infty$.

We conclude that what drives the elasticity of substitution of the GPF is the actual difference between the curvature of the LPF, described by θ, and the curvature of the technology frontier, described by $\alpha\delta$: the greater the difference, the lower the elasticity of substitution of the GPF.

The elasticity of substitution between capital (which is an accumulable factor) and labor (which cannot be accumulated) has sizeable effects on the long-run performance of the economy (see e.g. the two theorems by Klump and de La Grandville (2000)). In particular, $\sigma > 1$ might imply endogenous growth in the absence of technical progress and explosive growth when technical progress is present (see Jones and Manuelli 1990; Palivos and Karagiannis 2004). We shall discuss this issue in the following section.

4 Direction of technical change

Let us now embed our microfounded model in the neoclassical growth framework (Solow 1956) to draw conclusions about the long-run direction of technical change.

From now on, we shall assume exogenous technical progress in the form of a decline in $P(\tilde{a} > a, \tilde{b} > b)$: as time passes, more and more efficient technologies are invented. Due to Definition 1, so defined an R&D activity is equivalent to pushing the technology frontier further and further. The choice of an optimal technology pair (a, b) from the technology frontier that is available at each instant of time is again left to firms. This means that we shall explicitly take advantage of the fact that in our framework, the direction of technical change is endogenous.

Capital evolves according to the usual equation of motion:

$$\dot{K} = sY - \delta_K K, \tag{32}$$

where $\delta_K > 0$ is the capital depreciation rate, and $s \in (0, 1)$ is the exogenous savings rate. Labor force grows at a constant rate $\hat{L} \equiv \dot{L}/L = n \geq 0$. Equation (32) implies that along the balanced growth path (BGP) (if there exists one, which is a very stringent condition) the product-capital ratio stays constant.

In further derivations, we shall dwell on the three particular cases, $\delta = 0, \theta = 0$, and $\alpha = \beta$, that offer closed-form solutions only. They already give a taste of the vast multiplicity of long-run outcomes of our microfounded model. In particular, the CES case turns out to be the most revealing: it allows us to draw preliminary conclusions about the dynamic properties of Clayton–Pareto production functions in general.

4.1 Independent Pareto distributions

We start with the case of independent Pareto distributions ($\underline{\delta = 0}$), studied in detail by Jones (2005). Having assumed $\hat{N} = g$ and $y = Y/L$, and used $\hat{Y} = \hat{K}$, we find that the growth rate of the economy along the BGP is given by

$$\hat{Y} = \frac{g}{\alpha + \beta} + \frac{\beta}{\alpha + \beta}\hat{K} + \frac{\alpha}{\alpha + \beta}\hat{L} \quad \Rightarrow \quad \hat{y} = \frac{g}{\alpha}. \tag{33}$$

From (17), we also have that $\hat{a} = \frac{g}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\frac{g}{\alpha} = \frac{g}{\alpha}$ and $\hat{b} = \frac{g}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}\frac{g}{\alpha} = 0$. This is exactly the benchmark result of Jones of purely labor-augmenting technical change in the long run. Despite the fact that for each given K and L, technical progress is factor-neutral in this case, proportions of factors actually used in production change along the BGP in such a way that the unit productivity of capital (b) stays exactly constant.

4.2 Cobb–Douglas local production functions

We can now proceed to the case of Cobb–Douglas local production functions ($\underline{\theta = 0}$).

In such cases, we still obtain existence of a BGP and a Cobb–Douglas GPF. However, we observe that technical change ceases to be purely labor-augmenting in the long run. Namely, along the BGP,

$$\hat{Y} = \left(\frac{\psi}{\beta\delta} + \frac{1-\psi}{\alpha\delta}\right)g + \psi\hat{K} + (1-\psi)\hat{L} \quad \Rightarrow \quad \hat{y} = \left(\frac{\psi}{(1-\psi)\beta\delta} + \frac{1}{\alpha\delta}\right)g. \tag{34}$$

Now, the growth rate of the economy $\hat{y}$ depends not only on α, as in (33), but also on β (as well as ψ and δ). Redoing the same exercise as in the previous subsection, we show that in this case, along the BGP, $\hat{a} = \frac{g}{\alpha\delta} > 0$ and $\hat{b} = \frac{g}{\beta\delta} > 0$. This means that technical change augments *both* factors of production in the long run.

4.3 Constant elasticity substitution production function

The case of the CES global production function ($\underline{\alpha = \beta}$) turns out to be more revealing than the two Cobb–Douglas ones discussed just above, and it also provides a hint regarding what one can expect in the general Clayton–Pareto case.

First of all, a balanced growth path (with a positive growth rate of capital per worker) does not exist in the CES case. Therefore, to obtain some meaningful outcome concerning the long run, we have to dwell on the asymptotics: by "in the long run" we will now mean "as $t \to \infty$" rather than "along the balanced growth path." Moreover, we have to consider the two cases separately: $\xi > 0$ where capital and labor are gross substitutes, and $\xi < 0$ where they are gross complements.

Case $\xi > 0$ Capital and labor are gross substitutes if $0 > \alpha\delta > \theta$ or if $\alpha\delta > \theta > 0$. Simple calculations yield that in such a case,

$$\hat{k} = sA(\zeta + (1-\zeta)k^{-\xi})^{1/\xi} - \delta_K - n \quad \Rightarrow \quad \lim_{k\to\infty} \hat{k} = sA\zeta^{1/\xi} - \delta_K - n. \qquad (35)$$

Hence, there is endogenous growth via capital accumulation if $sA\zeta^{1/\xi} - \delta_K - n > 0$. Such endogenous growth is possible here thanks to the high elasticity of substitution of the analyzed GPF.[15] When capital and labor are gross substitutes, the marginal product of capital never declines to zero, and the Inada condition in infinity does not hold.[16]

Note that (35) makes economic sense only if $\delta < 0$ (so N is bounded) or if we exogenously fix N (assume out all R&D activity). If $A = A(N)$ were to grow exogenously on top of the elasticity-driven endogenous growth, we would have arrived at an implausible explosive case of an economy that reaches infinite production in finite time. What is more, in the gross-substitutes case $\xi > 0$, with $\delta > 0$ and positive growth in $A(N)$, the condition for endogenous growth $sA\zeta^{1/\xi} - \delta_K - n > 0$ will be satisfied sooner or later. Therefore, this economy is *bound* to explode to infinity in finite time. For obvious reasons, we rule this case out.

Within the case of an endogenously and yet non-explosively growing economy, individual factor productivities approach certain limits as $k \to \infty$ with time: $b^* \to N^{\frac{1}{\alpha\delta}}$, and $a^* \to 0$ if $\delta > 0$ or $a^* \to +\infty$ if $\delta < 0$. Hence, although there is no actual technical change, it is clearly a^*, the productivity of labor that drives long-run endogenous growth.

Case $\xi < 0$ If capital and labor are gross complements, endogenous growth is impossible. However, because $\xi < 0$ implies $\delta > 0$ and $\theta < 0$, exogenous R&D-driven growth (represented by perpetual growth in N) is both possible and plausible.

We find that in the gross-complements case there exists an asymptotic BGP: a BGP that cannot be reached in finite time but is gradually converged to as $k \to \infty$ with time. Precisely speaking, we have that

$$\hat{Y} = \frac{g}{\alpha\delta} + \varepsilon_K \hat{K} + (1-\varepsilon_K)\hat{L} \quad \Rightarrow \quad \hat{y} = \frac{g}{\alpha\delta} + \varepsilon_K \hat{k}. \qquad (36)$$

Let us now calculate the limit of $\hat{y}$ as $t \to \infty$. Because of exogenous growth in N, we have that $A \to \infty$ and $k \to \infty$. In consequence, the partial elasticity ε_K disappears in the limit: $\varepsilon_K = A^{\xi}\zeta(k/y)^{\xi} \to 0$. This suffices to show that $\hat{y} \to \frac{g}{\alpha\delta}$. By the virtue of the capital's equation of motion, we know that $\hat{k} \to \frac{g}{\alpha\delta}$ as well.

As for the growth rates of unit factor productivities, we use (24) to show that $\hat{a} \to \frac{g}{\alpha\delta}$ and $\hat{b} \to \frac{g}{\alpha\delta} + (\frac{\theta}{\alpha\delta-\theta})\frac{g}{\alpha\delta} = \frac{g}{\alpha\delta-\theta} > 0$. Hence, we see that in the CES case, technical change *augments both factors of production in the long run*, not only labor as in the Jones case.[17]

[15] $\xi > 0$ implies $\sigma > 1$. Further reasoning follows Palivos and Karagiannis (2004) and Jones and Manuelli (1990).

[16] In the general Clayton–Pareto case, σ is not constant. However, one can clearly expect that endogenous growth via capital accumulation can, and should, appear only if $\lim_{k\to\infty} \sigma(k) > 1$ (see Palivos and Karagiannis 2004).

[17] One should expect this result in the general Clayton–Pareto case as well, if only the long-run elasticity of substitution is less than unity (if $\lim_{k\to\infty} \sigma(k) < 1$).

There is one important twist to this reasoning, however. For $\theta = -\infty$ (so the LPFs are Leontief), we obtain that $\hat{b} \to 0$. Therefore, in the Leontief limit, technical change is purely labor-augmenting in the long run, a result that carries forward from Jones (2005) and Acemoglu (2003). The difference is that in their setups this property is general, and here it relies upon the assumption $\theta = -\infty$, which is both knife-edge and extreme.

5 Conclusion

In this article, we have derived from microfoundations the "Clayton–Pareto" class of GPF. We have discussed the properties of some of its most distinguished members. We have also analyzed the long-run implications of such production functions for the direction of technical change (i.e., we checked when it is labor-augmenting, capital-augmenting, or both).

Our "Clayton–Pareto" class of production functions has been obtained assuming that each of the unit factor productivities is Pareto-distributed, that dependence between these marginal distributions is captured by the Clayton copula, and that local production functions are CES. This class has been shown to nest both the Cobb–Douglas functions and the CES (both being empirically relevant: see Duffy and Papageorgiou 2000). Contrary to the presumptions in Jones (2005), the shape of the GPF typically depends on the shapes of local production functions.

Embedding our microfounded model in the neoclassical (Solow) growth framework, we have proven that technical progress tends to augment both factors of production in the long run. We could find only two exceptions to this rule, which would lead to purely labor-augmenting technical change: the case of independent marginal distributions, which implies a Cobb–Douglas GPF, and the case of Leontief local production functions, while the GPF is either CES or Cobb–Douglas.

We have also proven that in some cases, the neoclassical growth model with Clayton–Pareto production exhibits endogenous growth through capital accumulation. Moreover, assuming exogenous technological progress on top of it leads to explosivity.

Summing up, the goals achieved in the present paper were to critically re-examine the assumptions of Cobb–Douglas production functions and purely labor-augmenting technical change, ubiquitous in contemporary growth theory, and to show that even a slight departure from the original framework in Jones (2005) can lead to significantly different results. For instance, the CES production function can also be derived from Pareto distributions of unit factor productivities.

This research can be extended in the following ways. First, the setup can be generalized to include further factors of production, such as human capital or non-renewable and renewable resources. Second, alternative copulas may be used to capture the dependence between marginal idea distributions. Third, one may wish to analyze the behavior of our microfounded model under R&D-based endogenous growth. Fourth, one may want to relax the assumption that the LPFs are CES. Finally, empirical studies in this field would be extremely valuable.

Appendix

A Proof of proposition 1

Recall that the marginal idea distributions of $\tilde{a}$ and $\tilde{b}$ are Pareto, given by (1) and (2), respectively.

Because the Pareto distribution is nicely defined in terms of its survival function and not the cumulative distribution function (CDF) itself, we find it useful to substitute $X = -\tilde{a}$, $Y = -\tilde{b}$.[18]

CDFs of X and Y satisfy:

$$P(\tilde{a} > a) = P(X < -a) \equiv F_X(-a) = \left(\frac{\gamma_a}{a}\right)^{\alpha},$$

$$P(\tilde{b} > b) = P(Y < -b) \equiv F_Y(-b) = \left(\frac{\gamma_b}{b}\right)^{\beta},$$

where $a \geq \gamma_a > 0$ and $b \geq \gamma_b > 0$. Applying the Clayton copula formula given in (3) to the CDFs of X and Y yields:

$$F(a, b) = C(F_X(a), F_Y(b)) = \max\left\{0, \left(\left(\frac{\gamma_a}{-a}\right)^{-\alpha\delta} + \left(\frac{\gamma_b}{-b}\right)^{-\beta\delta} - 1\right)^{-\frac{1}{\delta}}\right\},$$

where $a \leq -\gamma_a < 0$ and $b \leq -\gamma_b < 0$.

Finally, we apply $P(\tilde{a} > a, \tilde{b} > b) = F(-a, -b)$ to obtain (4).

If $\delta = 0$, then

$$P(\tilde{a} > a, \tilde{b} > b) = F_X(-a)F_Y(-b) = \left(\frac{\gamma_a}{a}\right)^{\alpha}\left(\frac{\gamma_b}{b}\right)^{\beta},$$

which is directly (5). □

B Proof of proposition 3

We apply the Implicit Function Theorem to the Φ function, defined in (13). First of all, we notice that $\Phi \in C^1(\mathbb{R}^3_+)$. Second, we observe that

$$\frac{\partial \Phi}{\partial a} = \gamma\alpha\delta a^{\alpha\delta-1} + \left(\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}\right)c^{\beta\delta}k^{\frac{\theta\beta\delta}{\beta\delta-\theta}}a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}-1}, \tag{37}$$

and so, using $\alpha\delta - \theta > 0$ and $\beta\delta - \theta > 0$, we obtain $\frac{\partial\Phi}{\partial a} > 0$ if and only if $\delta > 0$, and $\frac{\partial\Phi}{\partial a} < 0$ if and only if $\delta < 0$. Hence (because $\delta \neq 0$),[19] clearly $\frac{\partial\Phi}{\partial a} \neq 0$ for all $k > 0$ and $N > 0$, so for all $k > 0$ and $N > 0$, there locally exists an implicit function a^* such that $\Phi(a^*(k; N), k; N) = 0$.

To obtain uniqueness of a^*, we shall note that for each given configuration of exogenous parameter values, and given $a, k, N > 0$, both partial derivatives of the implicit function, $\frac{\partial a}{\partial k}$ and $\frac{\partial a}{\partial N}$ have a constant sign. By the Implicit Function Theorem, we have $\frac{\partial a}{\partial k} = -\frac{\frac{\partial\Phi}{\partial k}}{\frac{\partial\Phi}{\partial a}}$ and $\frac{\partial a}{\partial N} = -\frac{\frac{\partial\Phi}{\partial N}}{\frac{\partial\Phi}{\partial a}}$. Constancy of the sign of $\frac{\partial\Phi}{\partial a}$ we have already proven above. $\frac{\partial\Phi}{\partial N} = -1$ so it is obviously always negative. Because

$$\frac{\partial\Phi}{\partial k} = \left(\frac{\beta\delta\theta}{\beta\delta-\theta}\right)c^{\beta\delta}a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}}k^{\frac{\beta\delta\theta}{\beta\delta-\theta}-1},$$

[18] Instead of inverting the random variables, we could also rotate the Clayton copula.

[19] The $\delta = 0$ (independence) case allows a unique, closed-form solution to the discussed problem. No sophisticated reasoning is necessary in this case. See Section 3.1.

we obtain $\frac{\partial \Phi}{\partial k} > 0$ if and only if $\delta < 0, \theta < 0$ or $\delta > 0, \theta > 0$; and $\frac{\partial \Phi}{\partial k} < 0$ if $\delta > 0$ and $\theta < 0$. Therefore, signs of $\frac{\partial a}{\partial k}$ and $\frac{\partial a}{\partial N}$ never change. Uniqueness of a^* follows from a juxtaposition of this fact with global differentiability of Φ.

From (12), we have that if a positive $a^*(k; N)$ exists and is unique, then automatically so is $b^*(k; N)$. □

C Optimality conditions

To save on algebraic manipulations, we shall simplify the competitive firm's optimization problem (10). Solving it is a task equivalent to finding extreme values of the following Lagrangian:

$$\mathcal{L}(a, b, \lambda) = \tilde{A}^{\theta}[\psi(bK)^{\theta} + (1-\psi)(aL)^{\theta}] - \lambda(\gamma a^{\alpha\delta} + b^{\beta\delta} - N).$$

The Lagrangian $\mathcal{L}$ should be maximized (if $\theta > 0$) or minimized (if $\theta < 0$).

We have

$$\begin{cases} \mathcal{L}_a(a, b, \lambda) = \tilde{A}(1-\psi)\theta a^{\theta-1}L^{\theta} - \lambda\gamma\alpha\delta a^{\alpha\delta-1} = 0, \\ \mathcal{L}_b(a, b, \lambda) = \tilde{A}\psi\theta b^{\theta-1}L^{\theta} - \lambda\beta\delta b^{\beta\delta-1} = 0, \\ \mathcal{L}_\lambda(a, b, \lambda) = \gamma a^{\alpha\delta} + b^{\beta\delta} - N = 0. \end{cases}$$

Moving the terms with λ in the first two equations to the right-hand side, and dividing sidewise yields (11).

In the optimum, we also have $\lambda = \frac{\tilde{A}^{\theta}(1-\psi)\theta L^{\theta} a^{\theta-\alpha\delta}}{\gamma\alpha\delta} = \frac{\tilde{A}^{\theta}\psi\theta K^{\theta} b^{\theta-\beta\delta}}{\gamma\beta\delta}$.

The second derivatives of the Lagrangian are (after inserting appropriate expressions for λ):

$$\begin{cases} \mathcal{L}_{aa}(a, b) = \tilde{A}^{\theta}(1-\psi)\theta a^{\theta-2}L^{\theta}(\theta - \alpha\delta), \\ \mathcal{L}_{ab}(a, b) = \mathcal{L}_{ba}(a, b) = 0, \\ \mathcal{L}_{bb}(a, b) = \tilde{A}^{\theta}\psi\theta b^{\theta-2}K^{\theta}(\theta - \beta\delta). \end{cases}$$

If $\mathcal{L}_{aa}$ and $\mathcal{L}_{bb}$ are both negative, then we have a maximum; and if they are both positive, we have a minimum. It is straightforward to see that we need $\alpha\delta > \theta$ and $\beta\delta > \theta$ for our optimization criteria to be satisfied. □

D Model of research

This section of the appendix presents a model of research that supports Definition 1.

We assume there is a continuum of researchers, located along the unit interval $I = [0, 1]$. Each researcher $i \in I$ draws one technology pair (a_i, b_i) from the joint idea distribution $(\tilde{a}, \tilde{b})$. The Law of Large Numbers implies that for all values of $a \geq \gamma_a$, $b \geq \gamma_b$, there must exist a researcher who has drawn a technology pair arbitrarily close to (a, b) in terms of the Pythagorean metric on $\mathbb{R}^2$.

Let us now define the dependence between all the individual draws of ideas and the technology that becomes the output of the whole research process. We assume that a technology pair (a, b) can be made available for production only after a given fraction $x \in (0, 1)$ of researchers understands it and helps incorporate it into the existing stock of knowledge. Put formally, a technology pair (a, b) becomes available only if a given fraction x of researchers have drawn a technology pair that is not inferior to (a, b).

Let us explain what we mean by *not inferior* in a 2-dimensional setup. We first consider a single ray from the origin: a semi-straight line $\{(a, b) \in \mathbb{R}^2_+ : b = \omega a\}$, where $\omega \in (0, +\infty)$ is given. Along such a ray, all points are indexed by a single number $a \geq \gamma_a$, and can be easily ordered because on the real line on which a lies, there exists a natural ordering. Using the Law of Large Numbers again, we obtain that there exists a unique $\bar{a} > \gamma_a$ such that for all $\gamma_a < a < \bar{a}$, technology $(a, \omega a)$ is understood by at least x percent of researchers. The "frontier" technology $(\bar{a}, \omega\bar{a})$ is characterized by $P(\tilde{a} > \bar{a}, \tilde{b} > \omega\bar{a}) = x$, and all technologies $(a, \omega a)$ for which $P(\tilde{a} > a, \tilde{b} > \omega a) > x$ are inferior to the technology $(\bar{a}, \omega\bar{a})$ because *both* their coordinates are lower. Therefore, the "frontier" technology pair $(\bar{a}, \omega\bar{a})$ is the best one attainable within the given ray.

We carry out this procedure for all rays from the origin, letting the index ω go from 0 to $+\infty$. In the final outcome, we obtain that the best attainable (frontier) technologies are located along the curve $\{(a, b) \in \mathbb{R}^2_+ : P(\tilde{a} > a, \tilde{b} > b) = x\}$. We refer to the x-th contour of the joint idea tail probability function as the *technology frontier*.

Until now, there has been no technological progress (and no time dimension either) in this model. In this respect, we are going to assume that as time goes on, researchers (whom we assume to be infinitely-lived for simplicity) gradually accumulate knowledge, and so it becomes easier and easier for them to understand and accomodate further innovations. Hence, further and further technology frontiers become attainable in the passage of time. As we want to simplify our analysis as much as possible, we do not model the knowledge accumulation process explicitly here. Instead, we proxy it with an exogenous decline in x, which yields equivalent results.

Employing our little model of research helps to simplify the original framework in Jones (2005). First, we do not have to use Fréchet distributions or Poisson processes to obtain our results. Second, we are able to achieve more generality, which would not be possible if we insisted on maintaining full stochasticity until the end of analysis: one can easily imagine that if the simplest case requires handling Fréchet distributions, then the more complex ones would simply be intractable.

References

Acemoglu, D. (2003), "Labor- and capital-augmenting technical change," *Journal of the European Economic Association* **1**, 1–37.

Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961), "Capital-labor substitution and economic efficiency," *Review of Economics and Statistics* **43**, 225–50.

Atkinson, A. B., and J. E. Stiglitz (1969), "A new view of technological change," *Economic Journal* **79**, 573–78.

Basu S., and D. N. Weil (1998), "Appropriate technology and growth," *Quarterly Journal of Economics* **113**, 1025–54.

Caselli F., and W. J. Coleman (2006), "The world technology frontier," *American Economic Review* **96**, 499–522.

de La Grandville, O. (1989), "In the quest of the Slutsky diamond," *American Economic Review* **79**, 468–81.

Duffy, J., and C. Papageorgiou (2000), "A cross-country empirical investigation of the aggregate production function specification," *Journal of Economic Growth* **5**, 87–120.

Gabaix, X. (1999), "Zipf's law for cities: An explanation," *Quarterly Journal of Economics* **114**, 739–67.

Houthakker, H. S. (1955–56), "The Pareto distribution and the Cobb–Douglas production function in activity analysis," *Review of Economic Studies* **23**, 27–31.

Jones, C. I. (2005), "The shape of production functions and the direction of technical change," *Quarterly Journal of Economics* **120**, 517–49.

Jones, L. E., and R. E. Manuelli (1990), "A convex model of equilibrium growth: Theory and policy implications," *Journal of Political Economy* **98**, 1008–38.

Klump, R., and O. de La Grandville (2000), "Economic growth and the elasticity of substitution: two theorems and some suggestions," *American Economic Review* **90**, 282–91.

Kortum, S. S. (1997), "Research, patenting, and technological change," *Econometrica* **65**, 1389–419.

Lotka, A. J. (1926), "The frequency distribution of scientific productivity," *Journal of the Washington Academy of Sciences* **16**, 317–23.

Nelsen, R. B. (1999), *An Introduction to Copulas*, New York: Springer.

Palivos, T., and G. Karagiannis (2004), "The elasticity of substitution in convex models of endogenous growth," mimeo. University of Macedonia at Thessaloniki.

Sattinger, M. (1975), "Comparative advantage and the distributions of earnings and abilities," *Econometrica* **43**, 455–68.

Solow, R. M. (1956), "A contribution to the theory of economic growth," *Quarterly Journal of Economics* **70**, 65–94.

Solow, R. M. (1960), "Investment and technological progress," K. J. Arrow, S. Karlin, and P. Suppes, eds, *Mathematical Methods in Social Sciences 1959*, 80–104, Stanford: Stanford University Press.

Solow, R. M., J. Tobin, C. von Weizsäcker, and M. Yaari (1966), "Neoclassical growth with fixed factor proportions," *Review of Economic Studies* **33**, 79–115.

Uzawa, H. (1961), "Neutral inventions and the stability of growth equilibrium," *Review of Economic Studies* **28**, 117–24.

Yuhn, K.-H. (1991), "Economic growth, technical change biases, and the elasticity of substitution: A test of the de La Grandville hypothesis," *Review of Economics and Statistics* **73**, 340–46.