Technological opportunity, long-run growth, and convergence

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We derive a R&D-based growth model where the rate of technological progress depends, *inter alia*, on the amount of technological opportunity. Incremental innovations provide direct increases to the knowledge stock but they reduce technological opportunity and thus the potential for further improvements. Technological opportunity is renewed by radical innovations, which have no direct impact on factor productivity. We study both the market equilibrium and the social planner allocation in this economy. Investigating the model for its implications on economic growth we find: (i) in the long run, a balanced growth path requires that the returns to radical innovations are at least as large as those of the incremental ones; (ii) the transition need not be monotonic. We show under which conditions our model generates endogenous cycles via complex dynamics without resorting to uncertainty; (iii) the calibrated model exhibits substantial quantitative differences between the market outcome and the social planner allocation.

JEL classifications: E32, O30, O41.

1. Introduction

Ideas are not alike. Some ideas provide answers to certain questions, solve some puzzles, or at least add qualifications to the answers already known; others pose more questions than they actually answer. Some inventions provide mankind with readily useful technologies; others need centuries of hard intellectual work to develop. Some ideas leave loose threads hanging—they open up the opportunity for further developments—while others tie these hanging ends together, creating useful knowledge by taking advantage of the opportunities created previously.

It is clear that this type of heterogeneity in ideas must have an impact on the rate of technological progress. Nonetheless, such heterogeneity has rarely been considered in the literature. Specifically, the mainstream R&D-based growth literature assumes that the amount of newly created knowledge depends primarily

on the current stock of knowledge and the number of active researchers (e.g. Romer, 1990; Aghion and Howitt, 1992; Jones, 1995; Li, 2000), with extensions to human capital (e.g., Strulik, 2005), spillovers (e.g., Howitt, 1999; Li, 2002), and R&D difficulty (e.g., Segerstrom, 2000). It is presumed that as long as all these ingredients are supplied in the required quantities, the opportunity for innovations is unlimited. There does exist however ample empirical evidence, some of which we shall review in the next section, which suggests otherwise and points at an important role of the heterogeneity in innovations for economic growth.

In our view, one of the serious attempts to provide a satisfying modelling approach going in this direction has been undertaken by Olsson (2000, 2005). Both articles lay the foundations for the distinction between incremental and radical innovations, and offer a formal description for the concept of technological opportunity. On the downside, these two papers are somewhat detached from standard theories of long-run economic growth and convergence because they focus primarily on developing a set-theoretic model of ideas and explain why technological breakthroughs tend to cluster in time. This creates a gap in the literature which needs to be filled.

The crucial contribution of the current article is to span a bridge between the intuitive Olsson's theory and the mainstream literature on R&D-based semi-endogenous growth (e.g., Romer, 1990; Jones, 1995). We provide a common denominator for both these frameworks: our model is general enough to make it possible to analyse the impacts of technological opportunity as well as incremental and radical innovations on long-run growth and convergence.

This article follows Olsson (2005) in assuming that the R&D sector of the economy produces incremental and radical innovations, related as follows. Incremental innovations provide direct increases to the economy's productivity and utilize technological opportunity opened up by radical innovations. Radical innovations extend the existing technological opportunity by combining previous discoveries (abstract ideas which are initially useless) with existing knowledge. Technological opportunity behaves like a renewable resource here: it is exhausted by incremental innovations and renewed by radical innovations. As opposed to Olsson (2005), however, our model does not rely on a linearity assumption that gives rise to bang-bang solutions and non-smooth dynamics. In Olsson's model, all researchers work either in the incremental or in the radical innovation sector, but never in both at the same time. It is clear that the assumption leading to this result is rather strong; we manage to weaken it considerably. Our analysis confirms, however, Olsson's results qualitatively by signifying the importance of R&D labour re-allocations for oscillatory dynamics. In addition, we demonstrate that:

(i)   technological opportunity must be continuously renewed in order to sustain technological progress over the long run;

(ii) if the flows of radical innovations are too small in comparison to the flows of incremental innovations, technological opportunity will be gradually depleted and economic growth will eventually come to a halt, and if radical

innovations are relatively abundant, then the volume of unused technological opportunity would tend to explode;

(iii) the transition to the balanced growth path may be oscillatory: in addition to the standard result of monotonic convergence, we obtain the possibility of complex dynamics for a wide variety of parameter values. These complex dynamics can come in the form of oscillations, converging, stable or diverging, and Andronov–Hopf bifurcations. Thus, our model generates endogenous cycles without uncertainty.

The current article can also be viewed in comparison to the contributions by Li (2000, 2001, 2002) as well as Young (1993), Aghion and Howitt (1996), Cozzi and Galli (2008, 2009). All these authors accounted for the heterogeneity of ideas by considering a distinction between horizontal and vertical innovations, basic and applied research, or between technological and scientific knowledge. They posited different production technologies in both types of R&D and acknowledged their diverse roles in driving long-run growth. However, none of these contributions captured Olsson's idea directly—that one of the types increases whereas the other type decreases the stock of technological opportunity, on a one-to-one basis.

Perhaps the closest related contribution from this literature is Li (2001) who, like us, builds an R&D-based growth model where one type of innovations (the 'scientific' one) accelerates technological progress whereas the other (the 'technological' one) decelerates it. Li (2001) has also analysed the patterns of oscillatory dynamics in his economy. He has, however, assumed this type of behaviour of the model directly, by positing that scientific breakthroughs arrive in discrete jumps. In contrast, by modeling both types of innovations as smooth functions of time, we are able to obtain the cyclicality result endogenously. The other important difference is that in Li's model, there is no variable with the properties of technological opportunity in the sense discussed here.

When introducing different kinds of innovations into growth models, several researchers have already suggested the possibility of cycles to occur (Jovanovic and Rob, 1990; Aghion and Howitt, 1992; Cheng and Dinopoulos, 1992; Bresnahan and Trajtenberg, 1995; Amable, 1996; Freeman *et al.*, 1999; Francois and Lloyd-Ellis, 2003; Alvarez-Cuadrado *et al.*, 2004; Phillips and Wrase, 2006; Bramoulle and Saint-Paul, 2010). However, their approaches and therefore their implied sources of cyclical growth differ from ours. In particular, in none of these approaches does technological progress depend on an exhaustible factor, such as technological opportunity is in our case.

The remainder of the article is structured as follows. In Section 2 we lay out the foundations of our framework. We introduce our principal concepts and present the system of equations determining the temporal evolution of knowledge. In Section 3 we set up and solve our growth model. We analyse both the market equilibrium and the social optimum. We then discuss both the balanced growth path and transition dynamics and do sensitivity analysis on the important parameters. Section 4 concludes.

2. Technological opportunity and the evolution of knowledge

New developments vary not only in their impact on factor productivity, but also in their impact on the opportunity for further developments. This difference between two innovations can manifest itself in the magnitudes of productivity increments, but also in the magnitudes and the direction of change in the extent and efficiency of follow-up research. There are radical developments that open new avenues of thought and there are incremental developments that only 'fish out' technological opportunity. Intuitively, one could look at this distinction through the lens of the dichotomy between basic and applied research. Basic research is often carried out in fields rather detached from day-to-day economic activity, such as plasma physics, neurobiology, or experimental psychology, and it would typically be of little business interest but potentially of immense interest to scientists working in related areas, who are able to build on these developments. The principal role of basic research is thus to facilitate further research, both basic and applied. Applied research, on the other hand, would typically be based upon previous scientific developments and bring about new opportunities to business rather than to science. It would tend to capture the opportunities created by basic research while offering relatively less scientific stimuli in return. Applied research is vacuous without basic research, but basic research alone cannot guarantee technological progress whose benefits everyone can experience: applied research is the necesseary means for making productive use of the conquests of basic science.

Radical innovation as viewed in this paper resembles basic research in the sense presented above, while incremental innovation could be associated with applied research.[1] Olsson's set-theoretic setup should be understood as a metaphor of the evolution of human thought in reality. Consequently, all the concepts defined precisely in this metaphorical world (such as technological opportunity) are supposed to have real-world analogues. A number of examples have been listed in the online Appendix of this paper to emphasize the grounds for making such an analogy.

2.1 The laws of motion

Let us now clarify the notation of variables which shall be used in our growth model. By B_t we shall denote the amount of technological opportunity at time t, by R_t the flow of radical innovations, by C_t the flow of incremental innovations, and by A_t the current stock of knowledge. Technological opportunity is increased by radical innovations, whereas incremental innovations add to the stock of knowledge but diminish technological opportunity. We write $\dot{A}_t = C_t$ and $\dot{B}_t = R_t - C_t$ (Olsson, 2005).[2]

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[1]For precise definitions and the relationship between all these terms and the set-theoretic setup, see Olsson (2000, 2005) as well as the online Appendix.

[2]A promising alternative, somewhat departing from the Olsson's set-theoretic setup but also plausible empirically, would be to write $\dot{B}_t = R_t/C_t$. We leave this possibility for further research.

The set-theoretic setup as well as an intuitive rationale[3] equip us with the basic understanding of how innovations and knowledge evolve over time, and we shall proceed to provide a more specific characterization which would then allow us to solve for the precise dynamic paths. To assure analytical tractability as well as comparability to the semi-endogenous growth literature which evolved from Jones (1995), we use standard Cobb-Douglas functional forms. They read:

$$\dot{A}_t = \delta(u_t \ell_{At} L_t)^\beta B_t^\mu, \tag{1}$$

$$\dot{B}_t = \underbrace{-\delta(u_t \ell_{At} L_t)^\beta B_t^\mu}_{\text{incremental innovations}} + \underbrace{\gamma((1-u_t)\ell_{At} L_t)^\beta A_t^\nu}_{\text{radical innovations}}, \tag{2}$$

We denote total population by L_t and assume it to be equal to the total amount of working time available in the economy at time t. Population is assumed to grow exogenously at a constant rate $n > 0$. Then, ℓ_{At} is the proportion of working time devoted to R&D, with $u_t \ell_{At}$ being the proportion of time spent on working in the incremental innovation sector and $(1 - u_t)\ell_{At}$ the time spent in the radical innovation sector. The parameter $\delta > 0$ is proportional to the rate at which incremental innovations come about, whereas $\gamma > 0$ relates to the rate at which radical innovations arrive. The exponents μ and ν are crucial for the dynamic behaviour of our model. From our preceding argumentation, we know that they ought to be strictly positive. We shall further assume that $\mu, \nu \in (0, 1)$ (less-than-proportional external effects in R&D) which assures non-explosive semi-endogenous growth in the long run. The relative size of these exponents is decisive for the long-run evolution of innovations, which we investigate in the next subsection. The parameter $\beta > 0$ measures the degree of returns to scale with respect to employment in R&D sectors. Our assumption that $\beta \in (0, 1)$ implies decreasing returns to scale in R&D activity which are required for positive shares of both kinds of innovation to be pursued in equilibrium and stands in contrast to Olsson (2005) who has $\beta = 1$ and thus constant returns to scale, which leads to bang-bang solutions.

Let us now compare this setup with several standard R&D-based models of (semi-) endogenous growth, on the one hand, and with Olsson (2000, 2005) on the other. We shall see shortly that this article spans a bridge between these two frameworks and can be reduced to either one by neglecting certain assumptions. Romer's (1990) specification of technical change would correspond to eq. (1) only, with $B_t = A_t$ and $\mu = 1$. This leads to the scale effect discussed by Jones (1995) and implies explosive dynamics if $n > 0$. However, even Jones's R&D equation, which avoids scale effects, is only a subcase of our specification, corresponding to eq. (1) only, with $B_t = A_t$, $\mu < 1$ and $n > 0$. Hence, the system (1)–(2) can be reduced to the standard specifications of R&D equations found throughout the economic growth literature by identifying technological opportunity with technology itself and restricting attention to eq. (1).

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[3]Available in the online Appendix.

The two-R&D-sector model with horizontal and vertical innovation, specified by Li (2002), would correspond to a modified version of eq. (1) – with an additional spillover term going from A to $\dot{A}$ but then with $\beta = 1$ – coupled with an equation of form $\dot{B}_t = (1 - u_t)\ell_{At}L_tA_t^{\phi_A}B_t^{\phi_B}$. This change requires a substantial reinterpretation of the B variable: it is then no longer a measure of technological opportunity, exhausted by incremental innovations and unrelated to current factor productivity; instead, it is a measure of average product quality, which provides direct increases to factor productivity, and which might be negatively affected by incremental innovations (Li assumes $\phi_A \leqslant 0$) but never depleted, thanks to the multiplicative character of this specification.

In comparison to Olsson (2005), we allow for both incremental and radical innovations happening at the same time. We neglect Olsson's linearity assumption $\beta = 1$ that gives rise to bang-bang solutions and thus abrupt reallocations of R&D workers between radical and incremental R&D. We are also more specific on discoveries fueling radical innovation by linking them to the current technology level A_t, with elasticity v.

2.2 The long-run properties

We now focus on the properties of the balanced growth path (BGP) under different relative sizes of the exponents in the radical and incremental innovations, μ and v.[4] We do this by finding the necessary conditions under which the growth rates of all economic variables are constant. These conditions imply in particular that the sectoral allocation of labour does not change over time.

Let us take eqs (1)–(2) and solve for the BGP. Incremental innovations grow at a constant rate if $\hat{A} = \delta(u\ell_A L)^\beta B^\mu / A$ is constant.[5] Assuming a constant u_t requires that $\beta n + \mu\hat{B} = \hat{A}$. Since we assume constant population growth $n > 0$, then $\hat{A}$ is constant if $\hat{B}$ is constant. A constant growth rate of technological opportunity, $\hat{B} = R/B - C/B$, requires that the ratios R/B and C/B are constant.

We know that $C/B = \delta(u\ell_A L)^\beta B^\mu / B$ is constant if $\beta n + \mu\hat{B} = \hat{B}$, and $R/B = \gamma((1 - u)\ell_A L)^\beta A^v / B$ is constant if $\beta n + v\hat{A} = \hat{B}$. Both these ratios can thus be constant simultaneously with $\hat{A} = \hat{B} > 0$ only if $\mu = v$. In consequence, if the external returns from technology to radical innovations and from technological opportunity to incremental innovations are equal, then this directly gives rise to standard semi-endogenous growth along the lines of Jones (1995).

If $\mu < v$, on the other hand, then we require that $\lim_{t\to\infty}\{L_t^\beta B_t^{\mu-1}\} = 0$, which implies an asymptotic BGP where technology grows at a rate $\hat{A} = \frac{(1+\mu)\beta n}{1-\mu v}$ and technological opportunity—at a rate $\hat{B} = \frac{(1+v)\beta n}{1-\mu v}$. Since $\mu < v$ we know that, on the asymptotic BGP, technological opportunity grows faster than technology. In other words, in the limit, technological opportunity will be driven by radical

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[4]When we refer to semi-endogenous growth, we mean R&D-based growth which is ultimately driven by population growth. See, e.g., Jones (1995).

[5]Throughout the article, we shall use the notation: $\hat{X} \equiv \dot{X}/X$.

innovations only. Furthermore, the growth rate of incremental innovations is proportional to that of radical innovations, and the flow of radical innovations grows faster than of incremental ones. If $\lim_{t\to\infty}\{L_t^\beta B_t^{\mu-1}\} > 0$ then we obtain that $\lim_{t\to\infty}\hat{B} = \frac{1+\nu}{1+\mu}\hat{A}$, suggesting again that technological opportunity B grows at a proportional and faster rate than technology A.

In the last case, where $\mu > \nu$, we obtain that $\lim_{t\to\infty}\dot{A} = \dot{B} = B = 0$. Thus, even though technological opportunity is renewed by radical innovations, the effect of its depletion due to more efficient incremental innovations is dominant in the long run. Growth comes to a halt in the limit.[6]

In consequence, we find that long-run predictions along the lines of Jones (1995), Kortum (1997), Segerstrom (1998), or Peretto (1998) may hold only in the case where the extents of external returns to incremental innovations are less or equal to those of radical innovations. Since the current literature compares the model predictions with historical time paths of R&D expenditures and factor productivities, with the single dynamical equation of technology in mind, our results suggest that one should be more cautious with those predictions. When technological opportunity is viewed as the driving force behind actual technology, then it could very well be that the empirical literature misses the influence of the evolution of technological opportunity on effective technological progress and thus runs into systematic error.

Conclusively, a necessary condition for obtaining semi-endogenous growth as in Jones (1995) within our model requires $\mu \geqslant \nu$. In the following section, we study the model under its most transparent parametrization $\mu = \nu$, consistent with a BGP. One of our main findings is that even the optimal allocation of such a model can be subject to oscillations. Our model is thus able to span a bridge from the cyclical paths of Olsson to the monotonic dynamics of Jones or Romer.

3. The model

We embed the dynamical equations analysed above in a semi-endogenous growth model and study both its decentralized equilibrium and its social planner solution. Our interest is to understand the ways in which consumption is allocated across time, and labour is divided between all sectors of the economy. We will also compare both allocations to see which of the resultant tradeoffs are 'generic' to the model, and which are specific to one of the considered allocations.

As far as the intertemporal consumption decision is concerned, our approach is based on the standard framework where the infinitely-lived representative agent obtains utility from the discounted stream of the unique consumption good. The utility function takes the usual CRRA form: $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, with $\theta > 0$ being the inverse of the intertemporal elasticity of substitution in consumption.[7]

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[6]These results are preserved if one allows for different β parameters in the two R&D sectors. We shall abstract from this unnecessary complexity throughout the remainder of the article.

[7]In the special case $\theta = 1$, the formula is replaced with $u(c) = \ln c$.

Moreover, we shall assume a standard Cobb-Douglas production function which takes as inputs: technology A with elasticity σ, physical capital K with elasticity α, and labour $(1 - \ell_A)L$ with elasticity $1 - \alpha$. Technology accumulates faster the more labour is allocated to research, but its underlying ability to accumulate is constrained by technological opportunity. The market equilibrium will be computed in an increasing variety framework similar to the one in Jones (2005), where the capital input in final goods production is composed of a continuum of measure A of imperfectly substitutable intermediate goods. The monopolistic profits accrued from producing those goods provide the incentive for R&D firms to pursue incremental research, resulting in inventing new varieties, $\dot{A}$.

Given the assumption that innovations are heterogenous, however, in the decentralized allocation we have to specify the markets which would assign prices to both incremental and radical innovations. The former provide direct increases to aggregate productivity by increasing the mass of capital goods varieties A, and therefore their pricing will be based on the discounted stream of monopoly profits attained by capital goods producers. Radical innovations, on the other hand, do not provide increases in aggregate productivity and are only indirectly useful for the economy. Hence, their pricing has to be based on the evaluation of these indirect impacts. We achieve this by decomposing the process of incremental innovation into three consecutive steps, introducing monopolistic competition in an analogous way as in the case of the assembly of physical capital. We shall assume that, to produce incremental innovations, one has to assemble an infinity of mass B of imperfectly substitutable intermediate ideas, each of them produced monopolistically by a technological opportunity explorer. Then, there is free entry to radical innovation, increasing the mass B of technological opportunities to be explored.

As suggested previously, we shall concentrate on the case of $\mu = \nu$ here. We impose this restrictive condition for analytical tractability and comparability to the semi-endogenous growth literature.

3.1 The decentralized equilibrium

The decentralized market equilibrium is shaped by the decisions of households, final goods producers, intermediate goods producers, and R&D firms. We shall discuss their optimization problems in that order and then define and compute the general equilibrium.

3.1.1 *Households*   Households maximize utility subject to wealth accumulation.

$$\max_{c_t} \int_0^\infty L_0 \frac{c_t^{1-\theta} - 1}{1 - \theta} e^{-(\rho - n)t} dt, \tag{3}$$

subject to

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t. \tag{4}$$

Labour is supplied inelastically (one unit of labour per person) and there is an *ex post* uniform wage rate w_t across all occupations due to wage arbitrage. a_t represents household wealth at time t, $\rho > 0$ is the discount rate, $\theta > 0$ is the coefficient of relative risk aversion, and L_0 the initial population amount that grows at rate $n > 0$. We take consumption to be the numeraire. Hence, the optimal consumption path follows the standard Euler equation:

$$g_c \equiv \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}. \tag{5}$$

3.1.2 *The final goods sector*   The final goods sector solves

$$\max_{x_{it}, \ell_{At}} \left(\int_0^A x_{it}^{\varphi} di \right)^{\frac{\alpha}{\varphi}} ((1 - \ell_{At})L_t)^{1-\alpha} - w_t((1 - \ell_{At})L_t) - \int_0^A q_{it}x_{it}di. \tag{6}$$

Here, x_{it} are intermediate inputs produced by firm i at time t, A refers to the number of varieties, q_{it} is the cost of using intermediate input x_{it}, and the parameter $\varphi \in (0, 1)$ describes the complementarity of intermediate inputs whereas $\alpha \in (0, 1)$ is the share of intermediate inputs in final output.

With symmetric inputs, $x_t = x_{it}$, and the total amount of intermediate inputs (or simply capital) is equal to $\int_0^A x_{it}di = K_t$. The total production of final goods is then $Y_t = A_t^{\sigma}K_t^{\alpha}((1 - \ell_{At})L_t)^{1-\alpha}$, where $\sigma = \alpha(1 - \varphi)/\varphi$. Maximization leads to the demand function for intermediate inputs x_{it} given by:

$$x(q_{it}) = \left(\alpha \frac{Y}{q_{it} \int_0^A x_{it}^{\varphi}di} \right)^{\frac{1}{1-\varphi}}. \tag{7}$$

Hence, unit price of x_{it} equals:

$$q_{it} = \alpha \frac{Y}{K}, \tag{8}$$

with $Ax = K$ being the capital resource constraint. Optimal wages are given by

$$w_t = (1 - \alpha)A_t^{\sigma}k_t^{\alpha}(1 - \ell_{At})^{-\alpha}. \tag{9}$$

3.1.3 *Capital goods producers*   Firms producing capital goods solve:

$$\max_{q_{it}} \Pi_{it} = (q_{it} - r_t - d)x(q_{it}), \tag{10}$$

where $d > 0$ denotes the depreciation rate of capital inputs, r_t the interest rate, and $x(q_{it})$ the demand given price q_{it}. We thus obtain that the optimal rental rate q_{it} equals, by symmetry,

$$q_{it} = q_t = \frac{r_t + d}{\varphi}. \tag{11}$$

Combining (8) with (11) gives the interest rate r_t:

$$r_t = \varphi \alpha \frac{Y_t}{K_t} - d. \tag{12}$$

Since $\varphi \in (0, 1)$, we know that interest is less than the marginal product of capital. By combining (7) with (11) we get an equilibrium demand of x given by

$$x = \left(\frac{\alpha \varphi Y}{A^{1-\varphi} K^{\varphi}(r+d)} \right)^{\frac{1}{1-\varphi}} = \frac{K}{A}. \tag{13}$$

Profits of the firm selling variety i are then

$$\Pi_{it}^{A} = \Pi_t = (1 - \varphi)(r_t + d)x/\varphi = (1 - \varphi)\alpha \frac{Y}{A}. \tag{14}$$

The no-arbitrage condition describes the equilibrium dynamics of the price of a patent p_{At}:

$$r_t = \frac{\Pi_{it}}{p_{At}} + \frac{\dot{p}_{At}}{p_{At}}. \tag{15}$$

The supply of patents is determined by the R&D sector.

3.1.4 *Incremental innovations*   As mentioned above, we shall assume that incremental innovations are worth p_{At} in patents for new intermediate capital goods to be produced monopolistically. They are in turn produced in a competitive market where R&D firms assemble an infinity of mass B_t of imperfectly substitutable intermediate ideas, whose quantities are denoted by χ_{it}, with $i \in [0, B]$.[8] For each intermediate idea used in the R&D process, incremental innovators are charged a licence fee in the form of a per-unit royalty z_{it}, paid to technological opportunity explorers. This assumption is consistent with real-world practice because per-unit royalties are the predominant means of making profit of ideas.[9]

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[8]As an example, consider an incremental innovator who designs a new type of software. As an intermediate input, he requires a programming code that is only a marginal innovation (i.e., requires a small quantity of this particular intermediate idea), or he could need one that is very complex and touches the bounds of current knowledge (i.e., requires a lot of this intermediate idea). A technological opportunity explorer, in this example a software programmer, would then write such a code and sell it to the incremental innovator. Nevertheless, by itself, the code is not directly useful. Thus, we suggest that intermediate ideas by themselves do not advance technology. In our model, incremental innovators perform the role of bundling together these intermediate ideas into an actual incremental innovation that is useful for the real sector. Analogous examples of intermediate ideas could be, e.g., designs of machine parts, production processes, or partial solutions to complex analytical or numerical problems.

[9]The theoretical literature suggests that it is optimal for an outside firm to charge a fixed fee for an idea (Kamien and Tauman, 2002). However, under asymmetric information (Beggs, 1992), or in case a cost-reducing innovation is non-drastic (Wang, 1998), per-unit royalties should be preferred. On the empirical side, Rostoker (1983) surveyed corporations and found that 39% demand a royalty, while 46% demand a mix between royalty and downpayment. Macho-Stadler *et al.* (1996) found that in 59% of all

Thus, incremental innovators solve

$$\max_{\chi_{it}}\left\{ p_{At}B_t^{\frac{\psi-1}{\psi}}\left(\int_0^{B_t}\chi_{it}^{\psi}di\right)^{\frac{1}{\psi}} - \int_0^{B_t}z_{it}\chi_{it}di \right\}, \qquad \psi \in (0,1). \qquad (16)$$

The iso-elastic demand curve for quantities of intermediate ideas χ_{it} is thus given as:

$$\chi(z_{it}) = \left(\frac{p_{At}B_t^{\frac{\psi-1}{\psi}}\left(\int_0^{B_t}\chi_{it}^{\psi}di\right)^{\frac{1-\psi}{\psi}}}{z_{it}} \right)^{1-\psi}. \qquad (17)$$

In the symmetric equilibrium, where $\chi_{it} = \chi_t$ for all $i \in [0, B_t]$, the above pricing scheme reduces to $z_{it} = z_t = p_{At}$. The total volume of incremental innovation equals $B_t\chi_t \equiv \dot{A}_t$.

Thus, technological opportunity explorers charge a licence fee that, in the symmetric equilibrium, is equivalent to a per-unit royalty on the production of incremental innovations, and takes all the rent away from incremental innovators.

3.1.5 *Technological opportunity explorers*   Each intermediate idea $i \in [0, B_t]$ in the technological opportunity set is explored monopolistically by a technological opportunity explorer who maximizes profits by setting an optimal licence fee z_{it} given the iso-elastic demand curve specified above. Technological opportunity explorers hire labour against the equilibrium wage w_t which they view as exogenous. Hence, the optimization problem becomes:

$$\max_{z_{it}} \Pi_{it}^B = z_{it}\chi(z_{it}) - \frac{w_tu_t\ell_{At}L_t}{B_t} = \chi(z_{it})\left(z_{it} - \frac{w_t}{\xi_tB_t}\right), \qquad (18)$$

where the last equality follows from the assumed specification of technology:

$$\chi_{it} = (u_t\ell_{At}L_t)\xi_t, \qquad (19)$$

with $\xi_t = \delta(u_t\ell_{At}L_t)^{\beta-1}B_t^{\mu-1}$ considered exogenous. At the level of the given technological opportunity explorer, the technology is thus viewed as linear, even though it will be concave and increasing with B_t in the aggregate. The optimal choice of the per-unit royalty z_{it} yields:

$$z_{it} = \frac{w_t}{\psi\xi_tB_t}. \qquad (20)$$

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contracts a royalty payment is demanded, while Bousquet *et al.* (1998) observed this in 78% of contracts. The main difference between our approach and this literature is that, in our case, the incremental idea is not a cost-reducing idea (Kamien and Tauman, 2002) but a necessary input of the R&D process.

Hence, under symmetry, each opportunity explorer's profit equals:

$$\Pi_{it}^B = \Pi_t^B = \frac{w_t}{\xi_t B_t}\left(\frac{1-\psi}{\psi}\right)\chi_t = \left(\frac{1-\psi}{\psi}\right)\frac{w_t u_t \ell_{At} L_t}{B_t}. \tag{21}$$

Combining this with the equilibrium wage set in (9), we obtain the following equation specifying the incremental innovation patent price p_{At}:

$$p_{At} = z_t = \frac{w_t}{\psi\xi_t B_t} = \frac{1}{\psi}\cdot\frac{w_t u_t \ell_{At} L_t}{\dot{A}_t} = \frac{1}{\psi}\cdot\frac{(1-\alpha)Y_t u_t \ell_{At}}{(1-\ell_{At})\dot{A}_t}. \tag{22}$$

3.1.6 *Radical innovations* We assume a competitive market for radical innovations, with firms hiring labour for the competitive wage w_t and selling patents for each newly produced technological opportunity $\dot{B}_t$ to technological opportunity explorers for the price p_{Bt}. The value of patents p_{Bt} comes from the fact that technological opportunity explorers exert monopoly power and earn positive profits in the form of licence fees. The discounted stream of these profits determines p_{Bt} in equilibrium. Assuming free entry, we have:

$$p_{Bt}\dot{B}_t = w_t(1-u_t)\ell_{At}L_t, \tag{23}$$

where $\dot{B}_t = -\delta(u_t\ell_{At}L_t)^\beta B_t^\mu + \gamma((1-u_t)\ell_{At}L_t)^\beta A_t^\mu$. For simplicity, we ignore the problem of net destruction of technological opportunity if $\dot{B}_t < 0$ here because at the BGP and in its vicinity, B_t will be growing over time. Therefore we do not have to consider the case where patents for technological opportunity explorers are finitely lived.

Combining eqs (22)–(23), we obtain:

$$\frac{u_t}{1-u_t} = \frac{\psi p_{At}\dot{A}_t}{p_{Bt}\dot{B}_t}, \tag{24}$$

and thus at the BGP, both p_A and p_B grow at the same rate $g + n - \hat{A}$.

Finally, patents on radical innovations have to follow the usual no-arbitrage condition:

$$r_t = \frac{\Pi_t^B}{p_{Bt}} + \frac{\dot{p}_{Bt}}{p_{Bt}}. \tag{25}$$

3.1.7 *The market equilibrium* The decentralized equilibrium consists of allocations $\{c_t, \ell_{At}, u_t, a_t, \{x_{it}\}, \{\chi_{it}\}, Y_t, K_t, \{\Pi_{it}^A\}, \{\Pi_{it}^B\}, L_t, A_t, B_t\}_0^\infty$ with prices $\{w_t, r_t, \{q_{it}\}, \{z_{it}\}, p_{At}, p_{Bt}\}_0^\infty$ such that we obtain for all t, that c_t, a_t solve the household's maximization problem; $\{x_{it}\}$ and ℓ_{At} solve the final good sector's problem; q_{it} and Π_{it}^A solve the capital goods sector problem; $\{\chi_{it}\}$ and $u_t\ell_{At}$ solve the incremental innovation sector's problem; z_{it} and Π_{it}^B solve the technological opportunity explorers' problem; there is free entry into the radical innovation sector; the capital market clears with $a_tL_t = K_t + p_{At}A_t + p_{Bt}B_t$; the labour market

clears with $u_t \ell_{At} + (1 - u_t)\ell_{At} + (1 - \ell_{At}) = 1$; the capital resource constraint satisfies $\int_0^A x_{it} di = K_t$; the intermediate idea resource constraint satisfies $\int_0^B \chi_{it} di = \dot{A}_t$; assets have equal returns given by $r_t = \frac{\Pi_{it}^A}{p_{At}} + \frac{\dot{p}_{At}}{p_{At}} = \frac{\Pi_{it}^B}{p_{Bt}} + \frac{\dot{p}_{Bt}}{p_{Bt}}$.

3.1.8 *The balanced growth path*   As it is usual with semi-endogenous growth models, growth rates of all major variables c, k, y, A, B along the BGP can be computed using the appropriate production functions only, and they will not depend on any endogenous variables. Hence, they will also necessarily be the same in the decentralized and in the social planner allocation.

Solving for the BGP yields the following long-run growth rate of the economy:

$$g \equiv \hat{y} = \hat{k} = \hat{c} = \frac{\sigma}{1 - \alpha}\frac{\beta n}{1 - \mu}, \tag{26}$$

whereas the long-run growth rates of technology and technological opportunity are

$$\hat{A} = \hat{B} = \frac{\beta n}{1 - \mu}. \tag{27}$$

As expected from the previous section, technological opportunity and technology grow at a common rate, and consumption, income, and capital grow at a rate being a multiple of this rate. The transversality condition of the optimization requires that $n < \rho + (\theta - 1)g$, and the R&D problem does not add any more conditions on top of that, provided that the economy is in the decentralized equilibrium.

While the differences between the decentralized equilibrium and the optimal allocation cannot be found in growth rates, they will certainly appear in the levels of (appropriately detrended) variables at the BGP. To compare these, we shall rewrite the system in terms of the six following variables which are stationary on the BGP: $\{c/k, u, \ell_A, y/k, A/B, L^\beta/A^{1-\mu}\}$. We shall denote $X \equiv A/B$ and $Y \equiv L^\beta/A^{1-\mu}$. The superscript d indicates that we are dealing with the decentralized equilibrium. The appropriate formulas for the BGP are the following:

$$\frac{1 - u^d}{u^d} = \frac{1 - \varphi}{1 - \alpha} \cdot \frac{1 - \psi}{\psi} \cdot G, \tag{28}$$

$$\frac{1 - \ell_A^d}{\ell_A^d} = u^d \cdot \frac{1 - \alpha}{(1 - \varphi)\alpha\psi G}, \tag{29}$$

$$X^d + 1 = \frac{\gamma}{\delta}\left(\frac{1 - u^d}{u^d}\right)^\beta (X^d)^{\mu+1}, \tag{30}$$

$$\left(\frac{y}{k}\right)^d = \frac{1}{\alpha\varphi}(\theta g + d + \rho), \tag{31}$$

$$\left(\frac{c}{k}\right)^d = \frac{1}{\alpha\varphi}(\theta g + d + \rho) - g - d - n, \tag{32}$$

$$Y^d = \frac{L^\beta}{A^{1-\mu}} = \frac{\beta n}{(1 - \mu)\delta(u^d \ell_A^d)^\beta}(X^d)^\mu. \tag{33}$$

We used the shorthand notation:

$$G \equiv \frac{\hat{A}}{\hat{A} + (\theta - 1)g - n + \rho} = \frac{\frac{\beta n}{1-\mu}}{\frac{\beta n}{1-\mu} + (\theta - 1)\frac{\sigma}{1-\alpha}\frac{\beta n}{1-\mu} - n + \rho}.$$

Equation (30) defines the unique solution for X^d since the left-hand side increases linearly from $\lim_{X\to 0}(X+1) = 1$ to $\lim_{X\to\infty}(X+1) = +\infty$. The right-hand side increases in a strictly convex manner from $\lim_{X\to 0} cX^{\mu+1} = 0$ to $\lim_{X\to\infty} cX^{\mu+1} = +\infty$. Since both sides are continuous, there necessarily exists a unique, positive point in which they intersect.

All other equations provide explicit formulas for all variables which are stationary along the BGP. We shall now compare these formulas to their counterparts in the social planner allocation.

3.2 The social planner problem

We will now proceed to a description of the optimal allocation in the considered economy.

3.2.1 Setup   The maximization problem of the social planner looks as follows.

$$\max_{\{c,u,\ell_A;k,A,B\}_{t=0}^\infty} \quad L_0 \int_0^\infty \frac{c^{1-\theta} - 1}{1 - \theta} e^{-(\rho-n)t} dt \qquad \text{subject to:} \tag{34}$$

$$y = A^\sigma k^\alpha (1 - \ell_A)^{1-\alpha}, \tag{35}$$

$$\dot{k} = y - c - (d + n)k, \tag{36}$$

$$\dot{A} = \delta(u\ell_A L)^\beta B^\mu, \tag{37}$$

$$\dot{B} = [-\delta u^\beta B^\mu + \gamma(1 - u)^\beta A^\mu](\ell_A L)^\beta, \tag{38}$$

$$L = L_0 e^{nt}, \qquad n > 0,$$

$$L_0, k_0, A_0, B_0 \quad \text{given.} \tag{39}$$

The parameter restrictions are $0 < n < \rho$, necessary to guarantee a positive effective discount rate, and $\sigma, \alpha, \beta, d \in (0, 1)$ as well as $\theta, \mu, \delta, \gamma > 0$.[10] $\mu < 1$ is required to guarantee positive semi-endogenous growth in the long-run. d is the instantaneous depreciation rate of physical capital.

3.2.2 The balanced growth path   Maximizing the Hamiltonian associated with the above optimization problem and solving for the BGP yields the following long-run growth rate of the economy:

$$g \equiv \hat{y} = \hat{k} = \hat{c} = \frac{\sigma}{1 - \alpha}\frac{\beta n}{1 - \mu}, \tag{40}$$

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[10]In the special case $\theta = 1$, the utility function is replaced with $u(c) = \ln c$.

whereas the long-run growth rates of technology and technological opportunity are (as in Jones, 1995)

$$\hat{A} = \hat{B} = \frac{\beta n}{1 - \mu}. \tag{41}$$

As anticipated, all these growth rates coincide with the ones obtained in the decentralized equilibrium. All transversality conditions boil down to the single requirement that $n < \rho + (\theta - 1)g$, the same one as in the decentralized economy.

Denoting the steady-state ratio of technology to technological opportunity A/B by X, we find that the optimal share of incremental research effort relative to radical research effort is:

$$\frac{1 - u^*}{u^*} = \mu G(X^* + 1) \tag{42}$$

where $X^\star \equiv (A/B)^\star$ solves the following implicit equation:[11]

$$\left(\frac{\gamma}{\delta}\right)^{\frac{1}{1-\beta}} (\mu G)^{\frac{\beta}{1-\beta}} (X^*)^{\frac{1+\mu}{1-\beta}} = 1 + X^*. \tag{43}$$

Equation (43) always has a unique positive solution. The asterisk indicates that we are dealing with the socially optimal allocation.

Furthermore, we find that the optimal share of labour allocated to R&D along the BGP solves

$$\frac{1 - \ell_A^*}{\ell_A^*} = \frac{(1 - \alpha)\mu}{\beta\sigma} \left((X^* + 1)\frac{u^*}{1 - u^*} - 1 \right). \tag{44}$$

The interpretation of these results is as follows. The optimal allocation of labour towards R&D is increasing in n, but decreasing in ρ and θ. A higher population growth rate n implies an optimally higher ratio of technology to technological opportunity which requires a greater proportion of workers to be allocated to the research sector. The less important the future or the larger the incentives for consumption smoothing the more labour will be diverted to the production sector.

The technology parameters α, σ, and μ increase ℓ_A^* if $\rho > (1 - \beta)n$. A sufficiently high β implies that this inequality holds and that the returns to adding more labour to R&D outweigh the costs of foregoing higher current consumption.

3.3 Comparing the market equilibrium to the social optimum

3.3.1 *Qualitative results* Several findings stand out when the market equilibrium is compared to the social optimum. First of all, eqs (31) and (32) differ with respect to their socially optimal counterparts (not shown) only by the φ parameter, measuring the extent of the proportional markup in the monopolistically

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[11]In the social planner problem, $G = \frac{\hat{A}}{\hat{A} + (\theta - 1)g - n + \rho}$ is interpreted as the BGP growth rate of the shadow value of (incremental or radical) innovations, $G = \hat{A} + \hat{\lambda}_A = \hat{B} + \hat{\lambda}_B$.

competitive market for intermediate capital goods, which does not appear in the optimal allocation. Hence, this distortion implies that both the c/k and y/k ratios are 'too high' in the market equilibrium, as the monopolistic profit margin tends to slow down the accumulation of capital.

Secondly, eq. (33) has an identical counterpart in the social planner allocation, but the actual values of Y will likely differ because of the possible differences in ℓ_A, u and X between both allocations.

Thirdly, eq. (30) has its social planner counterpart in the form of eq. (43). The solutions for X^d and $X^\star$ will only differ if there are differences in u^d and $u^\star$ or ℓ_A^d and ℓ_A^*.

Thus, the main differences between the social planner allocation and the market equilibrium arise through the allocation of labour across the three sectors: production, incremental R&D, and radical R&D. As regards ℓ_A, it is possible to derive from eq. (44) of the centralized equilibrium an equation analogous to (29). It takes the form:[12]

$$\frac{1 - \ell_A^*}{\ell_A^*} = \frac{(1-\alpha)\varphi}{(1-\varphi)\alpha\beta}\left(\frac{1}{G} - \mu\right), \tag{45}$$

while we can re-write eqs (28) and (29) as

$$\frac{1 - \ell_A^d}{\ell_A^d} = \frac{1-\alpha}{(1-\varphi)\alpha} \cdot \frac{1}{G} \cdot \frac{(1-\alpha)}{(1-\alpha)\psi + (1-\psi)(1-\varphi)G}. \tag{46}$$

In general, the ordering of ℓ_A^* and ℓ_A^d is ambiguous. However, the parameters defining the mark-ups turn out to be crucial for the ordering. The closer φ is to zero, the higher the returns to technology in production and the higher the mark-up in the production of the final goods. For sufficiently low φ, one obtains unambiguously $\ell_A^d < \ell_A^*$. Thus, a sufficiently high complementarity in intermediate inputs, effectively constraining physical capital accumulation in the decentralized allocation, guarantees that the social planner would allocate relatively more labour to the production of new varieties.

Furthermore, the parameter ψ plays an important role in the decentralized model as well. A high substitution parameter ψ leads to low profits from incremental innovations which drives equilibrium wages down and thus increases the incentive to have more workers in the final goods sector. The higher is ψ the higher is ℓ_A^d, if $\varphi > \alpha$. This last condition arises as φ drives down monopolistic profits while α increases them. This effect is absent in the social optimum because there are no monopoly rents to extract.

As regards the allocation of workers between incremental and radical R&D, from eqs. (28) and (42) we find that the term $\frac{1-\varphi}{1-\alpha} \cdot \frac{1-\psi}{\psi}$ present in the decentralized case is replaced by $\mu(X^\star + 1)$ in the socially planned economy. We interpret this in the following way: whereas in the decentralized equilibrium, the allocation of

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[12]Note that $\sigma = \alpha\frac{1-\varphi}{\varphi} \Leftrightarrow \varphi = \frac{\alpha}{\alpha+\sigma}$.

R&D workers across sectors is driven by the relative price of patents due to incremental and radical R&D, the optimal allocation takes directly the technological constraints as given by the current ratio of technology to technological opportunity into account. Furthermore, the term μ is absent from the decentralized case because opportunity explorers view their production function as linear and fail to notice their external negative impact on the aggregate R&D output.

3.3.2 *Calibration*   To provide our comparisons with a quantitative edge, let us now assign baseline values to all parameters in our model. The market equilibrium is calibrated on the basis of equations (28) through (33), whereas the social planner calibration is based on eqs (42), (43), (44), and three equations analogous to (31)–(33).[13] To bring our numerical example as close to reality as possible, we shall pick these values within a calibration exercise. Our approach is as follows.

First of all, in line with historical data and previous calibration exercises, we choose the benchmark parameters $n = 0.015$, $\rho = 0.03$, $d = 0.05$, $\theta = 1$, and $\alpha = 0.36$ (see Kydland and Prescott, 1982; Steger, 2005, and Jones and Williams, 2000). Furthermore, relying on the empirical evidence in Basu (1996), we choose $\varphi = 0.714$, which implies a mark-up of 1.4. This leads to $\sigma = \alpha \frac{1-\varphi}{\varphi} = 0.144$.

Secondly, the parameters β and μ are assumed to solve $g = \frac{\sigma}{(1-\alpha)} \frac{\beta n}{1-\mu}$, where $g = 0.017$ is the historical US real growth rate net of population growth.[14] This allows us to solve for one parameter as the function of the other. We then choose β to target the observed ratios of y/k and c/k (where we follow the literature and assume them to be good approximations of the steady state). We find that $\beta = 0.5$ leads to $\mu = 0.9$ and steady state ratios $c/k = 0.26$ and $y/k = 0.33$, which are reasonably close to their currently observed values ($c/k = 0.25$ and $y/k = 0.29$).

Thirdly, our calibration implies a steady-state savings rate of $s = 0.21$, precisely in line with the observed savings rate (see Barro and Sala-i-Martin, 2003). In addition, we choose $\psi = 0.9$, leading to a steady state labour allocation in the research sector of $\ell_A = 0.12$, which slightly overshoots the observed ratio of $\ell_A^d = 0.1$ (see Jones, 1995). Our calculations finally lead to an incremental R&D share of $u^d = 0.99$, and thus almost all researchers will be allocated to incremental research in the decentralized equilibrium.

Further parameters, γ and δ, have never been discussed in the literature. Thus, we pick them at arbitrary plausible values and then perform sensitivity analysis over these values. All the calibrated parameters are listed in Table 1. Since only γ and δ are chosen arbitrarily, we investigate their effects in the next section. However, as it can be seen in Table 2, neither of them drives the standard macroeconomic variables along the BGP and they only affect the allocation of labour between radical and incremental research in the social planner allocation.

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[13]They are not shown and correspond to the relevant equations characterizing the market solution, only with $\varphi = 1$.

[14]This is the average real GDP growth rate minus the population growth rate in 1947-2010, based on data from the US Department of Commerce, Bureau of Economic Analysis.

Table 1 The baseline calibration

δ	γ	β	n	θ	d	α	σ	ρ	μ	φ	ψ
2	2	0.5	0.015	1	0.05	0.36	0.144	0.03	0.9	0.714	0.9

Table 2 The market equilibrium and the social planner allocations

	g	$\hat{A}$	$\frac{y}{k}$	$\frac{c}{k}$	s	$\frac{A}{B}$	$\frac{L^{\beta}}{A^{1-\mu}}$	u	ℓ_A
ME	0.017	0.075	0.33	0.26	0.21	22.18	5.66	0.99	0.012
SP	0.017	0.075	0.12	0.05	0.6	2.82	1.09	0.78	0.01
θ	0/0	0/0	+/+	+/+	−/−	+/+	+/+	+/+	−/−
ρ	0/0	0/0	+/+	+/+	−/−	+/+	+/+	+/+	−/−
n	+/+	+/+	+/+	+/+	+/+	−/−	+/+	−/-	+/+
γ	0/0	0/0	0/0	0/0	0/0	−/−	−/−	0/+	0/0
δ	0/0	0/0	0/0	0/0	0/0	+/+	−/−	0/+	0/0
ψ	0/0	0/0	0/0	0/0	0/0	+/0	+/0	+/0	+/0

Notes: The market equilibrium (ME), and the social planner (SP) allocation – the first comparative statics denote the impact on ME, the second ones that on SP.

3.3.3 *Quantitative results*   The results of this baseline calibration are depicted in Table 2, which shows clear differences between the market equilibrium and the social planner case. Particularly significant are the differences in the steady-state saving rate, with the social planner's saving rate being almost three times higher than the one we see in the market equilibrium, leading to a much lower consumption-capital ratio than the market equilibrium suggests. Additionally, though the amount of labour delegated to the R&D sector is approximately the same in both cases, the social planner would allocate much less labour to incremental innovations and more to radical ones. This would lead to a much lower steady state ratio of incremental innovations to technological opportunity.

Additionally, we include comparative statics in Table 2. They are helpful to understand the workings of the analysed model. A '+' means that an increase in a given parameter raises the given steady-state value, whereas a '−' means it lowers it. Zero denotes no impact. These comparative statics hold in the vicinity of our baseline calibration. Also, we only changed those parameters that could be changed independently without changing the values of the other parameters (see above).

Several particularly noteworthy facts should be emphasized here:

(i) Technological opportunity plays a similar role to savings. In particular, increases in the discount rate ρ and reductions in the intertemporal elasticity of substitution in consumption $1/\theta$ reduce both the savings rate and technological opportunity relative to technology.

(ii) The β parameter, measuring returns to scale in R&D, influences the steady-state variables in the same way as the μ parameter, measuring the extent of external effects in R&D.

(iii) Increases in the population growth rate raise long-run growth rates, but reduce the level of de-trended technology. The ratio of technology to technological opportunity is decreased as well. Relatively more technological opportunity goes then together with an increase in overall R&D labour share.

(iv) Changes in γ and δ do not impact the optimal allocation of labour in the market equilibrium and only impact the allocation of labour between radical and incremental innovations in the social planner case.

(v) The parameter ψ, related to the mark-up in the incremental innovation sector, impacts labour allocations u^d and ℓ_A^d positively in the market equilibrium. This mark-up is absent in the social planner case.

3.4 The transition

In this section we shall analyse the transition dynamics of our model around the balanced growth path. We derive the dynamical equations for variables which are stationary along the BGP. Hence, our system is going to be rewritten in terms of the six following variables: $\{c/k, u, \ell_A, y/k, A/B, L^\beta/A^{1-\mu}\}$. We shall again denote $X \equiv A/B$, and use also the notation $Y \equiv L^\beta/A^{1-\mu}$. There are three choice-like variables, $c/k, u, \ell_A$, and three state-like variables: $y/k, X, Y$.

The transition dynamics are fully characterized by the following dynamical system.

$$\widehat{c/k} = \left(\frac{\alpha}{\theta} - 1\right)\frac{y}{k} - \frac{d+\rho}{\theta} + \frac{c}{k} + d + n \tag{47}$$

$$\widehat{y/k} = \sigma\delta(u\ell_A)^\beta X^{-\mu} Y + (\alpha - 1)(\frac{y}{k} - \frac{c}{k} - d - n) - (1-\alpha)\left(\frac{\ell_A}{1-\ell_A}\right)\hat{\ell}_A \tag{48}$$

$$\hat{X} = \delta(u\ell_A)^\beta X^{-\mu} Y + \delta(u\ell_A)^\beta X^{1-\mu} Y - \gamma((1-u)\ell_A)^\beta XY \tag{49}$$

$$\hat{Y} = \beta n - (1-\mu)\delta(u\ell_A)^\beta X^{-\mu} Y \tag{50}$$

$$\hat{u} = -\frac{(1-u)}{1-\beta}\left\{\mu\hat{X} + \left[\frac{\beta\sigma}{1-\alpha}\left(\frac{1-\ell_A}{u\ell_A}\right) - \mu X\left(1 + \frac{\gamma}{\delta}\left(\frac{1-u}{u}\right)^{\beta-1}X^\mu\right) + \right.$$
$$\left. + \mu\left(\frac{1-u}{u}\right)\right]\delta(u\ell_A)^\beta X^{-\mu} Y\right\} \equiv -\frac{(1-u)}{1-\beta}\cdot\Xi, \tag{51}$$

$$\hat{\ell}_A = \frac{1}{1-\beta+\alpha\frac{\ell_A}{1-\ell_A}}\left\{\alpha\frac{c}{k} - (1-\alpha)(d+n) + \beta n + (\mu - \sigma)\delta(u\ell_A)^\beta X^{-\mu} Y + \right.$$
$$\left. - \mu\left(\frac{u}{1-u}\right)\gamma((1-u)\ell_A)^\beta XY - u\cdot\Xi\right\}. \tag{52}$$

Table 3 Eigenvalues of the linearized system

Eigenvalues
$0.1233 + 0.0176i$
$0.1233 - 0.0176i$
0.0557
$-0.0811 + 0.0176i$
$-0.0811 - 0.0176i$
-0.0135

In the remainder of this section, we shall resort to numerical approximations because the implied analytical formulas, although readily attainable, are too large to be informative.

3.4.1 *Oscillatory dynamics*   One of Olsson's main results is that research evolves in cycles. We will now show that cycles may obtain in our model as well, despite us not relying on Olsson's key linearity assumption ($\beta = 1$). The oscillatory dynamics are obtained from the dynamic system (47) to (52) and using parameter values that calibrate the social planner allocation: $\delta = 2$, $\gamma = 2$, $\beta = 0.512$, $n = 0.015$, $\theta = 2$, $d = 0.04$, $\alpha = 0.36$, $\sigma = 0.3$, $\rho = 0.05$ and $\mu = 0.5$.[15] Table 3 presents the eigenvalues of the dynamical system after its linearization around the steady state. The two complex eigenvalues with negative real parts imply that under our benchmark calibration, dampened oscillations are observed.

Furthermore, the balanced growth path is saddle-path stable: there are three unstable eigenvalues – having positive real parts – and three stable eigenvalues. The number of unstable roots is exactly equal to the number of choice-like variables in the benchmark parametrization of our model, and hence there exists a unique time path of the economy approaching its balanced growth path (or in the case of a de-trended system, approaching its steady state).[16]

Furthermore, we confirm by the means of a sensitivity analysis that such oscillations indeed occur under a large variety of parameter choices. In contrast, occurrence of oscillatory dynamics is impossible in continuous-time setups with

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[15]This calibration brings the social planner allocation of our model close to the historical data. For details we refer the reader to Growiec and Schumacher (2007).

[16]Indeterminacy is unambiguously ruled out here despite the fact that y/k is a function of ℓ_A which is a choice variable. y/k is not a separate choice variable, though, since once ℓ_A is counted as a choice variable itself, there are no degrees of freedom left for choosing y/k. In sum, once c/k, u and ℓ_A are chosen, everything else is predetermined. This means that there are exactly three degrees of freedom. In other words: we could have used another variable in the de-trended system instead of y/k without altering any of the model implications and the count of stable roots, that variable being $Z = A^\sigma k^{\alpha-1} = (y/k)(1-\ell_A)^{\alpha-1}$. Z is a function of predetermined variables A and k only and is also stationary along the BGP. We would then have a system in six variables, c/k, ℓ_A, u, X, Y, Z, three choice-like variables and three state-like variables, which would be completely equivalent to the system at hand.

semi-endogenous growth but without technological opportunity, such as the one of Jones (1995).[17]

Conclusively, if one adopts the technological opportunity approach presented here, then this model can generate cyclical dynamics along the transition to the BGP. The source of these cycles is found in the relative rates at which radical and incremental innovations arrive.

The intuition for the oscillatory dynamics is as follows. Incremental innovations at the same time reduce technological opportunity and improve actual technology. This technology then feeds back into radical innovations which increase technological opportunity. If radical innovations come at amounts lower than that of incremental innovations, then the increases in technological opportunity are small, and smaller than the reductions through incremental innovations. This leads to a monotonic convergence of the optimal ratio A/B to the steady-state value from above. On the contrary, if radical innovations come at relatively large amounts, then the same mechanism leads to convergence of A/B from below. In the case radical innovations arrive at some intermediate number, then the improvements in technological opportunity will be counterbalanced by the reductions through incremental innovations at an amount which makes the ratio of actual technology to technological opportunity fluctuate and converge to the steady-state value of A/B in a non-monotonic manner.

We can therefore conclude that growth in our model is subject to R&D-induced fluctuations around a trend which is ultimately driven by population growth.

3.4.2 *A comparison to the literature*   Our result of oscillatory dynamics ought to be compared to the predictions found in previous literature. The following list of contributions is obviously not exhaustive;[18] it only serves to illustrate the differences between our approach and the other ones. We have also grouped together articles where cycles are generated through channels which are broadly the same. In these articles, the main sources of oscillatory dynamics are:

(i)   The relationship between wage costs, population growth, and profits (Goodwin, 1967; Francois and Lloyd-Ellis, 2003). If growth is high, then unemployment falls, raising wages and decreasing profits. When growth is lower than population growth, this leads to a recession, increases unemployment and starts the cycle again;

(ii) Creative destruction (Aghion and Howitt, 1992; Cheng and Dinopoulos, 1992; Amable, 1996; Francois and Lloyd-Ellis, 2003; Phillips and Wrase, 2006);

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[17]Haruyama (2009) shows however that in discrete time, endogenous cycles may arise as long-run equilibria even in very standard R&D-based models, both of fully endogenous and semi-endogenous growth.

[18]Our focus here is primarily on technology and innovations. Other sources of fluctuations include uncertainty (e.g. Kydland and Prescott, 1982), preferences (Alvarez-Cuadrado *et al.*, 2004), different production technologies, etc.

(iii) Monopoly profits accrued from the distinction between fundamental and secondary innovations (Jovanovic and Rob, 1990; Cheng and Dinopoulos, 1992). These models are based on a similar distinction as our model is. Effectively though, the fluctuations in monopoly profits are the actual source of cycles;

(iv) General Purpose Technologies—GPTs (Bresnahan and Trajtenberg, 1995; Helpman and Trajtenberg, 1998; Freeman *et al.*, 1999). GPTs are assumed to have the property that before they can be applied, costly adaptation to them is required. This reduces economic growth in the short run but increases it afterwards;

(v) Discrete time (Haruyama, 2009). Some continuous-time models leading to a uniform convergence to a steady state may start exhibit limit cycles when rewritten in discrete time. This applies in particular to the increasing-variety R&D-based endogenous growth model and the R&D-based semi-endogenous growth model;

(vi) Linearity and discrete scientific breakthroughs (Li, 2001; Olsson, 2005). Due to the Olsson's linearity assumption, labour is either fully allocated to radical or to incremental research. The interplay between these two types of innovations leads to cycles just like in our model, but it does so through the bang-bang labour allocation and discrete jumps in technological opportunity. In Li (2001) discrete jumps in scientific knowledge are simply assumed.

Unlike all these works, our model generates smooth endogenous cycles through the interplay between currently available knowledge and technological opportunity. Because of the sources as well as the nature of cycles in our model, it cannot be considered a member of any of the groups (i)-(vi).

3.5 Sensitivity analysis and Andronov–Hopf bifurcations

We wish to provide further analysis of the complex dynamics by doing some comparative statics with respect to the decisive parameters. As is visible in Fig. 1, complex eigenvalues and thus oscillations occur for an intermediate range of arrival rates of the radical innovations relative to incremental innovations (ceteris paribus). Moreover, the oscillation frequency and thus the length of the cycles depends on the γ/δ ratio, with a maximum frequency appearing around $\gamma/\delta = 1$ (in the figure, this corresponds to $\gamma = \delta = 2$). Even though not decisive for the growth rate, these two arrival rates are decisive for the transition.

To take the analysis of the model's dynamics a step further, we shall now discuss the consequences of varying the returns-to-scale parameter β in its range (0, 1], as presented in Fig. 2.[19] In such case, not only dampened oscillations appear but also limit cycles. We identify an Andronov–Hopf bifurcation.

As can be seen in Fig. 2, as long as $\beta < 0.96569$, greater returns to scale in R&D imply faster convergence to the BGP but also oscillations of higher frequency.

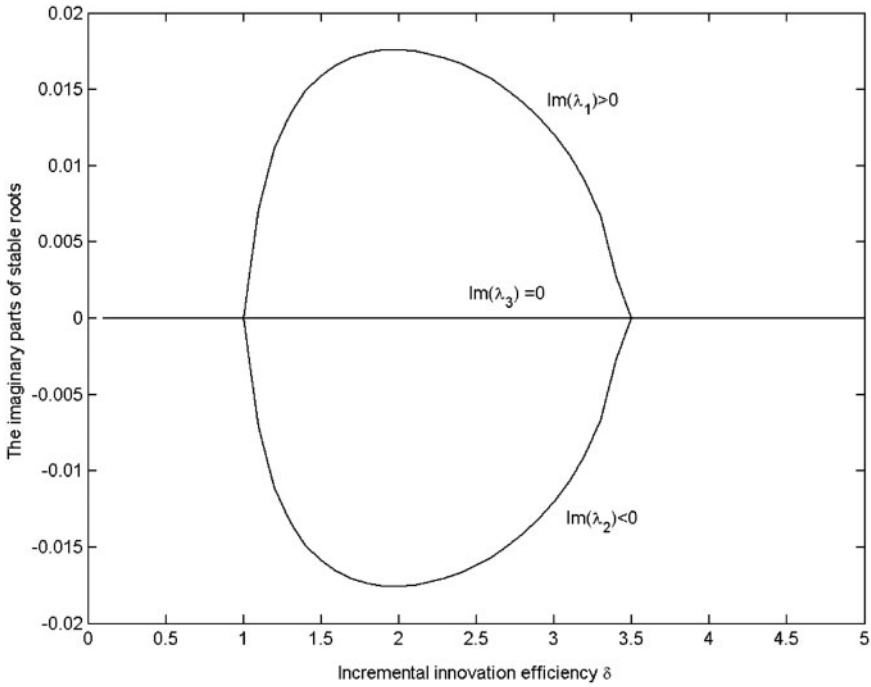[19]All other parameters are set at their benchmark values.

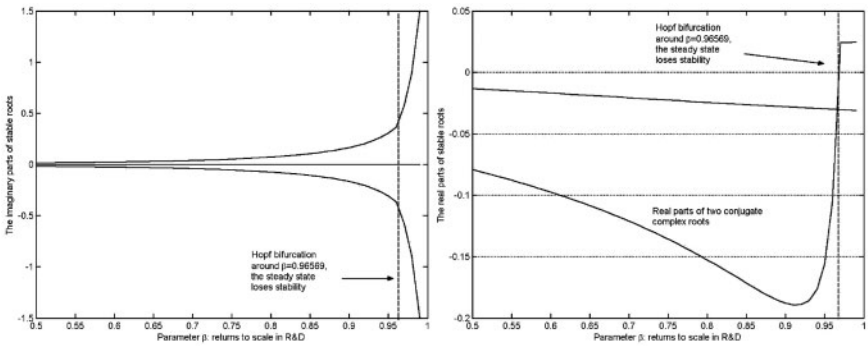**Fig. 1** Oscillatory dynamics. The effect of varying δ, holding $\gamma = 2$ fixed.

Fig. 2 Changes in dynamics following changes in β. The Andronov–Hopf bifurcation appears around $\beta \approx 0.96569$.

When β crosses the threshold value of 0.96569 from below, the relevant conjugate eigenvalues have their real parts crossing zero from below and there emerges a limit cycle. This means that we observe an Andronov–Hopf bifurcation. The intuition for this result is the following. If returns to scale in the R&D sector are reduced fast (low β), convergence to the BGP is monotonic, because R&D output is quickly
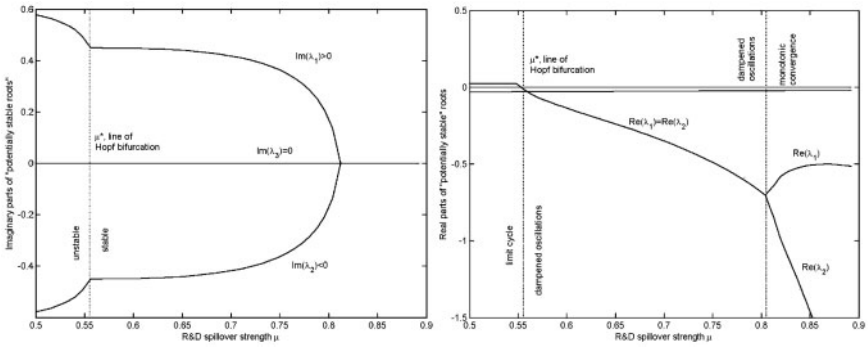
Fig. 3 Changes in dynamics following changes in μ. When $\beta = 0.97$, the Andronov–Hopf bifurcation appears around $\mu \approx 0.56$.

becoming less and less responsive to labour reallocations. This curbs the incentives to constantly reallocate researchers across the two R&D sectors and thus eliminates fluctuations. If β is larger then dampened oscillations appear, and their frequency increases with β. If returns to scale in R&D are high or close to constant, then incentives to reallocate labour are very strong, and oscillations become persistent. Obviously, as we noticed above, the intuition for cycles does not only depend on the returns to scale to labour in the R&D sector, but also requires the two kinds of innovations to come in approximately similar amounts.

We also notice that the Olsson's initial intuition for obtaining persistent cycles, namely the linearity assumption which he imposes (here it corresponds to $\beta = 1$), was correct. However, as we demonstrate, $\beta = 1$ is not a necessary condition for cycles in our setup. Indeed, optimal permanent cycles occur already for less-than-constant returns to scale in R&D, $\beta < 1$, although given our baseline parameters, β needs to be close to one.[20] Obviously, the higher is population growth, the lower the minimum value of β which leads to permanent cycles.

As suggested, even though oscillations may occur for rather low values of β, it seems that a necessary condition for limit cycles is a high value of β. It is not sufficient, however. To see that, let us assign β with a value of 0.97, close to the bifurcation value discussed above, and check the consequences of varying the R&D spillover parameter μ, as in Fig. 3. We find that the larger is the spillover parameter μ, the slower are the oscillations, and for sufficiently large values (here $\mu > 0.85$) oscillations disappear completely. The Andronov-Hopf bifurcation occurs again and is identified at $\mu \approx 0.56$. A smaller μ suggests a relatively higher importance of labour for the creation of innovations, which implies that the preferences of the social planner have a larger influence over the path of innovations (which we

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[20]This finding, of course, does not exclude the possibility that other parameter values lead to bifurcations for values of β significantly below one.
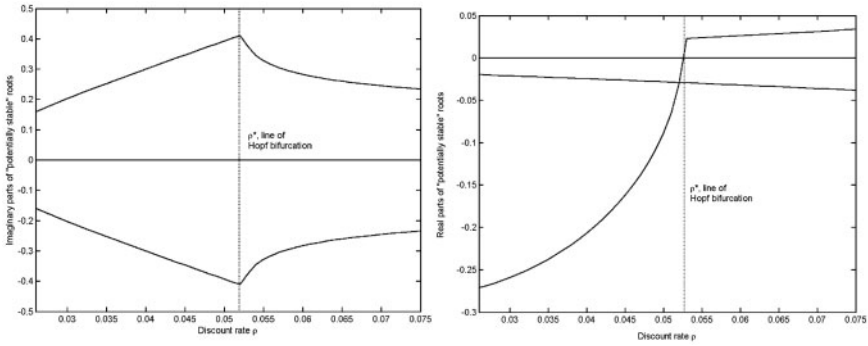
Fig. 4 Changes in dynamics following changes in the discount rate ρ. When $\beta = 0.97$, the Andronov–Hopf bifurcation appears around $\rho \approx 0.053$.

confirm below). On the contrary, the larger is μ the more important become the relative amounts of technology and technological opportunity.

We have also performed a similar analysis for changes in the discount rate, as in Fig. 4. At $\rho \approx 0.053$ we observe a similar Andronov–Hopf bifurcation. When $\rho < 0.053$, we have converging oscillations, and smaller values of ρ lead to oscillations of smaller frequency, as intuition would suggest. This result suggests that not only technology matters, but the preferences of the planner play a role, too. So, if the planner is sufficiently impatient (large ρ) then he will initially allocate more labour to incremental innovations, allowing faster growth now and therefore more consumption. However, because radical innovations will then come in smaller amounts, and technological opportunity will be gradually exhausted, at some point the planner will have to increase the number of researchers in the radical innovations sector in order to prevent economic stagnation. In the moment that enough technological opportunity will have been created, the planner will shift the workers again to the incremental innovations sector to satisfy his impatience for consumption. In case the discount rate is too large, the economy will never converge to the BGP because the planner will be too quick in reallocating labour across the two R&D sectors.

4. Conclusion

In this article we have analysed an R&D-based semi-endogenous growth model where technological advances depend on the available amount of technological opportunity. The model distinguishes two types of innovations which have different impacts on the evolution of knowledge: incremental innovations provide direct increases to the stock of knowledge but reduce the technological opportunity which is required for further incremental innovations, whereas radical innovations serve to renew this opportunity. Hence, technological opportunity behaves like a renewable resource. Even though the basic idea belongs to Olsson (2005), we have substantially generalized his framework in order to attain

direct comparability with established R&D-based growth models by Romer (1990) and Jones (1995).

We have analysed the model for its growth implications, leading to three key observations. First, it suggests that long-run growth along the lines of Jones (1995), Kortum (1997), and Segerstrom (1998) requires external returns to radical innovations to be at least as strong as those of incremental innovations. Furthermore, if one expects the returns to radical and incremental innovations to come with approximately the same output elasticity, then standard, analytically more tractable models (e.g., Jones, 1995) will be sufficient to estimate the growth effects of technological progress. If, however, one presumes that it is unlikely that this condition holds, then our model can help in deriving relevant long-run predictions. For example, as we have shown, economic growth can easily come to a halt if technological opportunity is not renewed sufficiently fast. This could be the case if technology spillovers in the radical innovations sector are too small.

The second novel result is related to the transition dynamics. Focusing on the case where returns to radical innovations are equal to those of incremental ones, we have demonstrated that, even though the long-run implications of this model will then be analogous to Jones (1995) and Segerstrom (1998), the transition need not. We have solved for the transition dynamics of the social planner allocation in our model, indicating the conditions when it need not be monotonic. Indeed, endogenous oscillations and limit cycles are obtained for a wide range of plausible parameter values. We therefore suggest that technological opportunity, as characterized here, can be a source of endogenous cycles even without uncertainty or an inefficient allocation.

Thirdly, we have also demonstrated that there are two reasons why the decentralized market outcome differs from the social planner allocation in our model. One, monopolistic mark-ups charged by intermediate goods producers slow down the accumulation of capital and imply a too high consumption to capital ratio (and income to capital ratio) in the decentralized equilibrium. Two, the allocation of labour between the innovation sector and the final goods sector may differ in the centralized and decentralized solution since (i) the final goods producers do not internalize their effect on the evolution of aggregate technology, (ii) the social planner, as opposed to intermediate goods producers, does not extract monopoly rents, and (iii) in the decentralized equilibrium, stronger complementarity between intermediate inputs reduces the wages in that sector and thus makes more workers willing to work in the final goods sector.

As a final note we would like to advocate the idea of technological opportunity as a concept which is extremely useful for the description of the evolution of technology. This is a rather novel, intuitive idea which still lacks sound empirical justification, and thus a thorough empirical analysis is certainly needed.

Supplementary material

Supplementary material (the Appendix) is available online at the OUP website.

Acknowledgements

Funding

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