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AUTOMATION, PARTIAL AND FULL

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When some steps of a complex, multi-step task are automated, the demand for human work in the remaining complementary sub-tasks goes up. In contrast, when the task is fully automated, the demand for human work declines. Upon aggregation to the macroeconomic scale, partial automatability of complex tasks creates a bottleneck of development, where further growth is constrained by the scarcity of essential human work. This bottleneck is removed once the tasks become fully automatable. Theoretical analysis using a two-level nested constant elasticity of substitution production function specification demonstrates that the shift from partial to full automation generates a non-convexity: humans and machines switch from complementary to substitutable, and the share of output accruing to human workers switches from an upward to a downward trend. This process has implications for inequality, the risk of technological unemployment, and the likelihood of a secular stagnation.

Keywords: Automation, Task, Complementarity, Economic Growth

1. INTRODUCTION

Thinking about the long-run impact of automation on the economy, the threat of technological unemployment, or the likelihood of an upcoming secular stagnation, it is important to account also for mechanisms that may not have been important in the past but are likely to intensify in the future. In this paper, I discuss one such mechanism: *a shift from partial to full automation of complex tasks*. The key theoretical insight of this paper is as follows: if a task is complex—that is, requires completion of at least two complementary sub-tasks—then it makes a crucial difference if only some or all of the sub-tasks are automatable. Automating some but not all sub-tasks increases the relative value of (and returns to) sub-tasks that have not been automated. Automating all sub-tasks undoes this effect. Partial automatability makes people and programmable machines complementary, whereas full automatability makes them substitutable. For this reason, growing wages and stable employment are safe only when full automation is technologically infeasible. For the very same reason, though, achieving

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full automatability of complex tasks can generate a substantial permanent boost to output growth.

Think, for example, of the photography industry, represented by companies of two technological eras: Kodak and Instagram. “Kodak was founded in 1880, and at its peak employed nearly 145,300 people, with many more indirectly employed via suppliers and retailers. Kodak’s founding family, the Eastmans, became wealthy, while providing skilled jobs for several generations of middle-class Americans. Instagram was founded in 2010 by a team of fifteen people. In 2012 it was sold to Facebook for over one billion dollars. Facebook, worth far more than Kodak ever was, employs fewer than 5000 people. At least ten of them have a net worth ten times that of George Eastman.” (https://www.newstatesman.com/politics/2014/01/kodak-vs-instagram-why-its-only-going-get-harder-make-good-living[access: 13.12.2019]) Gradual technological improvements and partial automation in the photography industry have been benefiting companies like Kodak for decades, increasing their employment and the overall wage bill. By contrast, full automation exemplified by Instagram reversed this trend. Once the entire multi-step task of providing the service—in this case sharing a certain visual experience with oneself and others—could be provided to the consumer without any human input, employment and the wage bill in the photography industry plummeted. What rose instead was the returns to automation technology (computer software) and, in effect, profits and shareholder value of companies like Instagram.

A similar case to consider is retail sales of books. “When Amazon.com sold its first book 20 years ago, Borders Books & Music had a thriving retail empire generating about $1.6 billion a year in sales. Today, Borders is nothing but a memory, ushered to the grave by an e-commerce revolution led by Amazon.” (https://www.sfchronicle.com/business/article/How-Amazon-factor-killed-retailers-like-6378619.php [access: 13.12.2019]) And again, it is not that Borders did not automate at all: they adopted electronic inventory management, invoicing systems, put up an online catalog, etc. But in their case, automation was only partial, and they did not make the final step which Amazon did: offering the entire service—putting a certain book in a certain person’s hand—without any human input. Just like in the photography industry, the real game changer here was the shift from partial to full automation. Amazon’s new technology was *disruptive*.

The key reason to believe that in the coming decades we will see many more industries disrupted by a shift from partial to full automation is that growth in the digital sphere, responsible for all progress in automation, is now an order of magnitude faster than growth in the global capital stock and gross domestic product (GDP): data volume, processing power, and bandwidth double every 2–3 years, whereas global GDP doubles every 20–30 years. In particular, since the 1980s “general-purpose computing capacity grew at an annual rate of 58%. The world’s capacity for bidirectional telecommunication grew at 28% per year, closely followed by the increase in globally stored information (23%)” [Hilbert and Lopez (2011)]. The costs of a standard computation have been declining by 53% per year on average since 1940 [Nordhaus (2017)]. The processing, storage, and communication of information have decoupled from the cognitive capacities of the human brain. “Less than 1% of information was in digital format in the mid-1980s, growing to more than 99% today” [Gillings et al. (2016)]. Preliminary evidence also suggests that since the 1980s the efficiency of computer algorithms has been improving at a pace that is of the same order of magnitude as accumulation of digital hardware [Grace (2013); Hernandez and Brown (2020)]. Corroborating this finding, in the recent decade we have witnessed a surge in artificial intelligence (AI) breakthroughs based on the methodology of *deep neural networks* [Tegmark (2017)], from autonomous vehicles and simultaneous language interpretation to self-taught superhuman performance at chess and Go [Silver et al. (2018)].

In the current paper, I formalize the consequences of a shift from partial to full automation with a simple model of tasks that consist of two sub-tasks. I assume that sub-tasks within a task are complementary, whereas within each sub-task people and machines are substitutable. I use this model to compare the “partial automation” scenario where some of the sub-tasks can be performed only by a human against a “full automation” scenario where all sub-tasks can be performed both by a human and a pre-programmed machine. I assume perfect competition at factor markets and perfect mobility of factors across the sub-tasks, so that their remuneration in both sub-tasks is equalized. In each of the considered scenarios, I compute the equilibrium allocation of factors across sub-tasks, factor shares, and wage rates. I then analyze how these numbers change with technological progress and the accumulation of programmable machines able to perform automatable tasks. (My results can be easily generalized to tasks consisting of an arbitrary number of sub-tasks.)

I find that when an essential sub-task of a complex task is not automatable, deepening of automation in other, automatable sub-tasks make the scarce human input increasingly valuable, thereby increasing wages and the labor share of output towards unity. Then, as the depth of automation becomes sufficiently high, human work becomes the bottleneck of further economic growth. In the absence of population growth and labor-augmenting technical change, the economy then eventually stagnates. When all tasks are automatable, in contrast, deepening of automation makes the scarce human input increasingly *less* valuable, decreasing the labor share of output towards zero. As automation technology becomes sufficiently advanced, human work becomes unimportant for production. Economic growth continues unabated even in the absence of population growth and labor-augmenting technical change; however, its fruit is then increasingly captured by (owners of) pre-programmed machines and their software, not the human workers.

The shift from partial to full automatability of tasks creates a non-convexity in economic development, where human and machine inputs switch from complementary to substitutable (in the sense of the aggregate elasticity of substitution, cf. Miyagiwa and Papageorgiou (2007); Xue and Yip (2013)). While boosting growth, it also generates a secular upward trend in inequality by gradually redirecting income from a wide population of workers to a relatively narrow group of owners of programmable machines and their software.

The theoretical results obtained in the current study are also helpful in answering the important question whether automation will bring technological unemployment. Will a robot take your job? Will humans go the way of horses? Does technological progress destroy fewer or more jobs than it creates? [Brynjolfsson and McAfee (2014); Frey and Osborne (2017); Autor and Solomons (2018)] On past evidence, the overall balance has been positive thus far: even if routine jobs were succumbing to automation [Autor and Dorn (2013)], these falls have been compensated—and in aggregate value terms, more than compensated—by the rise of high-skill, non-routine cognitive tasks and occupations [(“frontier jobs”, Autor, 2019) as well as the auxiliary low-skilled ones (“wealth work” and “last mile jobs”). However, this conclusion is not guaranteed to persist: as more and more sectors become fully automated, even these jobs may eventually disappear.

The current paper also informs the debate on the future of global economic growth—whether we should expect secular stagnation [Jones (2002); Gordon (2016)], further exponential growth, or a technological singularity [Kurzweil (2005)]. The key finding here is that secular stagnation requires setting a firm limit to automatability: for such a scenario to materialize there must always exist an essential task in the economy, complementary to all others, which cannot be fully automated. Otherwise, production will get increasingly automated and aggregate output will gradually decouple from human work, becoming instead proportional to the work done by pre-programmed machines.

The current study is related more broadly to studies focusing on automation and its impacts on productivity, employment, wages, and factor shares [Acemoglu and Autor (2011); Autor and Dorn (2013); Graetz and Michaels (2018); Acemoglu and Restrepo (2018); Andrews et al. (2016); Arntz et al. (2016); Frey and Osborne (2017); Barkai (2020); Autor et al. (2020); Jones and Kim (2018); Hemous and Olsen (2018); Prettner (2019)]. It also touches the nascent literature on macroeconomic implications of development of “digital labor,” AI, and autonomous robots [Yudkowsky (2013); Graetz and Michaels (2018); Sachs et al. (2015); Benzell et al. (2015); DeCanio (2016); Acemoglu and Restrepo (2018); Aghion et al. (2019); Berg et al. (2018); Benzell and Brynjolfsson (2019); Lu (2020)]. In particular, Benzell and Brynjolfsson (2019) consider a model where automation replaces capital and labor but is complementary to a scarce factor “genius.” The predictions of this model are very similar to the “partial automation” scenario, with “genius” acting as the human work necessary for carrying out the essential non-automatable sub-task.

In contrast to the voluminous literature assuming aggregate Cobb-Douglas or constant elasticity of substitution (CES) production, the considered model generates secular changes in the aggregate elasticity of substitution between human and machine inputs σ [Growiec and Schumacher (2008); Kemnitz and Knoblach (2020)]. Specifically, the shift from partial to full automatability of tasks is predicted to eventually reverse the trend in σ from a downward to an upward one. Accordingly, detailed empirical analysis provided by Growiec and Mućk (2020) suggests a protracted decline in the US capital-labor substitution elasticity σ since 1980s, in line with predictions of the "partial automation" scenario and the early stage of the "full automation" scenario. In contrast, estimates obtained by Cantore et al. (2017) and Kemnitz and Knoblach (2020) suggest a growing σ, predicted for the late stage of the "full automation" scenario, which is very unlikely to have happened yet.

The current paper is also complementary to a highly influential line of work by Acemoglu and Restrepo (2018, 2019a,b). These papers highlight that automation has two opposing effects on labor demand: the *displacement effect* (people are displaced by machines in performing certain tasks and sub-tasks) and the *productivity effect* (increased overall productivity raises demand for all factors of production, including human labor). The latter effect operates particularly effectively through increasing demand for human work in complementary (sub-)tasks which have not been automated yet. Acemoglu and Restrepo (2019*a*) also consider the possibility of *deepening of automation*, understood as increases in the productivity of machines employed in tasks already automated. Finally, these authors also show that creation of new tasks or sub-tasks which cannot yet be automated brings forward a *reinstatement effect*, creating additional demand for human work in these new (sub-)tasks. The current study highlights, though, that Acemoglu and Restrepo's analysis hinges on the assumptions that (i) there will always be some non-automatable (sub-)tasks, and (ii) new (sub-)tasks are created as (at least temporarily) non-automatable. In consequence, the "full automation" scenario, where *all* sub-tasks are automatable and the potential new sub-tasks are automatable, too, is not covered by these investigations. And this makes a difference because in the "full automation" scenario the positive productivity effect of deepening of automation will no longer translate into increased demand for human work, and neither will the creation of new (sub-)tasks reinstate labor. For this reason, I believe that when thinking about the long-run impact of automation on the economy, the "full automation" scenario deserves some attention as well.

Last but not least, the current paper can also be viewed in conjunction with my other one [Growiec (2019)] in which I formalize the distinction between mechanization and automation with the hardware–software model. What I refer to as "human and machine work" in the current paper is equivalent to "human cognitive work and pre-programmed software" within the *software* factor discussed there. Keeping this in mind, it is straightforward to observe that mechanization initiated in the Industrial Revolution had vastly different implications for factor shares than automation which began with the Digital Revolution: the former featured replacement of humans with machines in the hardware factor (brawn) whereas the latter pertains to the software factor (brains). Mechanization raised demand for human cognitive work, automation replaces it. Amidst advancing automation, demand for human cognitive work can go up only to the extent it is complementary to the automated (sub-)tasks, that is, only as long as automation is partial.

2. MODEL OF PARTIAL AND FULL AUTOMATION

2.1. Setup

Consider a task T consisting of two complementary sub-tasks T_1 and T_2. Output of task T is modeled with a normalized CES function with constant returns to scale [Klump and de La Grandville (2000); Klump et al. (2012); McAdam and Willman (2013)] and gross complementarity of the two sub-tasks. This functional form is a convenient analytical device allowing me to capture the key requisite property of sub-tasks: that each of them is essential in producing T and cannot be fully replaced by the other sub-task, no matter how large the discrepancy is between their output. Formally,

$$T = T_0 \left(\pi_0 \left(\frac{T_1}{T_{01}} \right)^{\varepsilon} + (1 - \pi_0) \left(\frac{T_2}{T_{02}} \right)^{\varepsilon} \right)^{\frac{1}{\varepsilon}}, \qquad \varepsilon < 0, \tag{1}$$

where the parameter $\varepsilon < 0$, signifying gross complementarity, links to the elasticity of substitution via $\sigma = \frac{1}{1-\varepsilon} \in (0, 1)$. The constant parameter $\pi_0 \in (0, 1)$ is the share of sub-task 1 in the total output of task T at the point of normalization. Other variables with subscript 0 are additional (positive) normalization constants [Klump and de La Grandville (2000)].

Output of each of the sub-tasks is also modeled using the normalized CES form, but with gross substitutability of machine and human inputs:

$$T_i = T_{0i} \left(\pi_{0i} \left(\psi_i \frac{K}{K_0} \right)^{\theta} + (1 - \pi_{0i}) \left(n_i \frac{L}{L_0} \right)^{\theta} \right)^{\frac{1}{\theta}}, \quad i = 1, 2, \quad \theta \in (0, 1]. \tag{2}$$

The elasticity parameter $\theta \in (0, 1]$, implying the elasticity of substitution within sub-tasks $\sigma_i = \frac{1}{1-\theta} > 1$, signifies gross substitutability of humans and machines in each of the two sub-tasks. In the polar case $\theta = 1$ ($\sigma_i = +\infty$), both inputs are perfectly substitutable in production; if $\theta \in (0, 1)$, then there is a little complementarity, but nevertheless neither people nor machines are essential for producing T_i, that is, positive output can be produced also without them.

In this context, by *automation* I will mean replacing human work L with machine work K within sub-tasks—assuming that human work L follows the instructions coming from the worker's brain, whereas machine work K follows the instructions included in pre-programmed machine code. In the ensuing analysis, I will compare two scenarios: (i) *partial automation*, where sub-task 2 is not automatable and can be performed by humans only, and (ii) *full automation*, where both tasks can be performed both by humans L and programmable machines K. I will also discuss the consequences of a shift from scenario (i) to (ii).

It must be emphasized that in this model K really denotes *programmable machines* (e.g., computers or robots) and not just any kind of physical capital—and by the same token, growth in K/L over time represents the accumulation of programmable machines per worker and not capital deepening. The difference is that traditional, non-programmable machines cannot store and run their code, and therefore are dependent upon humans to operate them, invalidating our key assumption of gross substitutability of people and machines within sub-tasks.

$K > 0$ is the total supply of programmable machines in the economy, whereas $L > 0$ is total employment of people. Both quantities are taken as given. (Later on, in Section 2.4 I will endogenize labor supply to allow for a discussion of the possible emergence of technological unemployment under partial vs. full automation.)

The parameter ψ_i captures the (productivity-adjusted) share of machines employed in performing sub-task i, such that $\psi_1 + \psi_2 = \psi$, where $\psi > 0$ represents the overall unit productivity of machines. Analogously, n_i captures the (productivity-adjusted) share of people employed in performing sub-task i, such that $n_1 + n_2 = n$, where $n > 0$ represents the overall unit productivity of human work.

Since the number ψ represents the unit productivity of programmable machines K, increases in ψ over time represent progress in efficiency of machine work, stemming, for example, from improved machine architecture or improved algorithms. In turn, because n represents the number of productivity-adjusted hours worked per human worker, increases in n may thus represent either increases in average hours worked or in the average unit productivity of an hour worked.

In the partial automation scenario, it is assumed that $\psi_2 = 0$ and thus $\psi_1 = \psi$, that is, all machines must be allocated to sub-task 1. In the full automation scenario, by contrast, $\psi_2 \in [0, \psi]$. The partial automation scenario is therefore a constrained variant of the more general full automation scenario.

Finally, observe that the current setup can be understood as encompassing any finite number of sub-tasks: "sub-task 1" is a catch-all term covering all sub-tasks which are automatable, and "sub-task 2" includes all sub-tasks which are non-automatable under partial automation but automatable under full automation.

2.2. Notation and Preliminary Results

Given perfect competition and constant returns to scale, the shares of sub-tasks 1 and 2 in output sum up to unity. The shares of humans and machines in each of the sub-tasks sum up to unity, too. Under the normalized CES specification, the shares are computed as follows:

$$\pi = \pi_0 \left(\frac{T_1}{T_{01}} \frac{T_0}{T} \right)^{\varepsilon}, \quad \text{share of sub-task 1,} \tag{3}$$

$$\pi_1 = \pi_{01} \left(\frac{\psi_1 K}{K_0} \frac{T_{01}}{T_1} \right)^{\theta}, \quad \text{machines share in sub-task 1,} \tag{4}$$

$$\pi_2 = \pi_{02} \left(\frac{\psi_2 K}{K_0} \frac{T_{02}}{T_2} \right)^{\theta}, \quad \text{machines share in sub-task 2.} \tag{5}$$

The overall machines share of output is $\pi_K = \pi\pi_1 + (1 - \pi)\pi_2$, and the human labor share is $\pi_L = 1 - \pi_K = \pi(1 - \pi_1) + (1 - \pi)(1 - \pi_2)$.

Using normalized intensive units, $k=\frac{K}{K_0}\frac{L_0}{L}, t_i=\frac{T_i}{T_{0i}}\frac{L_0}{L}, t=\frac{T}{T_0}\frac{L_0}{L}$, we obtain $\pi=\pi_0(t_1/t)^{\varepsilon}$, and for $i=1,2$, $\pi_i=\pi_{0i}(\psi_i k/t_i)^{\theta}$.

In any interior solution, equalization of wages across sub-tasks 1 and 2 ($w_1=w_2$) yields

$$w_1=\frac{\partial T}{\partial(n_1L)}=\frac{\partial T}{\partial(n_2L)}=w_2 \qquad \Longleftrightarrow \qquad \frac{n_1}{n_2}=\frac{\pi}{1-\pi}\frac{1-\pi_1}{1-\pi_2}. \tag{6}$$

Furthermore, in the full automation scenario where both sub-tasks are automatable, equalization of rental rates of machines across sub-tasks 1 and 2 ($r_1=r_2$) yields

$$r_1=\frac{\partial T}{\partial(\psi_1K)}=\frac{\partial T}{\partial(\psi_2K)}=r_2 \qquad \Longleftrightarrow \qquad \frac{\psi_1}{\psi_2}=\frac{\pi}{1-\pi}\frac{\pi_1}{\pi_2}. \tag{7}$$

In the partial automation scenario where sub-task 2 is not automatable, equation (7) ceases to hold and all machines are allocated to sub-task 1 where they are remunerated according to their marginal product.

By the *depth of automation*, I will consequently mean the ratio $\psi k/n$, proportional to $\frac{\psi K}{nL}$. Furthermore, when discussing long-run implications of the partial and full automation scenarios I will identify the impact of *deepening of automation*, which I will understand as an exogenous increase in the ratio $\psi k/n$ over time. Deepening of automation, that is, growth in unit productivity of machines relative to human work, can be achieved either at the extensive margin (increase in k—accumulation of programmable hardware per worker in the form of, for example, raw computing power) or at the intensive margin (increase in ψ/n—relatively faster increase in unit productivity of machines than people). The latter channel represents, among other possibilities, computer software development and deployment of increasingly capable AI algorithms which learn from data. (Substitution parameters θ and ε are assumed constant over time, although in principle one could also extend the analysis and consider their dynamics. In particular, in the *full automation* scenario exogenous growth in θ would further reduce the importance of human work for overall output and thereby augment the effects of deepening of automation.)

2.3. Results under Perfect Substitutability ($\theta = 1$)

To simplify the exposition, in the following paragraphs I will assume perfect substitutability of people and machines within each sub-task, $\theta=1$. This special case is particularly transparent insofar as it implies that in equilibrium, output at the level of the whole task follows a piecewise CES function. In Appendix A, I relax the assumption of perfect substitutability of people and machines within each sub-task, allowing for $\theta\in(0,1)$. Analyzing this more general case I find that although aggregate output no longer follows a piecewise CES function then, and therefore the respective formulas are somewhat less transparent, all key findings remain intact.

2.3.1. Partial automation: Sub-task 2 not automatable ($\psi_2 = 0$). When $\theta = 1$, from wage equalization (6) we get

$$\frac{t_1}{t_2} = \left(\frac{\pi_0}{1-\pi_0} \frac{1-\pi_{01}}{1-\pi_{02}} \right)^{\frac{1}{1-\varepsilon}}. \tag{8}$$

Hence, the interior equilibrium requires that output from sub-tasks 1–2 must come in a fixed proportion. In the following analysis, I will denote this ratio as $\xi > 0$.

Equilibrium allocation. Assuming $\psi_2 = 0$ and thus $\psi_1 = \psi$, and therefore $t_1 = \pi_{01}(\psi k) + (1-\pi_{01})n_1$ and $t_2 = (1-\pi_{02})n_2$, I obtain that employment in sub-task 1 equals

$$n_1 = \begin{cases} \dfrac{\xi(1-\pi_{02})n - \pi_{01}(\psi k)}{\xi(1-\pi_{02}) + (1-\pi_{01})}, & \text{if } \dfrac{\psi k}{n} \leq \dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \\ 0, & \text{if } \dfrac{\psi k}{n} > \dfrac{\xi(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{9}$$

If the depth of automation $\psi k/n$ is low (below a certain exogenous threshold), human work is used in both sub-tasks. The economy is then in an interior equilibrium. If the depth of automation is high, though, the economy finds itself in a corner equilibrium where human work is used only in sub-task 2 which is not automatable.

Intuitively, as long as people are employed in both sub-tasks, their wages must be equalized in equilibrium (otherwise they could move to the other sub-task and earn more there, and such allocation would not constitute an equilibrium). When the depth of automation is sufficiently high, though, firms operating the automatable sub-task 1 will prefer to employ machines only. Human workers will then be remunerated proportionally to their marginal product in the scarce non-automatable sub-task 2.

Factor shares. Assuming $\psi_2 = 0$, the relative factor share in the economy, that is, the ratio of the machines share π_K to the human labor share π_L equals

$$\Pi = \frac{\pi_K}{\pi_L} = \frac{\pi\pi_1}{\pi(1-\pi_1) + (1-\pi)} = \frac{\pi_0\pi_{01}t_1^{\varepsilon-1}(\psi k)}{\pi_0(1-\pi_{01})t_1^{\varepsilon-1}n_1 + (1-\pi_0)t_2^{\varepsilon}}. \tag{10}$$

Hence,

$$\Pi = \begin{cases} \left(\dfrac{\pi_{01}}{1-\pi_{01}}\right)\dfrac{\psi k}{n}, & \text{if } \dfrac{\psi k}{n} \leq \dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \\ \left(\dfrac{\pi_{01}}{1-\pi_{01}}\right)\left(\dfrac{\pi_{01}\psi k}{(1-\pi_{02})n}\right)^{\varepsilon}, & \text{if } \dfrac{\psi k}{n} > \dfrac{\xi(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{11}$$

Equation (11) signifies that the relative factor share is crucially determined by the depth of automation, $\psi k/n$. The relationship is not monotone, though: if the depth of automation $\psi k/n$ is low and people are employed in both sub-tasks, it is positive, whereas if the depth of automation is high and people are employed only in the non-automatable sub-task 2, it is negative.

It is also instructive to compute the equilibrium wage rate, which is equal to:

$$w = w_2 = (1-\pi)(1-\pi_2)\frac{T}{n_2 L} = (1-\pi_0)(1-\pi_{02})\left(\frac{t_2}{t}\right)^{\varepsilon-1}\frac{T_0}{L_0}, \tag{12}$$

and thus is inversely related to the contribution of the non-automatable sub-task 2 to overall output, t_2/t. Over time, assuming that deepening of automation will gradually reduce the relative contribution of sub-task 2, wages are predicted to grow.

Aggregate elasticity of substitution. In line with equation (11), the following values of the aggregate elasticity of substitution between people and machines σ are obtained

$$\sigma = \begin{cases} +\infty, & \text{if } \frac{\psi k}{n} \leq \frac{\xi(1-\pi_{02})}{\pi_{01}}, \\ \frac{1}{1-\varepsilon}, & \text{if } \frac{\psi k}{n} > \frac{\xi(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{13}$$

This result has very important implications: as long as the depth of automation is low, human and machine work are perfectly substitutable at the level of the whole task because they are substitutable at the level of each sub-task and there is a degree of freedom to keep the ratio of both sub-tasks fixed in equilibrium. If the depth of automation is high, though, this degree of freedom is no longer present. When all human work is allocated to the non-automatable sub-task 2, it becomes complementary to machines because the human-operated sub-task 2 is complementary to the machine-operated sub-task 1.

Aggregate production. Inserting the equilibrium allocation of human work into final task output, we obtain a piecewise normalized CES production function with human and machine work as inputs, with a non-convexity around the threshold automation level $\frac{\xi(1-\pi_{02})}{\pi_{01}}$, where factor inputs switch from being substitutes to complements:

$$t = \begin{cases} (\pi_0 + (1-\pi_0)\xi^{-\varepsilon})^{\frac{1}{\varepsilon}}(\pi_{01}(\psi k) + (1-\pi_{01})n_1), & \text{if } \frac{\psi k}{n} \leq \frac{\xi(1-\pi_{02})}{\pi_{01}}, \\ (\pi_0(\pi_{01}\psi k)^{\varepsilon} + (1-\pi_0)((1-\pi_{02})n)^{\varepsilon})^{\frac{1}{\varepsilon}}, & \text{if } \frac{\psi k}{n} > \frac{\xi(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{14}$$

Impact of automation. Assuming that the depth of automation $\psi k/n$ will go up over time (reflecting technological progress and the accumulation of programmable machines able to perform sub-task 1), eventually it must cross the threshold $\frac{\xi(1-\pi_{02})}{\pi_{01}}$. From that moment on we arrive at a corner solution where all human work is allocated to the non-automatable sub-task 2, making human and machine work complementary (with a low elasticity of substitution $\frac{1}{1-\varepsilon}$). The human labor share of output grows, eventually to unity as $\psi k/n \to \infty$. Wages grow in negative sync with the declining contribution of sub-task 2 to overall output (t_2/t), mirroring the increasing scarcity of human work but eventually converge to a firm upper bound.

Long-run steady state. In the long-run steady state (in which $\psi k/n \to \infty$), all human work is allocated to the non-automatable sub-task 2 and the human labor share of output π_L is one. Output per worker approaches the upper limit:

$$t_{\max} = (1-\pi_0)^{\frac{1}{\varepsilon}}(1-\pi_{02})n. \tag{15}$$

In consequence, wages approach their respective upper limit:

$$w_{\max} = (1-\pi_0)^{\frac{1}{\varepsilon}}(1-\pi_{02})\frac{T_0}{L_0}. \tag{16}$$

In the long run, scarcity of human work is a bottleneck of development. In the absence of population growth and labor-augmenting technical change, total output is bounded above and further growth is impossible. The only way to circumvent this "underdevelopment trap" is to make all sub-tasks automatable, rendering the human input no longer essential for production.

2.3.2. Full automation: Both sub-tasks automatable ($\psi_2 > 0$). When both sub-tasks are automatable, both people and machines can be freely allocated to either of the two sub-tasks. In any interior equilibrium, wages and rental rates of machines must be equalized across the sub-tasks (equations (6) and (7)), implying that

$$\frac{n_1}{n_2} = \frac{\psi_1}{\psi_2}\left(\frac{\pi_2}{\pi_1}\frac{1-\pi_1}{1-\pi_2}\right). \tag{17}$$

However, further inspection reveals that with perfect substitutability of people and machines within sub-tasks ($\theta = 1$), equation (17) is either trivially satisfied if $\pi_{01} = \pi_{02}$ or otherwise leads to a contradiction, that is, an interior equilibrium does not exist.

Analyzing the typical case $\pi_{01} \neq \pi_{02}$, without loss of generality we may assume $\pi_{01} > \pi_{02}$, so that sub-task 1 is relatively more machine-intensive than sub-task 2. In this case, instead of an interior equilibrium three types of corner equilibria are possible: (i) with machines in both sub-tasks and humans only in sub-task 2, (ii) with machines only in sub-task 1 and humans only in sub-task 2, and (iii) with machines only in sub-task 1 and humans in both sub-tasks. The choice of equilibrium will depend critically on the depth of automation, $\psi k/n$.

Equilibrium allocation. The following allocation of human work across sub-tasks is derived:

$$n_1 = \begin{cases} \dfrac{\xi(1-\pi_{02})n - \pi_{01}(\psi k)}{\xi(1-\pi_{02}) + (1-\pi_{01})}, & \text{if } \dfrac{\psi k}{n} \leq \dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \\ 0, & \text{if } \dfrac{\psi k}{n} > \dfrac{\xi(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{18}$$

In turn, the allocation of machines is

$$\psi_1 k = \begin{cases} \psi k, & \text{if } \frac{\psi k}{n} \leq \frac{\zeta(1-\pi_{02})}{\pi_{01}}, \\ \frac{\zeta((1-\pi_{02})n + \pi_{02}(\psi k))}{\zeta\pi_{02} + \pi_{01}}, & \text{if } \frac{\psi k}{n} > \frac{\zeta(1-\pi_{02})}{\pi_{01}}, \end{cases} \tag{19}$$

where $\xi = \left(\frac{\pi_0}{1-\pi_0}\frac{1-\pi_{01}}{1-\pi_{02}}\right)^{\frac{1}{1-\varepsilon}}$, $\zeta = \left(\frac{\pi_0}{1-\pi_0}\frac{\pi_{01}}{\pi_{02}}\right)^{\frac{1}{1-\varepsilon}}$ and $\xi < \zeta$ because $\pi_{01} > \pi_{02}$.

If the depth of automation is low ($\psi k/n$ below the lower threshold), human work is used in both sub-tasks and machines are used only in sub-task 1. For intermediate values of $\psi k/n$ there is perfect specialization, so that sub-task 1 employs only machines, and sub-task 2 employs only people. (Observe that both these possibilities were present also in the case where sub-task 2 was not automatable.) Finally, if the depth of automation is high ($\psi k/n$ above the higher threshold), human work is used only in sub-task 2 and machines are employed in both sub-tasks. In other words, when machine production is sufficiently efficient, both sub-tasks get automated.

If $\pi_{01} = \pi_{02}$, then $\xi = \zeta$ and the intermediate case disappears.

Factor shares. When both tasks are automatable, the relative factor share Π equals:

$$\Pi = \frac{\pi_K}{\pi_L} = \frac{\pi\pi_1 + (1-\pi)\pi_2}{\pi(1-\pi_1) + (1-\pi)(1-\pi_2)} \tag{20}$$

$$= \frac{\pi_0\pi_{01}t_1^{\varepsilon-1}(\psi_1 k) + (1-\pi_0)\pi_{02}t_2^{\varepsilon-1}(\psi_2 k)}{\pi_0(1-\pi_{01})t_1^{\varepsilon-1}n_1 + (1-\pi_0)(1-\pi_{02})t_2^{\varepsilon-1}n_2}. \tag{21}$$

Hence,

$$\Pi = \begin{cases} \left(\frac{\pi_{01}}{1-\pi_{01}}\right)\frac{\psi k}{n}, & \text{if } \frac{\psi k}{n} \leq \frac{\xi(1-\pi_{02})}{\pi_{01}}, \\ \left(\frac{\pi_{01}}{1-\pi_{01}}\right)\left(\frac{\pi_{01}\psi k}{(1-\pi_{02})n}\right)^{\varepsilon}, & \text{if } \frac{\psi k}{n} \in \left(\frac{\xi(1-\pi_{02})}{\pi_{01}}, \frac{\zeta(1-\pi_{02})}{\pi_{01}}\right), \\ \left(\frac{\pi_{02}}{1-\pi_{02}}\right)\frac{\psi k}{n}, & \text{if } \frac{\psi k}{n} \geq \frac{\zeta(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{22}$$

Equation (22) signifies that in the full automation scenario the relationship between the relative factor share and the depth of automation $\psi k/n$ changes its direction in two threshold points. When the depth of automation $\psi k/n$ is low, the relationship is positive, for intermediate values of $\psi k/n$ (implying perfect factor specialization) it is negative, and when the depth of automation is high, it becomes again positive.

The equilibrium wage rate still follows equation (12) and thus is inversely related to the contribution of sub-task 2 to overall output, t_2/t. Following the results above, however, in the full automation scenario this percentage contribution is constant regardless of factor endowments, and so are wages:

$$w = (1-\pi_0)(1-\pi_{02})\left(\pi_0\zeta^{\varepsilon} + (1-\pi_0)\right)^{\frac{1-\varepsilon}{\varepsilon}} \frac{T_0}{L_0}. \tag{23}$$

This exact constancy of wages, irrespective of depth of automation, requires the assumption that human and machine work are perfectly substitutable within sub-tasks ($\theta = 1$). In Appendix A, I show that if this latter substitutability is imperfect ($\theta \in (0, 1)$), even in the full automation scenario wages will be positively related to the depth of automation (albeit the relation will be less than proportional).

Aggregate elasticity of substitution. In line with equation (22), the following values of the aggregate elasticity of substitution between people and machines σ are obtained:

$$\sigma = \begin{cases} +\infty, & \text{if } \dfrac{\psi k}{n} \leq \dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \\ \dfrac{1}{1-\varepsilon}, & \text{if } \dfrac{\psi k}{n} \in \left(\dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \dfrac{\zeta(1-\pi_{02})}{\pi_{01}}\right), \\ +\infty, & \text{if } \dfrac{\psi k}{n} \geq \dfrac{\zeta(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{24}$$

This result has the following implications: when both tasks are automatable, human and machine work are perfectly substitutable at the level of the whole task both when the depth of automation is low and when it is high. Complementarity occurs only in the intermediate case of full specialization, where all human work is allocated to sub-task 2 and all machines operate in sub-task 1. Unlike the partial automation scenario, though, this result is however reversed once the depth of automation exceeds the upper threshold $\zeta(1-\pi_{02})/\pi_{01}$. From that moment onwards, a new degree of freedom is opened—machines are then freely allocated across both tasks, and in equilibrium they are set such that the contribution of each sub-task is fixed (t_1/t and t_2/t are constant).

Aggregate production. Inserting the equilibrium allocation of people and machines into the final task output, we again obtain a piecewise normalized CES production function with human and machine work as inputs—but this time with two non-convexities ("kinks") around the two threshold levels of depth of automation. The elasticity of substitution is infinite if $\psi k/n$ is low or high, and equal to a low value of $\frac{1}{1-\varepsilon} < 1$ if $\psi k/n$ takes intermediate values:

$$t = \begin{cases} \left(\pi_0 + (1-\pi_0)\xi^{-\varepsilon}\right)^{\frac{1}{\varepsilon}} (\pi_{01}(\psi k) + (1-\pi_{01})n_1), & \text{if } \dfrac{\psi k}{n} \leq \dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \\ \left(\pi_0(\pi_{01}\psi k)^{\varepsilon} + (1-\pi_0)((1-\pi_{02})n)^{\varepsilon}\right)^{\frac{1}{\varepsilon}}, & \text{if } \dfrac{\psi k}{n} \in \left(\dfrac{\xi(1-\pi_{02})}{\pi_{01}}, \dfrac{\zeta(1-\pi_{02})}{\pi_{01}}\right), \\ \left(\pi_0\zeta^{\varepsilon} + (1-\pi_0)\right)^{\frac{1}{\varepsilon}} (\pi_{02}(\psi_2 k) + (1-\pi_{02})n), & \text{if } \dfrac{\psi k}{n} \geq \dfrac{\zeta(1-\pi_{02})}{\pi_{01}}. \end{cases} \tag{25}$$

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Impact of automation. Assuming that the depth of automation $\psi k/n$ will go up over time, it will first cross the lower threshold and eventually the upper threshold $\zeta(1-\pi_{02})/\pi_{01}$. From that moment onwards, all human work will be allocated in equilibrium to the relatively less machine-intensive sub-task 2 whereas machines will be used in both sub-tasks. Thanks to the new degree of freedom—allocation of machines across sub-tasks—human and machine work will then be perfectly substitutable.

Balanced growth path. In the long run, assuming that the depth of automation $\psi k/n$ will grow exogenously at an exponential rate g, so will grow the final output and output of each sub-task:

$$g = g_t = g_{t_1} = g_{t_2}. \tag{26}$$

Hence, *full automation unpins economic growth from the capacity of human workers and instead pins it to the capabilities of programmable machines*. This has a few important consequences. First, it fends off the risk of a secular stagnation driven by zero-to-negative population growth rates and the prospect that new labor-augmenting technologies may soon approach the limits of human cognition. Second, when scarcity of human work is no longer a growth bottleneck, the relative share of machines in generating output will grow exponentially, too ($g_\Pi = g$), and with a fixed wage rate the human labor share of output will decline, eventually to zero as $\psi k/n \to \infty$. The fraction of machines allocated to sub-task 1 will gradually decline from 1 to the fixed limit

$$\lim_{\psi k \to \infty} \frac{\psi_1}{\psi} = \frac{\zeta \pi_{02}}{\pi_{01} + \zeta \pi_{02}}. \tag{27}$$

Third, as humans and machines will eventually become perfect substitutes, the human input will no longer be essential for production.

2.4. Technological Unemployment?

"No longer essential for production" does not immediately imply "unemployed", though. Indeed in the discussion so far there was no technological unemployment because employment L was considered fixed. People supplied their labor inelastically for any wage. If this assumption is relaxed, though, leading to an upward sloping labor supply curve, results change. Under the partial automation scenario, deepening of automation increases employment in equilibrium; under full automation the effect is exactly opposite. Technological unemployment, it turns out, occurs only when tasks are fully automatable.

To see this, note that in the full automation scenario of the model discussed above, when $\psi k/n \to \infty$ final output becomes proportional to output of either of the two sub-tasks, both of which follow a CES production function with gross substitutability between inputs ($\theta \in (0, 1]$). In particular in the perfect substitutability case ($\theta = 1$) final output becomes linear in the human and machine input once the depth of automation $\psi k/n$ exceeds a certain finite threshold. The human labor share of output tends to zero. Under partial automation, in contrast, as $\psi k/n \to \infty$ final output is driven exclusively by the scarce non-automatable sub-task 2 which constitutes a growth bottleneck. The elasticity of substitution between people and machines tends to the low value of $\frac{1}{1-\varepsilon} < 1$, and the human labor share of output tends to unity. Under full automation, human work is expendable; under partial automation, it is essential.

As a simple example illustrating how technological unemployment emerges for high $\psi k/n$ under full but not partial automation, consider the static problem of a representative household which maximizes utility from consumption and leisure subject to the constraint that all output is immediately consumed, taking ψk as given:

$$\max_{n\in[0,\bar{n})} u(c, \bar{n}-n) = \alpha \ln c + (1-\alpha)\ln(\bar{n}-n), \qquad \alpha \in (0,1), \tag{28}$$

where

$$c = t = t_0 \left(\pi(\psi k)^{\gamma} + (1-\pi)n^{\gamma}\right)^{\frac{1}{\gamma}}, \qquad t_0 > 0, \pi \in (0,1), \tag{29}$$

and $\gamma \in (-\infty, 0) \cup (0, 1]$. The first-order condition is

$$(1-\pi)n^{\gamma-1}(\alpha\bar{n} - n) = (1-\alpha)\pi(\psi k)^{\gamma}. \tag{30}$$

To reflect the full automation scenario with a large $\psi k/n$, one should take $\gamma = \theta \in (0, 1]$, so that people and machines are gross substitutes in aggregate production. In particular in the linear case $\theta = 1$, we obtain an explicit solution:

$$n = \begin{cases} \alpha\bar{n} - (1-\alpha)\dfrac{\pi}{1-\pi}(\psi k), & \text{if } \dfrac{\psi k}{\bar{n}} \le \dfrac{\alpha}{1-\alpha}\dfrac{1-\pi}{\pi}, \\ 0, & \text{if } \dfrac{\psi k}{\bar{n}} > \dfrac{\alpha}{1-\alpha}\dfrac{1-\pi}{\pi}. \end{cases} \tag{31}$$

Hence, when the depth of automation is sufficiently large, the equilibrium wage becomes so low relative to the returns to programmable machines that no one chooses to work. In the less-than-linear case $\theta \in (0, 1)$ labor supply is never quite zero, but nevertheless systematically declining with deepening of automation: using the implicit function theorem from (30) it is obtained that $n^*(\psi k)$ decreases with ψk, ultimately to 0 as $\psi k \to \infty$.

Hence, under the full automation scenario with endogenous labor supply the decline in labor demand following from deepening of automation translates not only into a sub-par increase in wages (when output grows at a rate g, wages grow at a rate $(1-\theta)g$, see Appendix A for the derivations in the case $\theta \in (0, 1)$), but also into an overall decline in employment. Full automatability of complex tasks begets technological unemployment.

This result is a polar opposite to the partial automation scenario, which is obtained by taking $\gamma = \varepsilon < 0$ in equation (29), so that machines and people are gross complements, not substitutes, in production, and human work is essential for producing final output. In such a scenario, human work becomes increasingly scarce, and following the equation (30) labor supply $n^*(\psi k)$ increases with ψk, ultimately to $\alpha\bar{n}$ as $\psi k \to \infty$. All hands on deck for the essential non-automatable sub-task 2!

3. DISCUSSION AND CONCLUSIONS

The current paper has discussed a new mechanism that may strongly affect the economic consequences of automation over the long run: *a shift from partial to full automatability of complex tasks*. If tasks generating value added are complex—that is, consist of at least two complementary sub-tasks—it makes a big difference if they are partially or fully automatable. The critical question in this regard is whether all sub-tasks can be automated or at least one sub-task cannot.

A shift from partial to full automatability of complex tasks is disruptive for at least four reasons. First, once a task in fully automated, people and machines switch from being complementary to substitutable in production and long-run trends in factor shares are reversed.

Second, while both partial and full automation increase inequality relative to the scenario with no automation at all, full automation does so more strongly and through a different channel. Partial automation leads to increases in the skill premium and polarization in the labor market [Autor and Dorn (2013); Autor and Solomons (2018)]: low- and middle-skilled routine occupations are replaced with machines and pre-programmed algorithms while high-skilled jobs complementary to the automated routine occupations thrive and increase their output share. In contrast, full automation leads to a declining output share of all types of human work, whether skilled or unskilled, physical or cognitive. What rises instead is the share of output accruing to (the owners of) programmable machines and their software [Barkai (2020); Autor et al. (2020)]. Whether this increases inequality relative to the partial automation scenario depends on the dispersion of high cognitive skills (which benefit most under partial automation) in the population relative to ownership of programmable machines and their software (which benefit most under full automation). In my perception, partially corroborated by the analysis by Benzell and Brynjolfsson (2019), ownership of programmable machines is likely to be more concentrated than ownership of high cognitive skills, because (i) human skills are to some extent naturally dispersed (each of us has one brain and cannot freely accumulate brainpower), (ii) in contrast, computer hardware (data processing power, data storage capacity, bandwidth) is accumulable per capita, (iii) computer software can almost costlessly scale up to the available hardware, and (iv) there are increasing returns to scale in the global digital economy. If my assumptions are correct, full automation should then increase inequality more strongly than partial automation.

Third, partial automation increases the demand for human cognitive work (in complementary occupations) whereas full automation decreases it. Therefore only full, but not partial automation is conducive to technological unemployment.

Fourth, full automation undoes the bottleneck of development ("underdevelopment trap") created by the relative scarcity of human cognitive work under partial automation. Full automation allows economic growth to decouple from the capacity of human workers and instead pins it to the capabilities of programmable hardware. Thus, a secular stagnation—caused by the limited combined cognitive capacity of human workers—accrues only when automation is partial but not when it is full.

The current analysis has a few limitations that may be addressed in the future. First, I consider people and machines as homogeneous inputs. I assume in particular that all workers possess the requisite skills to perform all sub-tasks. In reality, however, some of the non-automatable sub-tasks complementary to the automated routine sub-tasks may be so demanding in terms of cognitive skills that a fraction of human population may be unable to contribute to them regardless of the offered wage. In such a scenario, technological unemployment may arise already with partial automation. (Note, however, that technological unemployment would *not* arise under partial automation if there were only quantitative productivity differences between workers.) Moreover, relaxing the assumption that all workers possess the requisite skills to perform all sub-tasks calls for the modeling of endogenous human capital decisions of the workers. This call is also prompted by the fact that automation tends to be accompanied with increased multi-tasking on behalf of the workers, further raising skill requirements in parallel to the advances of automation [Boucekkine and Crifo (2008)]. More generally, this raises the issue of transition dynamics from partial to full automation under endogeneity of human capital decisions.

Second, I disregard the creation of new tasks over time. If the process of creating new tasks is non-automatable [(as in Acemoglu and Restrepo, 2018)], then even with full automation of the existing tasks people may occupy themselves by inventing and performing new tasks until these new tasks eventually become automated as well. This could potentially reduce the risk of technological unemployment and reinstate some labor.

Third, I abstract from the increasing complexity of tasks. In reality, newly created tasks tend to have more and more sub-tasks, a number of which may be non-automatable. In my analysis, I merely lump all the sub-tasks into two representative ones, an automatable and a non-automatable one.

Fourth, in reality some of the progress in automation may be taking place by partitioning the non-automatable sub-tasks into finer sub-tasks and automating some of them while leaving an ever smaller percentage of the overall task to be performed by people. Both latter trends may be included in the model by allowing the normalization constants, in particular the shares $\pi_0, \pi_{01}, \pi_{02}$, to shift over time.

Last but not least, I disregard the fact that in reality deepening of automation, represented in the model by increases in the ratio $\psi k/n$, is endogenous and can be affected by policy interventions.

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APPENDIX A: RESULTS UNDER IMPERFECT SUBSTITUTABILITY ($\theta \in (0, 1)$)

In contrast to the case with $\theta = 1$, the case of imperfect substitutability of people and machines within each sub-task ($\theta \in (0, 1)$) excludes corner equilibria: there will be smooth transitions instead of discrete jumps. Moreover, output at the level of the whole task will no longer follow a piecewise CES function, making the derived formulas somewhat less transparent. Nevertheless, all key properties of both scenarios remain intact also when people and machines are less than perfectly substitutable within sub-tasks.

A.1. PARTIAL AUTOMATION: SUB-TASK 2 NOT AUTOMATABLE ($\psi_2 = 0$)

If $\theta \in (0, 1)$, from wage equalization across sub-tasks (6), we get

$$\frac{n_1}{n_2} = \left(\frac{\pi_0}{1-\pi_0} \frac{1-\pi_{01}}{1-\pi_{02}} \right)^{\frac{1}{1-\theta}} \left(\frac{t_1}{t_2} \right)^{\frac{\varepsilon-\theta}{1-\theta}}. \tag{A1}$$

Let me now assume that $\psi_2 = 0$ and thus $\psi_1 = \psi$, so that sub-task 2 is not automatable.

Equilibrium allocation. From the assumption that $\psi_2 = 0$ it is obtained that $t_1 = (\pi_{01}(\psi k)^\theta + (1-\pi_{01})n_1^\theta)^{1/\theta}$ and $t_2 = (1-\pi_{02})^{1/\theta} n_2$. Hence, employment in sub-task 1 solves the implicit equation:

$$\left(\pi_{01}(\psi k)^\theta + (1-\pi_{01})n_1^\theta\right)^{\frac{\varepsilon-\theta}{\theta}} n_1^{\theta-1}(n-n_1)^{1-\varepsilon} = \frac{1-\pi_0}{\pi_0} \frac{(1-\pi_{02})^{\frac{\varepsilon}{\theta}}}{(1-\pi_{01})}. \tag{A2}$$

As the left-hand side of (A2) is strictly decreasing in both n_1 and ψk, from the implicit function theorem it is easily obtained that (i) the solution for n_1 is unique, and (ii) the fraction of people employed in the automatable sub-task 1 gradually declines with the depth of automation. As $\psi k \to \infty$, the equilibrium share $n_1^*(\psi k)$ must fall to zero, and in the limit all employment will eventually become concentrated in the non-automatable sector. (Recall that in contrast to the case $\theta = 1$, the equilibrium is always an interior one when $\theta \in (0, 1)$.)

Intuitively, people's wages must be equalized across both sub-tasks in equilibrium. Deepening of automation makes the machine input relatively cheaper, causing firms operating the automatable sub-task 1 to gradually replace people with machines in production.

Factor shares. If $\psi_2 = 0$, the relative factor share Π equals:

$$\Pi = \frac{\pi_K}{\pi_L} = \frac{\pi\pi_1}{\pi(1-\pi_1)+(1-\pi)} = \frac{\pi_0\pi_{01}t_1^{\varepsilon-\theta}(\psi k)^\theta}{\pi_0(1-\pi_{01})t_1^{\varepsilon-\theta}n_1^\theta + (1-\pi_0)t_2^\varepsilon}. \tag{A3}$$

Hence, after some algebra,

$$\Pi = \left(\frac{\psi k}{n}\right)^\theta \left(\frac{n_1}{n}\right)^{1-\theta} \frac{\pi_{01}}{1-\pi_{01}}, \tag{A4}$$

where $n_1 = n_1^*(\psi k)$ is the solution to equation (A2).

Equation (A4) means that the relative factor share is crucially determined by the depth of automation, $\psi k/n$. The relationship is not monotone, though, because there are two opposing forces at play. On the one hand, there is a direct effect of depth of automation (one may call it a "productivity effect"), increasing the share of machines at the expense of human work. On the other hand, there is also an indirect effect through reallocation of human work across sub-tasks ("reallocation effect"), working in the other direction. The former one dominates when the depth of automation is small, whereas the latter dominates when the depth of automation is large. The overall relationship is inverted U-shaped.

It is also instructive to compute the equilibrium wage rate:

$$w = w_2 = (1-\pi)(1-\pi_2)\frac{T}{n_2 L} = (1-\pi_0)(1-\pi_{02})^{\frac{1}{\theta}}\left(\frac{t_2}{t}\right)^{\varepsilon-1}\frac{T_0}{L_0}. \quad \textbf{(A5)}$$

Thus, the wage rate is negatively related to the contribution of the non-automatable sub-task 2 to overall output, t_2/t. Over time, assuming that deepening of automation will gradually reduce the relative contribution of sub-task 2, wages are predicted to grow.

Aggregate elasticity of substitution. The aggregate elasticity of substitution is found to gradually decline with the depth of automation, from a high value of $\frac{1}{1-\theta} > 1$ when $\psi k/n = 0$ to a low value of $\frac{1}{1-\varepsilon} < 1$ as $\psi k/n \to \infty$, crossing unity in the process. While initially highly substitutable, for sufficiently large $\psi k/n$, human and machine work become gross complements at the level of the whole task, because the exclusively human-operated sub-task 2 is complementary to the mostly machine-operated sub-task 1. Specifically, the aggregate elasticity of substitution follows the equation [Xue and Yip (2013)]:

$$\sigma = \frac{\Sigma}{1-\theta} + \frac{1-\Sigma}{1-\varepsilon}, \quad \textbf{(A6)}$$

where $\Sigma = (1-\pi_1) + \pi_1\left(\frac{n_1}{n}\right)$, using the definition of factor shares within sub-tasks (4). (Note that as $\psi k/n \to \infty$, not only $n_1 \to 0$ but also $\pi_1 \to 1$ and thus $\Sigma \to 0$.)

Aggregate production. Inserting the equilibrium allocation of human work $n_1^*(\psi k)$, satisfying the implicit equation (A2), into final task output, we obtain the following aggregate production function with human and machine work as inputs:

$$t = \left(\pi_0(\pi_{01}(\psi k)^\theta + (1-\pi_{01})n_1^*(\psi k)^\theta)^{\frac{\varepsilon}{\theta}} + (1-\pi_0)(1-\pi_{02})^{\frac{\varepsilon}{\theta}}(n - n_1^*(\psi k))^\varepsilon\right)^{\frac{1}{\varepsilon}}. \quad \textbf{(A7)}$$

Impact of automation. Assuming that the depth of automation $\psi k/n$ goes up over time, human work is increasingly allocated to the non-automatable sub-task 2. From a certain moment onwards, human and machine work become gross complements (with an aggregate elasticity of substitution converging to the low value of $\frac{1}{1-\varepsilon} < 1$ as $\psi k/n \to \infty$). The human labor share of output grows, eventually to unity. Wages grow in negative sync with the declining contribution of sub-task 2 to overall output (t_2/t), mirroring the increasing scarcity of human work, but eventually converge to a firm upper bound.

Long-run steady state. In the long-run steady state (in which $\psi k/n \to \infty$), all human work is allocated to the non-automatable sub-task 2 and the human share of output π_L is one. Output approaches the upper limit:

$$t_{\max} = (1-\pi_0)^{\frac{1}{\varepsilon}}(1-\pi_{02})^{\frac{1}{\theta}} n. \quad \textbf{(A8)}$$

In consequence, wages approach their respective upper limit, too:

$$w_{\max} = (1-\pi_0)^{\frac{1}{\varepsilon}}(1-\pi_{02})^{\frac{1}{\theta}}\frac{T_0}{L_0}. \quad \textbf{(A9)}$$

In the long run, scarcity of human work is a bottleneck of development. In the absence of population growth and labor-augmenting technical change, total output is bounded above and further growth is impossible. The only way to circumvent this trap is to make all sub-tasks automatable, so that the human input could be no longer essential for production.

A.2. FULL AUTOMATION: BOTH SUB-TASKS AUTOMATABLE ($\psi_2 > 0$)

When both sub-tasks are automatable, both human work and machines can be freely allocated to either of them. Then an interior equilibrium solution is obtained where both humans and machines are employed in both sub-tasks, and wages and rental rates of machines are equalized across both sub-tasks (equation (17)):

$$\frac{n_1}{n_2} = \frac{\psi_1}{\psi_2} \left(\frac{\pi_{02}}{\pi_{01}} \frac{1-\pi_{01}}{1-\pi_{02}} \right)^{\frac{1}{1-\theta}}. \tag{A10}$$

In this equilibrium, both factors of production are always reallocated in unison, counteracting the complementarity between sub-tasks. We denote their ratio $\frac{n_1}{n_2}\frac{\psi_2}{\psi_1} = \left(\frac{\pi_{02}}{\pi_{01}} \frac{1-\pi_{01}}{1-\pi_{02}} \right)^{\frac{1}{1-\theta}}$ as $\mu > 0$. (Recall that in contrast to the case $\theta = 1$, the equilibrium is always an interior one when $\theta \in (0, 1)$.)

Equilibrium allocation. Assuming without loss of generality that $\pi_{01} \geq \pi_{02}$ so that sub-task 1 is relatively more machine-intensive, the allocation of workers across sub-tasks solves the implicit equation:

$$\frac{n_1}{n_2} = \left(\frac{\pi_0}{1-\pi_0} \frac{1-\pi_{01}}{1-\pi_{02}} \right)^{\frac{1}{1-\theta}} \left(\frac{t_1}{t_2} \right)^{\frac{\varepsilon-\theta}{1-\theta}}, \tag{A11}$$

where

$$\frac{t_1}{t_2} = \left(\frac{\pi_{01}(\psi_1 k)^\theta + (1-\pi_{01}) n_1^\theta}{\pi_{02}(\psi_2 k)^\theta + (1-\pi_{02}) n_2^\theta} \right)^{\frac{1}{\theta}} = \frac{n_1}{n_2} \left(\frac{\pi_{01}\left(\frac{\psi_2 k}{\mu n_2}\right)^\theta + (1-\pi_{01})}{\pi_{02}\left(\frac{\psi_2 k}{n_2}\right)^\theta + (1-\pi_{02})} \right)^{\frac{1}{\theta}}, \tag{A12}$$

and

$$\frac{\psi_2 k}{n_2} = \frac{\psi k}{n} \frac{\frac{n_1}{n_2}+1}{\frac{n_1}{\mu n_2}+1}. \tag{A13}$$

Application of the implicit function theorem implies that (i) a unique solution $n_1^*(\psi k)$ exists, and (ii) as long as $\pi_{01} > \pi_{02}$, allocation of human work in the first (more machine-intensive) sub-task $n_1^*(\psi k)$ declines with ψk. As $\psi k \to \infty$, n_1^* converges from above to a positive constant. With $n_1^*(\psi k)$ in hand, the allocation of machines $\psi_1^*(\psi k)$ is calculated from (A10).

The particular case $\pi_{01} = \pi_{02}$ implies $\mu = 1$. The division of factors across sub-tasks then does not depend on relative factor endowments:

$$\frac{\psi_1}{n_1} = \frac{\psi_2}{n_2} = \frac{\psi}{n}, \qquad \frac{t_1}{t_2} = \frac{n_1}{n_2} = \frac{\psi_1}{\psi_2} = \left(\frac{\pi_0}{1-\pi_0} \right)^{\frac{1}{1-\varepsilon}}. \tag{A14}$$

In such a case, the problem simplifies greatly and the aggregate production function retains the normalized CES form with a high elasticity of substitution $\frac{1}{1-\theta} > 1$.

Assuming that the depth of automation $\psi k/n$ grows over time, in the case $\pi_{01} > \pi_{02}$ (where sub-task 1 is relatively more machine-intensive) human work is gradually reallocated towards sub-task 2, and machines—towards sub-task 1. This is intuitive: each factor is reallocated towards the sector where it has a greater comparative advantage. In the case $\pi_{01} = \pi_{02}$, the division of factors between sub-tasks is fixed. In both cases, reallocation of factors across sub-tasks helps circumvent the fact that the sub-tasks are mutually complementary. In result, the high degree of substitutability between people and machines is passed from the level of sub-tasks to the level of the entire task.

Factor shares. When both tasks are automatable, the relative factor share in the economy Π equals

$$\Pi = \frac{\pi_K}{\pi_L} = \frac{\pi\pi_1 + (1-\pi)\pi_2}{\pi(1-\pi_1) + (1-\pi)(1-\pi_2)} \tag{A15}$$

$$= \frac{\pi_0\pi_{01}t_1^{\varepsilon-\theta}(\psi_1 k)^\theta + (1-\pi_0)\pi_{02}t_2^{\varepsilon-\theta}(\psi_2 k)^\theta}{\pi_0(1-\pi_{01})t_1^{\varepsilon-\theta}n_1^\theta + (1-\pi_0)(1-\pi_{02})t_2^{\varepsilon-\theta}n_2^\theta}. \tag{A16}$$

Hence, after some algebra,

$$\Pi = \left(\frac{\psi k}{n}\right)^\theta \left(\frac{1+\frac{n_1}{n_2}}{\mu+\frac{n_1}{n_2}}\right)^\theta \left(\frac{\frac{n_1}{n_2}\frac{\pi_{01}}{1-\pi_{01}} + \frac{\pi_{02}}{1-\pi_{02}}\mu^\theta}{\frac{n_1}{n_2}+1}\right), \tag{A17}$$

where n_1/n_2 solves equation (A11). If also $\pi_{01} = \pi_{02}$, equation (A17) simplifies to:

$$\Pi = \left(\frac{\psi k}{n}\right)^\theta \frac{\pi_{01}}{1-\pi_{01}}. \tag{A18}$$

The relative factor share is again crucially determined by the depth of automation, $\psi k/n$. There are now three forces at play. First, a direct effect of depth of automation ("productivity effect"), increasing the share of machines at the expense of human work. Second, an indirect effect through reallocation of human work across sub-tasks, counteracting it, and third, an indirect effect through reallocation of machines, supporting it. In the special case $\pi_{01} = \pi_{02}$ the latter two effects disappear and deepening of automation monotonically increases the machines share of output.

The equilibrium wage rate is now

$$w = w_2 = (1-\pi)(1-\pi_2)\frac{T}{n_2 L} = (1-\pi_0)(1-\pi_{02})\left(\frac{t_2}{t}\right)^{\varepsilon-1}\left(\frac{t_2}{n_2}\right)^{1-\theta}\frac{T_0}{L_0}, \tag{A19}$$

and thus is shaped by two factors: (i) the contribution of sub-task 2 to overall output t_2/t—which is constant in the full automation scenario—and (ii) output per worker in the more labor-intensive sub-task 2, t_2/n_2, which is rising with the deepening of automation. Overall, wages are positively related to the depth of automation, but the relation is less than proportional because of the exponent $1-\theta$.

Aggregate elasticity of substitution. Equation (A17) implies that when both tasks are automatable, human and machine work are generally highly substitutable at the level of the whole task. The complementarity between both sub-tasks is counteracted by reallocating the factors accordingly. Specifically, the aggregate elasticity of substitution satisfies the following equation [Xue and Yip (2013)]:

$$\sigma = \frac{\Sigma}{1-\theta} + \frac{1-\Sigma}{1-\varepsilon}, \tag{A20}$$

where

$$\Sigma = (1-\pi_1)\left(\frac{\psi_1}{\psi}\right) + \pi_1\left(\frac{n_1}{n}\right) + (1-\pi_2)\left(\frac{\psi-\psi_1}{\psi}\right) + \pi_2\left(\frac{n-n_1}{n}\right), \qquad \textbf{(A21)}$$

using the sub-task factor share definitions (4)–(5).

In the special case $\pi_{01} = \pi_{02}$, we obtain $\mu = 1$ and hence $\frac{n_1}{n} = \frac{\psi_1}{\psi}$, implying $\Sigma = \left(\frac{n_1}{n}\right) + \left(\frac{n-n_1}{n}\right) = 1$ and finally $\sigma = \frac{1}{1-\theta}$ at all times.

Aggregate production. Inserting the equilibrium allocation of human work $n_1^*(\psi k)$ and machines $\psi_1^*(\psi k)$, satisfying the implicit equations (A10)–(A13), into final task output we obtain the following aggregate production function with human and machine work as inputs:

$$t = \Bigg(\pi_0\left(\pi_{01}(\psi_1^*(\psi k)k)^\theta + (1-\pi_{01})n_1^*(\psi k)^\theta\right)^{\frac{\varepsilon}{\theta}} + \qquad \textbf{(A22)}$$

$$+ (1-\pi_0)\left(\pi_{02}((\psi-\psi_1^*(\psi k))k)^\theta + (1-\pi_{02})(n-n_1^*(\psi k))^\theta\right)^{\frac{\varepsilon}{\theta}}\Bigg)^{\frac{1}{\varepsilon}}.$$

In the particular case $\pi_{01} = \pi_{02}$, equation (A22) simplifies and the aggregate production function retains the normalized CES form with a high elasticity of substitution $\frac{1}{1-\theta} > 1$:

$$t = (1-\pi_0)^{\frac{1}{\varepsilon}}\left(\left(\frac{\pi_0}{1-\pi_0}\right)^{\frac{1}{1-\varepsilon}} + 1\right)^{\frac{1-\varepsilon}{\varepsilon}}\left(\pi_{01}(\psi k)^\theta + (1-\pi_{01})n^\theta\right)^{\frac{1}{1-\theta}}. \qquad \textbf{(A23)}$$

Impact of automation. Assuming that the depth of automation goes up over time, the human labor share of output declines, eventually to zero as $\psi k/n \to \infty$. Wages continue to grow indefinitely, albeit slower than output because of the falling labor intensity of sub-task 2. In the long run, total output grows proportionally to the depth of automation, $\psi k/n$.

Balanced growth path. In the long run, assuming that the depth of automation $\psi k/n$ will grow exogenously at an exponential rate g, so will grow the final output and output of each sub-task:

$$g = g_t = g_{t_1} = g_{t_2}. \qquad \textbf{(A24)}$$

Hence, again we see that *full automation unpins economic growth from the capacity of human workers and instead pins it to the capabilities of programmable machines*. This has a few important consequences. First, it fends off the risk of a secular stagnation driven by zero-to-negative population growth rates and the prospect that new labor-augmenting technologies may soon approach the limits of human cognition. Second, when scarcity of human work is no longer a growth bottleneck, the relative share of machines in generating output will grow exponentially, too ($g_\Pi = g$), and the human labor share of output will decline, eventually to zero as $\psi k/n \to \infty$.

Third, the fraction of people allocated to the respective sub-tasks will gradually converge to a finite limit:

$$\lim_{(\psi k/n)\to\infty} \frac{n_1}{n_2} = \left(\frac{\pi_0}{1-\pi_0}\right)^{\frac{1}{1-\varepsilon}}\left(\frac{1-\pi_{01}}{1-\pi_{02}}\right)^{\frac{1}{1-\theta}}\left(\frac{\pi_{01}}{\pi_{02}}\right)^{\frac{\varepsilon-\theta}{(1-\varepsilon)\theta(1-\theta)}} \equiv \nu, \qquad \textbf{(A25)}$$

and so will the fraction of machines, $\frac{\psi_1}{\psi_2} = \frac{n_1}{\mu n_2} \to \frac{\nu}{\mu}$, and the proportion t_1/t_2:

$$\lim_{(\psi k/n)\to\infty} \frac{t_1}{t_2} = \left(\frac{\pi_{01}}{\pi_{02}}\right)^{\frac{1}{\theta}} \frac{\nu}{\mu}. \quad \textbf{(A26)}$$

Consequently, factor shares in both sectors will tend to equalize, $\frac{\pi_1}{\pi_2} \to 1$.

Fourth, wages will eventually set on an exponential growth path:

$$g_w = (1-\theta)g. \quad \textbf{(A27)}$$

The positive but sub-par growth rate of wages ($0 < g_w < g$) comes from the fact that with $\theta \in (0, 1)$, there is a little complementarity between human and machine inputs, and thus a part of the productivity increase due to the deepening of automation spills over to the workers.

As $\psi k/n \to \infty$, overall output will become proportional to the output of each of the two tasks t_1, t_2 and the elasticity of substitution between people and machines will converge to $\frac{1}{1-\theta} > 1$, so that people and machines will be gross substitutes and the human input will no longer be essential for production. (Equation (A21) elucidates that as π_1 becomes equal to π_2, $\Sigma \to 1$.)