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Factor-specific technology choice[☆]

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ABSTRACT

This paper analyzes the properties of a two-dimensional problem of factor-specific technology choice subject to a technology menu — understood as the choice of the degree of factor augmentation by a producing firm or the choice of quality of goods demanded by a consumer. The considered general setup encompasses the benchmark cases of Cobb–Douglas, CES and Leontief (minimum) functions. It is shown that the technology menu and the global function (envelope of local functions) are dual objects, in a well-defined generalized sense of duality. In the optimum, partial elasticities of (i) the local function, (ii) the technology menu and (iii) the global function are all equal and there exists a clear-cut, economically interpretable relationship between their curvatures. In particular, the elasticity of substitution of the global function is always above that of the local function. The paper also invokes Bergson's theorem to comment on the consequences of assuming homogeneity or homotheticity, with a particular focus on technology menus constructed as level curves of idea (unit factor productivity) distributions.

1. Introduction

The current paper deals with decision problems faced by firms which contemplate not just about the demand for production factors – such as capital and labor – but also about the degree of their technological augmentation (see e.g. Atkinson and Stiglitz, 1969; Basu and Weil, 1998; Acemoglu, 2003; Jones, 2005; Caselli and Coleman, 2006). Consider, for example, a firm owner planning to set up a new plant. She may build a traditional manufacture where production is labor-intensive, or a highly automated plant where all routine work is carried out by robots. Imagine that both factory designs require capital and labor inputs to be fed into the production process in essentially fixed, though vastly different proportions (Jones, 2005). Which of the available technologies should she choose? Logically, the higher is the market wage relative to the capital rental rate, the more inclined will she be to choose an automated plant — a technology which runs at higher capital–labor ratios, and thus is (under gross complementarity) both labor-augmenting and labor-saving. A similar problem is faced by a corporation which chooses between setting up web-based customer support service and a telephone-based one (León-Ledesma and Satchi, 2018): the former one embodies labor-saving technologies allowing the firm to employ less labor relative to capital, so it is more likely to be chosen if wage costs are relatively higher. By the same token, a declining trend in the price of computers relative to other equipment should induce firms to gradually re-organize production to be more ICT-intensive, and increasing oil prices — to adopt oil-saving production techniques, such as replacing diesel engines with electric ones.

Mathematically equivalent problems are also faced by consumers who are allowed to decide both about the quantity and quality of the demanded goods.[1] Consider, for example, a person who is about to buy a TV set and a piano. Next she will decide how much time to spend on watching TV and playing the piano. Both leisure activities provide utility and – in contrast to the previous examples – are substitutable, not complementary. Assume that an offer comes along, however, that the person could earn some money by playing the piano in public. This makes the time spent on TV viewing relatively more expensive. In such a scenario, compared to the initial one, she should be relatively more willing to spend more hours playing the piano, and in consequence — she will be wore willing to buy a better, more expensive piano rather than a higher quality TV set: a "piano-augmenting" technology choice. Under gross substitutability, however, this choice is accompanied

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1 Alternative interpretations of the factor-specific technology choice problem include also workers (or managers) who allocate their limited endowments of time/effort across two alternative tasks, and consumers who decide over their demand for two *goods characteristics* (Lancaster, 1966) and are also allowed to choose their attitudes towards these characteristics optimally from a behavior menu (Matveenko, 2016). The last interpretation suggests that the considered problem may also be viewed as a special case of multi-attribute utility theory (Dyer, 2005), as long as the consumer is allowed to optimize over the weights of attributes.

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0304-4068/

by increasing, not decreasing share of utility accruing to the augmented quantity: playing the piano.

The purpose of this paper is to provide a detailed treatment of a static, two-dimensional problem of optimal factor-specific technology choice, where the decision maker faces a menu of local technologies which depend on the quantity of the two factors and their respective quality (i.e., unit productivity). The menu features a trade-off insofar as choosing higher quality of one factor comes at the cost of reducing the quality of the other one. The decision maker is allowed to select her preferred technology, in order to maximize total output/profit/utility, for all configurations of factor quantities. The global function is then constructed as an envelope of local functions, as in Fig. 1. We concentrate on the technology choice problem only, setting the determination of factor demand and market equilibrium aside (see Appendix for discussion).

This paper has been motivated primarily by the earlier contributions to the theory of economic growth and factor-augmenting technical change (e.g., Basu and Weil, 1998; Acemoglu, 2003; Jones, 2005; Caselli and Coleman, 2006) but its appeal is much broader. The class of problems which we solve here can be applied potentially both in micro- and macroeconomics, and they can be viewed both as producer and consumer problems. Factor-specific technology choice problems could be useful, in particular, for addressing issues related to natural resources,[2] human capital and capital–skill complementarity,[3] industrial organization, international trade, labor markets, sectoral change, consumption patterns, social welfare, and so on. Unfortunately, the associated applied literature has remained rather scarce thus far because of the lack of direct empirical measures of firms' technology choices and the difficulty of their indirect identification from the data. In particular, in the dynamic setup the problem consists in delineating changes in technology choices (movements along a given technology menu) from factor-augmenting technological progress (shifts of the technology menu), Witajewski-Baltvilks (2015) and León-Ledesma and Satchi (2018).

Solutions to some specific cases of the factor-specific technology choice problem are well known. First, when the technology menu has the Cobb–Douglas form (which may arise, among other cases, if factor-specific ideas are independently Pareto-distributed; Jones, 2005) or if the local function is of such form (Growiec, 2008a), then the global function must also be of the Cobb–Douglas type. Second, combining a local function of a CES or a minimum (Leontief) form with a CES technology menu yields a global CES function (Growiec, 2008b; Matveenko, 2010; Growiec, 2013; León-Ledesma and Satchi, 2018).[4] Third, detailed treatment of the properties of factor-specific technology choice problems with a minimum (Leontief) local function, including their intriguing duality properties, has been provided by Rubinov and Glover (1998), Matveenko (1997, 2010) and Matveenko and Matveenko (2015).[5] The minimum function is however an extreme case, which may be viewed as both instructive and problematic. Fourth, several important results for the general factor-specific technology choice problem with an implicitly specified technology menu have been provided in section 2.3 of León-Ledesma and Satchi (2018).

Notwithstanding these results, the literature thus far has not devised a general theoretical framework allowing for a systematic treatment of the factor-specific technology choice problem in its generality. This paper fills this gap. Its contribution to the literature is four-fold.

First, we solve the generally specified static, two-dimensional problem of factor-specific technology choice. We find that a unique optimal factor-specific technology choice exists for any homothetic local function F and technology menu G. Plugging this choice into the local function F leads to a unique homogeneous (constant returns to scale) global function Φ, which may then be transformed to a homothetic form by an arbitrary monotone transformation. We also find that (i) the shape of the global function Φ depends non-trivially both on F and G unless one of them is of the Cobb–Douglas form, and (ii) the global function Φ offers more substitution possibilities (i.e., has less curvature) than the local function F unless the optimal technology choice is independent of factor endowments, which happens only if F is Cobb–Douglas or G follows a maximum function.

Second, we construct and solve the dual problem (in a well-defined generalized sense of duality) where, for every technology, the decision maker maximizes output/profit/utility subject to a requirement of producing a predefined quantity with the global technology Φ. Then, by plugging these optimal factor choices into the local function F, we obtain the technology menu G as an envelope. The results are fully analogous to the results of the primal problem.

Third, we exploit the duality property to find that in the optimum, partial elasticities of all three objects – the local function F, the technology menu G and the global function Φ – are all equal. We then identify a clear-cut, economically interpretable relationship between their curvatures, giving rise to interesting qualitative implications on concavity/convexity and gross complementarity/substitutability along the three functions. In particular, we find that when factors are gross complements along the local function (the typical case under the production function interpretation), then either the technologies are gross substitutes along the technology menu but the factors are gross complements along the global function, or vice versa, the technologies are gross complements along the technology menu and the factors are gross substitutes along the global function.

Fourth, we observe that the assumption of homotheticity which we make throughout the analysis, while shared by bulk of the associated literature, does not come without costs. The key limitation is Bergson's theorem (Bergson{Burk}, 1936): every homothetic function that is also additively separable (either directly or after a monotone transformation) must be either of the Cobb–Douglas or CES functional form. We apply this theorem, in particular, to probabilistic frameworks where the technology menu G is viewed as a level curve of a joint distribution of ideas (unit factor productivities), with the marginal idea distributions being either independent (Jones, 2005; Growiec, 2008b) or dependent following a certain copula (Growiec, 2008a). We show that such construction of the technology menu places a restriction on the considered class of functions G, often reducing them to the Cobb–Douglas or CES form. To demonstrate this, we adapt Bergson's theorem (Bergson{Burk}, 1936) to the case of copulas.

The closest related work is by León-Ledesma and Satchi (2018) who, among other objectives, address the general factor-specific technology choice problem (1). Thus there is some overlap of analytical results between both papers (as clearly indicated in the text below). There are also major differences, though. The current paper provides additional results for the primal problem (1), defines and solves the dual problem (2), and draws additional insights from duality (Theorems 4–5). Additional symmetries are uncovered by using an explicit, rather than implicit specification of the technology menu. Finally, it also extends the framework

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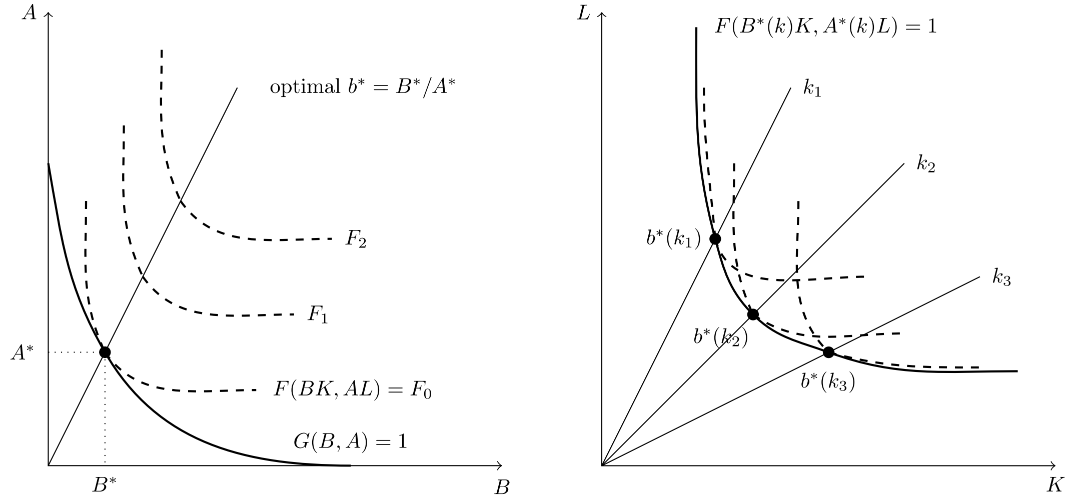
[2] Factor-specific technology choice problems arise naturally when studying the substitutability between exhaustible resources and accumulable physical capital (or renewable resources, cf. Dasgupta and Heal, 1979; Bretschger and Smulders, 2012) as well as human capital (or quality-adjusted labor, cf. Smulders and de Nooij, 2003).

[3] The choice of degree of factor augmentation becomes an important issue once one acknowledges that skilled and unskilled labor are imperfectly substitutable (e.g., Caselli and Coleman, 2006; Witajewski-Baltvilks, 2015) and potentially complementary to capital (Krusell et al., 2000; Duffy et al., 2004).

[4] The implications of factor-specific technology choice in the CES case have been also studied by Nakamura and Nakamura (2008) and Nakamura (2009).

[5] See also the book by Rubinov (2000).



Step 1. Optimal choice of factor-specific technologies B and A for given quantities K and L

Step 2. Construction of the global function $\Phi(K, L)$ as an envelope of local functions $F(BK, AL)$

Fig. 1. Construction of the global function from local functions by incorporating the optimal factor-specific technology choices.

to the case of homothetic functions and explains the key role of Bergson's theorem in the prevalence of Cobb–Douglas and CES functions, and Pareto and Weibull distributions, in the associated literature. León-Ledesma and Satchi (2018), in contrast, embed the factor-specific technology choice problem in a dynamic general equilibrium model allowing to differentiate between the short-run and long-run elasticity of substitution. Their model is then calibrated, estimated and taken to the data. It is the first framework allowing to reconcile gross complementarity of factors in the short run, a general form of factor-augmenting technical change, and long-run balanced growth, cleverly circumventing the Steady State Growth Theorem (Uzawa, 1961).

The paper is structured as follows. Section 2 presents the setup of the considered problem. In Section 3 we derive the optimal technology choice. In Section 4 we plug it into the local production function and thus build the envelope. Section 5 discusses the most instructive special cases known from the literature. Section 6 presents the similarities and differences between the homogeneous and the homothetic case. Section 7 studies the link between the technology menu and distributions of ideas. Section 8 concludes. A discussion of the relationships between our setup and the problem of output/utility maximization subject to a budget constraint, as well as the literature on factor-augmenting technical change, can be found in Appendix.

2. The primal and dual optimization problem

For the clarity of exposition, we shall first consider the case where the local function F, the technology menu G and the global function Φ are homogeneous. A generalization to the homothetic case is delegated to Section 6.

2.1. The primal problem

In the primal problem, the decision maker (the output- or profit-maximizing firm, the utility-maximizing consumer) maximizes a local function $F(BK, AL)$ with respect to the technology pair (B, A) taken from a level curve of the technology menu $G(B, A)$, taking $K > 0$ and $L > 0$ as given.[6] The global function $\Phi(K, L)$ is obtained as an envelope, by plugging the optimal choices $(B^*(K, L), A^*(K, L))$ into the local function. Formally, we write:

$$\Phi(K, L) = \max_{(B,A)\in\Omega_G} F(BK, AL) \qquad s.t.$$
$$\Omega_G = \{(B, A) \in \mathbb{R}^2_+ : G(B, A) = 1\}. \tag{1}$$

In the basic treatment of the static problem (1), it is assumed that the local function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous (constant-returns-to-scale, CRS). Homogeneity permits to rewrite F in its intensive form, $F(BK, AL) = F\left(\frac{BK}{AL}, 1\right) AL = f(bk)AL$, where $b = B/A$ and $k = K/L$. The local function F is interpreted as the local (short-run, exogenous-technology) production function faced by a firm or utility function of a consumer. Each of its arguments is a product of a quantity (K or L) and its quality multiplier, i.e., unit factor productivity (B or A, respectively). Finally, while mathematically this is not necessary, economic interpretation of the local function implies that in typical applications, it should be concave in each of its arguments.

Symmetrically, we also assume that the technology menu $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous. Analogously, we rewrite G in its intensive form, $G(B, A) = G\left(\frac{B}{A}, 1\right) A = g(b)A$. The technology menu G is a function which maps factor-specific quality levels to a scalar, interpreted as an overall "technology level" of the economy as faced by the decision maker. Under the production function interpretation, we say that the larger is the value of G, the more can be produced from given inputs; under the utility function interpretation, the value of G scales total utility attainable from the given endowment of goods.

2.2. The dual problem

In the dual problem, the decision maker maximizes a local function $F(BK, AL)$ with respect to the quantities (K, L), subject to maintaining a predefined level of output/utility given by the global function $\Phi(K, L)$, and taking the factor-specific technologies $B > 0$

[6] We denote the quantities K and L so that they are easily recognized as "capital" and "labor", in line with the production function interpretation of the discussed framework. However, this is done only to keep the discussion close to the associated literature. In fact, the theory can be applied just as well to utility maximization problems, where K and L are understood as quantities of two goods demanded by a consumer, as well as to production functions with any other pair of inputs.

and $A > 0$ as given. The technology menu $G(B, A)$ is obtained as an envelope, by plugging the optimal choices $(K^*(B, A), L^*(B, A))$ into the local function. Formally, we write:

$$G(B, A) = \max_{(K,L)\in\Omega_\Phi} F(BK, AL) \qquad s.t.$$
$$\Omega_\Phi = \{(K, L) \in \mathbb{R}^2_+ : \Phi(K, L) = 1\}. \tag{2}$$

In the basic treatment of the static problem (2), it is assumed that – alike the local function F – the global function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous. Due to homogeneity, we may rewrite Φ in its intensive form, $\Phi(K, L) = \Phi\left(\frac{K}{L}, 1\right) L = \phi(k)L$.

The difference between the local function F and the global function Φ is that the former maps the quantities of inputs into output keeping factor-specific technologies fixed, whereas the latter allows them to be chosen optimally. Under the production function interpretation, it is therefore natural to think of the local function as a short-run production function, and of the global function – as a long-run one (León-Ledesma and Satchi, 2018). Analogously, under the utility function interpretation the local function is a short-run utility function which takes attitudes towards goods characteristics as given, whereas the global function is a long-run utility function which also accounts for endogenous behavior formation (Matveenko, 2016). Again, economic interpretation of the global function implies that in typical applications, it should be concave in each of its arguments.

2.3. Homotheticity, additive separability and Bergson's theorem

Homotheticity of the considered functions has profound consequences. Importantly, ever since (Bergson{Burk}, 1936) we know that every homothetic and additively separable function must be either of the Cobb–Douglas or of the CES form. In the symbols of our current study, Bergson's theorem can be stated as follows:

Theorem 1 (*Bergson{Burk}, 1936*). *Let $F_h : \mathbb{R}^2_+ \to \mathbb{R}$ be a homothetic function which can be written as additively separable after a monotone transformation:*

$$\exists (f_h : \mathbb{R}_+ \to \mathbb{R},\ F : \mathbb{R}^2_+ \to \mathbb{R}_+) \quad F_h(x, y) = f_h(F(x, y)), \tag{3}$$

$$\exists (f_s : \mathbb{R}_+ \to \mathbb{R},\ D_x, D_y : \mathbb{R}_+ \to \mathbb{R}) \quad F_h(x, y) = f_s(D_x(x) + D_y(y)), \tag{4}$$

where f_h, f_s, D_x, D_y are monotone differentiable functions and F is an increasing, twice differentiable homogeneous function. Then either

$$D_x(x) = \alpha \ln x + c_x,\ D_y(y) = \beta \ln y + c_y$$
$$\Rightarrow F(x, y) = c \cdot x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}}, \tag{5}$$

where α, β, c_x, c_y are arbitrary constants and $c = \exp\left(\frac{c_x+c_y}{\alpha+\beta}\right)$, or

$$D_x(x) = \alpha x^\rho + c_x,\ D_y(y) = \beta y^\rho + c_y$$
$$\Rightarrow F(x, y) = (\alpha x^\rho + \beta y^\rho)^{\frac{1}{\rho}}, \tag{6}$$

where α, β, c_x, c_y are arbitrary constants and $\rho \neq 0$.

Proof. See Bergson{Burk} (1936) or Rader (1972), Theorem 8, page 212. ■

Bergson's theorem is the fundamental analytical cause why studies aiming at generalizing the CES framework must either give up homotheticity (e.g., Zhelobodko et al., 2012) or additive separability (e.g., Revankar, 1971; Growiec and Mućk, 2016 and this paper). It follows that in all the non-CES cases covered by the current study, the functions F, G and Φ cannot be written down as additively separable after any monotone transformation, a property shared among others by isoelastic elasticity of substitution (IEES) functions defined in Growiec and Mućk (2016).

2.4. Discussions and clarifications

Let us clarify a few important concepts before we present our main results.

Generalized duality. Problems (1) and (2) are dual to one another, although not in the standard, linear sense of duality (Diewert, 1993). Instead they are dual when taking the local function F as a (typically non-linear) linking function. This generalized form of duality ("F-duality") encompasses linear duality as a special case (after a switch from maximization to minimization in the dual problem). At the same time, it also generalizes *idempotent duality*, where the linking function is a minimum function (Rubinov and Glover, 1998; Matveenko and Matveenko, 2015).[7] The latter case can be viewed as a limiting case of F-duality.

Partial elasticities. Partial elasticities of homogeneous functions F, G and Φ with respect to their first arguments are defined as:

$$\pi_F(bk) = \frac{\partial F}{\partial (BK)}(BK, AL)\frac{BK}{F(BK, AL)} = \frac{f'(bk)bk}{f(bk)} > 0, \tag{7}$$

$$\pi_G(b) = \frac{\partial G}{\partial B}(B, A)\frac{B}{G(B, A)} = \frac{g'(b)b}{g(b)} > 0, \tag{8}$$

$$\pi_\Phi(k) = \frac{\partial \Phi}{\partial K}(K, L)\frac{K}{\tilde{\Phi}(K, L)} = \frac{\phi'(k)k}{\phi(k)} > 0. \tag{9}$$

Homogeneity implies that $\pi \in [0, 1]$ for all three functions and that partial elasticities with respect to their second arguments are equal to $1 - \pi$. It is also useful to define the *relative elasticities* Π, strictly increasing in π, as

$$\Pi_F(bk) = \frac{\pi_F(bk)}{1 - \pi_F(bk)} > 0, \qquad \Pi_G(b) = \frac{\pi_G(b)}{1 - \pi_G(b)} > 0,$$
$$\Pi_\Phi(bk) = \frac{\pi_\Phi(k)}{1 - \pi_\Phi(k)} > 0. \tag{10}$$

If one also assumed that factor markets were perfectly competitive, partial elasticities π and $1-\pi$ would also be equal to the respective factors' shares of output.

Curvature. We define the *curvature* of homogeneous functions F, G and Φ as:

$$\theta_F(bk) = -\frac{f''(bk)bk}{f'(bk)}, \qquad \theta_G(bk) = -\frac{g''(b)b}{g'(b)},$$
$$\theta_\Phi(k) = -\frac{\phi''(k)k}{\phi'(k)}. \tag{11}$$

Hence, our measure of curvature is the Arrow–Pratt coefficient of relative risk aversion, also called the relative love of variety (Zhelobodko et al., 2012). The curvature $\theta(x)$ is positively linked to the partial elasticity $1 - \pi(x)$ and inversely linked to the elasticity of substitution $\sigma(x)$, as in

$$\theta(x) = \frac{1 - \pi(x)}{\sigma(x)}. \tag{12}$$

As compared to the elasticity of substitution, the curvature $\theta(x)$ is relatively better suited to the simultaneous analysis of concave as well as convex functions: the curvature is always positive ($\theta(x) > 0$ for all x) for concave functions, always negative ($\theta(x) < 0$ for all x) for convex functions, and the curvature of linear functions is zero.[8]

Normalization. We carry out our analysis in *normalized units*. Production function normalization has been shown to be crucial for

[7] The term "idempotent duality" belongs to the realm of *tropical mathematics*. I am grateful to Matveenko and Matveenko (2015) for acquainting me with this notion. I was however deeply disappointed when I learned that *tropical mathematics* has nothing to do with *polar coordinates*.

[8] See Matveenko and Matveenko (2014) for a more detailed discussion of the relationship between $\theta(x)$ and $\sigma(x)$.

obtaining clean identification of the role of each parameter of the CES function (de La Grandville, 1989; Klump and de La Grandville, 2000; Klump et al., 2012). Its usefulness has also been demonstrated beyond the CES class (Growiec and Mućk, 2016) as well as for factor-specific technology choice problems (Growiec, 2013).

To maintain normalization while economizing on notation, we assume that K, L, B, A, k and b are already given in normalized units[9]:

$$K=\frac{\tilde{K}}{\tilde{K}_0},\qquad L=\frac{\tilde{L}}{\tilde{L}_0},\qquad B=\frac{\tilde{B}}{\tilde{B}_0},\qquad A=\frac{\tilde{A}}{\tilde{A}_0},\qquad k=\frac{\tilde{k}}{\tilde{k}_0},\qquad b=\frac{\tilde{b}}{\tilde{b}_0}. \tag{13}$$

Output is normalized in the same way as the inputs. We posit that $\tilde{G}(\tilde{B}_0,\tilde{A}_0)=G_0 \iff G(1,1)=1$ as well as $\tilde{\Phi}(\tilde{K}_0,\tilde{L}_0)=\Phi_0 \iff \Phi(1,1)=1$. Thus the level curves are

$$\Omega_G=\{(B,A)\in\mathbb{R}^2_+ : G(B,A)=1\} =\{(\tilde{B},\tilde{A})\in\mathbb{R}^2_+ : \tilde{G}(\tilde{B},\tilde{A})=G_0\}, \tag{14}$$

$$\Omega_\Phi=\{(K,L)\in\mathbb{R}^2_+ : \Phi(K,L)=1\} =\{(\tilde{K},\tilde{L})\in\mathbb{R}^2_+ : \tilde{\Phi}(\tilde{K},\tilde{L})=\Phi_0\}. \tag{15}$$

We also normalize the partial elasticities of the considered functions F, G and Φ:

$$\pi_{0F}\equiv\frac{\partial\tilde{F}}{\partial(\tilde{B}\tilde{K})}(\tilde{B}_0\tilde{K}_0,\tilde{A}_0\tilde{L}_0)\frac{\tilde{B}_0\tilde{K}_0}{\tilde{F}(\tilde{B}_0\tilde{K}_0,\tilde{A}_0\tilde{L}_0)}=\frac{f'(1)\cdot 1}{f(1)}=f'(1), \tag{16}$$

$$\pi_{0G}\equiv\frac{\partial\tilde{G}}{\partial\tilde{B}}(\tilde{B}_0,\tilde{A}_0)\frac{\tilde{B}_0}{\tilde{G}(\tilde{B}_0,\tilde{A}_0)}=\frac{g'(1)\cdot 1}{g(1)}=g'(1), \tag{17}$$

$$\pi_{0\Phi}\equiv\frac{\partial\tilde{\Phi}}{\partial\tilde{K}}(\tilde{K}_0,\tilde{L}_0)\frac{\tilde{K}_0}{\tilde{\Phi}(\tilde{K}_0,\tilde{L}_0)}=\frac{\phi'(1)\cdot 1}{\phi(1)}=\phi'(1). \tag{18}$$

In our discussion of examples, we will pay special attention to the case $\pi_{0F}=\pi_{0G}=\pi_{0\Phi}$. Such coincidence cannot be guaranteed for arbitrary functions, but it leads to particularly transparent outcomes whenever it happens to hold.

3. Optimal technology choice

To solve the primal optimization problem for a given pair (K, L), we set up the following Lagrangian $\mathcal{L}_P$:

$$\mathcal{L}_P(B,A)=F(BK,AL)+\lambda(G(B,A)-1). \tag{19}$$

We find that as long as the curvature of the local function F exceeds the curvature of the technology menu G (i.e., there are relatively few substitution possibilities along the local function), there exists a unique interior solution to the problem which equalizes partial elasticities of the local function and the technology menu. We also find that the optimal technology choice is biased towards the abundant factor ($\frac{\partial b^*(k)}{\partial k}>0$) if factors are gross substitutes along a concave local function or if the local function is convex ($1-\pi_F(bk)-\theta_F(bk)>0$, which requires that $\sigma_F(bk)>1$ or $\sigma_F(bk)<0$). Otherwise, optimal technology choice is biased towards the scarce factor ($\frac{\partial b^*(k)}{\partial k}<0$). Then factors are gross complements along a concave local function ($\sigma_F(bk)\in(0,1)$). In the intermediate, knife-edge case where the local technology is Cobb–Douglas ($\sigma_F(bk)=1$), optimal technology choice does not depend on factor endowments, i.e., $b^*(k)$ is constant.

[9] In empirical studies, variables are often normalized around sample means (Klump et al., 2007, 2012).

Theorem 2. *Let $F, G:\mathbb{R}^2_+\to\mathbb{R}_+$ be increasing, twice continuously differentiable homogeneous functions satisfying $\theta_F(b^*(k)k)>\theta_G(b^*(k))$ for a given pair $(K,L)\in\mathbb{R}^2_+$, and excluding the case where both of them are Cobb–Douglas functions. Then the problem (1) allows a unique interior maximum where*

$$\Pi_F(b^*(k)k)=\Pi_G(b^*(k)), \tag{20}$$

and

$$B^*(k)=\frac{b^*(k)}{g(b^*(k))},\qquad A^*(k)=\frac{1}{g(b^*(k))}. \tag{21}$$

The partial elasticity of the optimal technology choice $b^*(k)$ equals:

$$\frac{\partial b^*(k)}{\partial k}\frac{k}{b^*(k)}=\frac{1-\pi_F(bk)-\theta_F(bk)}{\theta_F(bk)-\theta_G(b)}. \tag{22}$$

Proof. See Appendix. ■

Eq. (20) is equivalent to eq. (2.5) in León-Ledesma and Satchi (2018) and eq. (7) in Jones (2005). Furthermore, the restriction that $\theta_F(b^*(k)k)>\theta_G(b^*(k))$ is equivalent to eq. (2.10) in León-Ledesma and Satchi (2018).

The construction of the dual problem is similar to its primal counterpart, so the results are alike as well. Proof of the following theorem is fully symmetric to the proof of Theorem 2 and therefore has been omitted.

Theorem 3. *Let $F, \Phi:\mathbb{R}^2_+\to\mathbb{R}_+$ be increasing, twice continuously differentiable homogeneous functions satisfying $\theta_F(bk^*(b))>\theta_\Phi(k^*(b))$ for a given pair $(B,A)\in\mathbb{R}^2_+$, and excluding the case where both of them are Cobb–Douglas functions. Then the problem (2) allows a unique interior maximum where*

$$\Pi_F(bk^*(b))=\Pi_\Phi(k^*(b)), \tag{23}$$

and

$$K^*(b)=\frac{k^*(b)}{\phi(k^*(b))},\qquad L^*(b)=\frac{1}{\phi(k^*(b))}. \tag{24}$$

The partial elasticity of the optimal factor choice $k^*(b)$ equals:

$$\frac{\partial k^*(b)}{\partial b}\frac{b}{k^*(b)}=\frac{1-\pi_F(bk)-\theta_F(bk)}{\theta_F(bk)-\theta_\Phi(k)}. \tag{25}$$

Having identified the optimal choices in the primal and dual problem, we are now in a position to insert them into the local function and thus to construct the appropriate envelopes.

4. The global function and the technology menu: construction, duality, and curvature

The global function Φ is constructed as an envelope of local functions by inserting the optimal technology choices from the primal problem, as derived in Theorem 2, into the local function F. Symmetrically, the technology menu G is constructed as an envelope of local functions by inserting the optimal factor choices from the dual problem, as derived in Theorem 3. The domains of both envelopes include all arguments for which interior optimal choices exist, i.e., all arguments for which the curvature of the local function exceeds the one of the constraint. The resultant envelopes have the following properties.

Theorem 4. *Let $\mathcal{D}_\Phi=\{(K,L)\in\mathbb{R}^2_+ : \theta_F(b^*(k)k)>\theta_G(b^*(k))\}$ and $\mathcal{D}_G=\{(B,A)\in\mathbb{R}^2_+ : \theta_F(bk^*(b))>\theta_\Phi(k^*(b))\}$ where $b^*(k)$ solves (20) and $k^*(b)$ solves (23). Then there exists a unique increasing homogeneous* global function *$\Phi:\mathcal{D}_\Phi\to\mathbb{R}_+$ solving problem (1) as well as a unique increasing homogeneous* technology menu

$G : \mathcal{D}_G \to \mathbb{R}_+$ solving problem (2). Their respective intensive forms are given by:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))}, \qquad g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))}. \tag{26}$$

Proof. See Appendix. ■

Please note that while both functions g and ϕ are increasing in their arguments, the optimal choices $b^*(k)$ and $k^*(b)$ do not have to be monotone (and hence, bijective). Therefore the mutual duality ("F-duality") of the global function and the technology menu must be limited to the domain where $b^*(k)$ and $k^*(b)$ are monotone, and thus can be inverted, so that $k = k^*(b^*(k))$ and $b = b^*(k^*(b))$, as well as:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \frac{f(b^*(k)k^*(b^*(k)))}{g(b^*(k))} = \frac{f(b^*(k)k^*(b^*(k)))}{\frac{f(b^*(k)k^*(b^*(k)))}{\phi(k^*(b^*(k)))}} = \phi(k^*(b^*(k))), \tag{27}$$

$$g(b) = \frac{f(bk^*(b))}{\phi(k^*(b))} = \frac{f(b^*(k^*(b))k^*(b))}{\phi(k^*(b))} = \frac{f(b^*(k^*(b))k^*(b))}{\frac{f(b^*(k^*(b))k^*(b))}{g(b^*(k^*(b)))}} = g(b^*(k^*(b))). \tag{28}$$

Interestingly, however, these intervals coincide precisely with the domain in which both inputs are either (i) gross complements along a concave local function (with $\sigma_F(bk) \in (0, 1)$ and $1 - \pi_F(bk) - \theta_F(bk) < 0$), or (ii) gross substitutes along a concave local function with the additional possibility of a convex local function (i.e., the case where $1 - \pi_F(bk) - \theta_F(bk) > 0$, requiring that either $\sigma_F(bk) > 1$ or $\sigma_F(bk) < 0$). Most of the production function studies thus far concentrated on the former possibility (e.g., Rubinov and Glover, 1998; Jones, 2005; Growiec, 2013; Matveenko and Matveenko, 2015) and assumed that factors are always gross complements along the local production function. We generalize these studies by accommodating both variants.

Theorem 5. *Let $\Omega = \{(k, b) \in \mathcal{D}_\Phi \times \mathcal{D}_G : 1-\pi_F(bk)-\theta_F(bk) \neq 0\}$. Then for each connected subset of Ω, both equalities in (26) hold simultaneously and partial elasticities of F, G and Φ are equal:*

$$\pi = \pi_F(bk) = \pi_G(b) = \pi_\Phi(k). \tag{29}$$

For all $(k, b) \in \Omega$ it also holds that $1 - \pi - \theta_G(b) = 0 \iff 1-\pi-\theta_\Phi(k) = 0$ and otherwise the curvatures of the three functions are linked[10] *via*

$$\frac{1}{1-\pi-\theta_F(bk)} = \frac{1}{1-\pi-\theta_G(b)} + \frac{1}{1-\pi-\theta_\Phi(k)}. \tag{30}$$

Proof. See Appendix. ■

Eq. (30), equivalent to eq. (2.15) in León-Ledesma and Satchi (2018), is a precise quantitative description of the relationship between the curvatures of the local function, the technology menu and the global function. It also has some very intuitive properties. First, $\theta_F(bk)$ always exceeds both $\theta_G(b)$ and $\theta_\Phi(k)$. Hence, factor-specific technology choice always adds more flexibility to the local function, thereby decreasing its curvature (and thus, under concavity, increasing its elasticity of substitution, Growiec, 2013; León-Ledesma and Satchi, 2018).

Second, it is instructive to evaluate the signs of both sides of (30). If the left-hand side is negative, meaning that factors are gross complements along the local function (by all means the usual case in the production function literature), then $1 - \pi - \theta_G(b)$ and $1 - \pi - \theta_\Phi(k)$ must be of opposing signs. Hence, it must be that either the technologies are gross substitutes along the technology menu but the factors are gross complements along the global function, or vice versa, the technologies are gross complements along the technology menu and the factors are gross substitutes along the global function. Intuitively, if the *technologies b* are easily substituted with one another then their choice is of relatively minor importance for the effective input ratio bk; then the substitutability of *inputs k* must remain low. Conversely, if the technologies come in almost fixed proportions then even small changes in b will exert a major impact on bk. In such a case, optimal technology choice is a potent force, able to make inputs k easily substitutable along the global function. For example, the intriguing case where factors are gross complements in the short run but gross substitutes over the long run ($\sigma_F(bk) < 1 < \sigma_\Phi(k)$), deliberated by León-Ledesma and Satchi (2018), can be straightforwardly obtained by assuming that the technologies are gross complements along the technology menu, $1 - \pi - \theta_G(b) < 0$ with $\theta_F(bk) > \theta_G(b)$.

The remaining possibility is that the left-hand side of (30) is positive, so that the factors are gross substitutes already along the local function. In such a situation, both $1 - \pi - \theta_G(b)$ and $1 - \pi - \theta_\Phi(k)$ must be positive as well, encompassing the cases of gross substitutability and convexity.

Additional findings follow from considering the factor-specific technology choice problem jointly with the problem of output/utility maximization subject to a budget constraint (see Appendix for details). Importantly, meaningful and potentially applicable results are obtained here also for the case of gross substitutability along the local function – which is in fact quite natural if K and L are interpreted e.g. as skilled and unskilled labor in the production function (Caselli and Coleman, 2006), or as two similar consumption goods in the utility function. An interior maximum (e.g., a competitive equilibrium) is obtained in the joint optimization problem as long as the marginal rate of substitution crosses the $-\frac{w}{r}$ ratio (i.e., minus the relative price of L) for some configuration of factors (K, L) along the isoquant/indifference curve of both the local and global function. This additional condition is relatively easy to verify: for example, it is automatically verified for all CES functions, both with gross complementarity and gross substitutability of factors (Growiec and Mućk, 2016).

In contrast, an interior maximum of the joint problem cannot be sustained in the case of a convex local or global function ($\theta_F(bk) < 0$ or $\theta_\Phi(k) < 0$): it would then be optimal to specialize completely in only one factor. Thus the framework's ability to deliver a convex global function – even from a concave local function – should be treated rather as a theoretical curiosity than a legitimate option to be used in model building.

5. Notable special cases

Several special parametrizations of the above general setup have been discussed in the literature. We shall now provide an overview of these cases, thus underscoring the wide applicability of our general theorems. Most notably, under certain assumptions they can be used as microfoundation for global Cobb–Douglas and CES production/utility functions.

5.1. The Cobb–Douglas function

The primal problem with a Cobb–Douglas technology menu has been studied by, among others, Jones (2005) and León-Ledesma and Satchi (2018), Lemma 1. Its variant with a Cobb–Douglas local function has been reviewed as an example in Growiec (2008a). The appendix to Growiec (2013) has also considered the case of a continuum of factors. Here we reproduce these results as special

[10] For more intuition, note that Ω could also be understood as the set of arguments for which F has non-unitary elasticity of substitution, $\sigma_F(bk) \neq 1$. Moreover, the statement $1 - \pi - \theta_G(b) = 0 \iff 1 - \pi - \theta_\Phi(k) = 0$ is equivalent to $\sigma_G(b) = 1 \iff \sigma_\Phi(k) = 1$, representing the case where both G and Φ have a locally unitary elasticity of substitution.

cases of our general theory as well as elucidate certain important problems which may arise in the primal and dual problems under this particular parametrization.

Cobb–Douglas local function. If the local function is of the homogeneous, normalized Cobb–Douglas form, then:

$$F(BK, AL) = (BK)^{\pi_{0F}}(AL)^{1-\pi_{0F}}, \qquad f(bk) = (bk)^{\pi_{0F}}. \tag{31}$$

Assuming that G is not Cobb–Douglas and that $\theta_G(b) < 1 - \pi_{0F}$, from (20) we obtain that the optimal technology choice is independent of k:

$$\Pi_{0F} = \Pi_G(b) \Rightarrow b^*(k) \equiv b^* = \Pi_G^{-1}(\Pi_{0F}). \tag{32}$$

Inserting this choice for all $(K, L) \in \mathbb{R}^2_+$, from (26) we obtain:

$$\phi(k) = \frac{f(b^*k)}{g(b^*)} = \left(\frac{(b^*)^{\pi_{0F}}}{g(b^*)}\right) k^{\pi_{0F}} \Rightarrow \Phi(K, L) = \left(\frac{(b^*)^{\pi_{0F}}}{g(b^*)}\right) K^{\pi_{0F}} L^{1-\pi_{0F}}. \tag{33}$$

It means that irrespective of the shape of G, the global function must be Cobb–Douglas with the same exponent π_{0F} as the local function. The shape of G affects only the multiplicative constant, i.e., total factor productivity (TFP). If, additionally, $\pi_{0F} = \pi_{0G}$ then $b^* = 1$ and hence the constant becomes equal to unity, implying $\Phi(K, L) = K^{\pi_{0F}} L^{1-\pi_{0F}}$.

A fully symmetric result is obtained when solving the dual problem with a Cobb–Douglas local function.

While intuitive, the case of Cobb–Douglas local functions is *pathological* in the sense that the technology menu and the global function cannot be viewed as dual objects because the optimal choice is constant and thus not invertible. Indeed, trying to solve the primal problem when F and G are both Cobb–Douglas functions with the same exponent π_{0F}, immediately leads to indeterminacy:

$$\max_{(B,A)\in\mathbb{R}^2_+} F(BK, AL) = (BK)^{\pi_{0F}}(AL)^{1-\pi_{0F}} \qquad s.t. \qquad G(B, A) = B^{\pi_{0F}}A^{1-\pi_{0F}} = 1 \tag{34}$$

implies maximizing $K^{\pi_{0F}}L^{1-\pi_{0F}}$ which does not depend on B and A. Indeterminacy would also follow if we tried to solve the dual problem when F and Φ are both Cobb–Douglas with the same exponent π_{0F}.

This pathological outcome is a direct consequence of violation of the curvature assumption in Theorem 2 (when solving the primal problem while assuming that F and G are Cobb–Douglas functions with the same exponent) or in Theorem 3 (when making this assumption for F and Φ in the dual problem).

Cobb–Douglas technology menu. Let us now consider the case where the technology menu G is Cobb–Douglas with an exponent π_{0G}:

$$G(B, A) = B^{\pi_{0G}}A^{1-\pi_{0G}}, \qquad g(b) = b^{\pi_{0G}} \tag{35}$$

and the local function exhibits more curvature, $\theta_F(bk) > 1 - \pi_{0G}$. In this case, the optimal technology choice is monotone and thus duality is present again. From (20) we obtain

$$\Pi_F(b^*(k)k) = \Pi_{0G} \Rightarrow b^*(k) = \frac{\Pi_F^{-1}(\Pi_{0G})}{k}. \tag{36}$$

Inserting this choice for all $(K, L) \in \mathbb{R}^2_+$, from (26) we obtain:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \left(\frac{f(\Pi_F^{-1}(\Pi_{0G}))}{(\Pi_F^{-1}(\Pi_{0G}))^{\pi_{0G}}}\right) k^{\pi_{0G}} \Rightarrow \Phi(K, L) = \left(\frac{f(\Pi_F^{-1}(\Pi_{0G}))}{(\Pi_F^{-1}(\Pi_{0G}))^{\pi_{0G}}}\right) K^{\pi_{0G}} L^{1-\pi_{0G}}. \tag{37}$$

It means that irrespective of the shape of F, the global function must be Cobb–Douglas with the same exponent π_{0G} as the technology menu. The shape of F affects only the multiplicative constant, i.e., total factor productivity (TFP). If, additionally, $\pi_{0F} = \pi_{0G}$ then $b^*(k) = 1/k$ and hence the constant becomes equal to unity, implying $\Phi(K, L) = K^{\pi_{0G}} L^{1-\pi_{0G}}$.

Cobb–Douglas global function. The dual problem for a Cobb–Douglas global function $\Phi(K, L) = K^{\pi_{0\Phi}} L^{1-\pi_{0\Phi}}$ is solved analogously. Irrespective of the shape of F, the technology menu must be Cobb–Douglas with the same exponent $\pi_{0\Phi}$ as the global function. The shape of F affects only the multiplicative constant, i.e., the overall technology level in the economy. If, additionally, $\pi_{0F} = \pi_{0\Phi}$ then $k^*(b) = 1/b$ and hence the constant becomes equal to unity, implying $G(B, A) = B^{\pi_{0\Phi}} A^{1-\pi_{0\Phi}}$.

5.2. The CES function

The primal problem with a CES local function and a CES technology menu has been analyzed by, among others, Growiec (2008b, 2013). The former study also touched upon the dual problem, whereas the appendix to the latter considered the more general case of a continuum of factors.

It turns out that with a CES (or Leontief) local function, a CES technology menu is dual to a CES global function — and vice versa. Let us now briefly review this case as a specific application of our general theory.

Formally, for the primal problem let us assume that

$$F(BK, AL) = (\pi_{0F}(BK)^{\rho} + (1 - \pi_{0F})(AL)^{\rho})^{\frac{1}{\rho}}, \qquad G(B, A) = (\pi_{0G}B^{\alpha} + (1 - \pi_{0G})A^{\alpha})^{\frac{1}{\alpha}}, \tag{38}$$

with $\rho \neq 0$ and $\alpha \neq 0$ as well as $\rho < \alpha$ which implies $\theta_F(bk) > \theta_G(b)$. From (20) we obtain:

$$\Pi_{0F}(bk)^{\rho} = \Pi_{0G}b^{\alpha} \Rightarrow b^*(k) = \left(\frac{\Pi_{0F}}{\Pi_{0G}}\right)^{\frac{1}{\alpha-\rho}} k^{\frac{\rho}{\alpha-\rho}}. \tag{39}$$

Inserting this choice for all $(K, L) \in \mathbb{R}^2_+$, from (26) we obtain:

$$\phi(k) = \frac{f(b^*(k)k)}{g(b^*(k))} = \frac{\left(\pi_{0F}(\frac{\Pi_{0F}}{\Pi_{0G}})^{\frac{\rho}{\alpha-\rho}} k^{\frac{\alpha\rho}{\alpha-\rho}} + (1 - \pi_{0F})\right)^{\frac{1}{\rho}}}{\left(\pi_{0G}(\frac{\Pi_{0F}}{\Pi_{0G}})^{\frac{\alpha}{\alpha-\rho}} k^{\frac{\alpha\rho}{\alpha-\rho}} + (1 - \pi_{0G})\right)^{\frac{1}{\alpha}}} = \zeta \cdot \left(\pi_{0\Phi}k^{\xi} + (1 - \pi_{0\Phi})\right)^{\frac{1}{\xi}}, \tag{40}$$

where $\xi = \frac{\alpha\rho}{\alpha-\rho}$ denotes the elasticity parameter of the resultant global function (linked to its elasticity of substitution via $\sigma_{\Phi} = \frac{1}{1-\xi}$), the multiplicative constant equals $\zeta = (1 - \pi_{0F})^{\frac{1}{\rho}}(1 - \pi_{0G})^{-\frac{1}{\alpha}}(1 - \pi_{0\Phi})^{-\frac{1}{\xi}}$, and $\pi_{0\Phi}$ is the partial elasticity of the global function at the point of normalization which satisfies:

$$\Pi_{0F}^{\frac{1}{\rho}} = \Pi_{0G}^{\frac{1}{\alpha}}\Pi_{0\Phi}^{\frac{1}{\xi}}. \tag{41}$$

We also observe that in the special case where $\pi_{0F} = \pi_{0G} = \pi_{0\Phi}$, the optimal technology choice simplifies to $b^*(k) = k^{\frac{\rho}{\alpha-\rho}}$ with $\zeta = 1$. The dual problem is solved analogously.

Hence, we find that indeed the global function is CES, and its curvature is indeed lower than that of the local function (cf. Growiec, 2013; León-Ledesma and Satchi, 2018). In line with eq. (30), we obtain the following relationship between the three functions' elasticity parameters:

$$\frac{1}{\rho} = \frac{1}{\alpha} + \frac{1}{\xi}. \tag{42}$$

Hence, if factors are gross complements along the local function ($\rho < 0$), then α and ξ must be of opposing signs, meaning that either the technologies are gross substitutes along the technology menu but the factors are gross complements along the global function, or vice versa, the technologies are gross complements along the technology menu and the factors are gross substitutes along the global function. If, in contrast, factors are gross substitutes along the local function ($\rho > 0$), then both α and ξ must be positive as well, encompassing the cases of gross substitutability and convexity.

5.3. The minimum and maximum functions

Mutual duality between the technology menu and the global function subject to a minimum (Leontief) local function, along which the factors are perfectly complementary (i.e., *idempotent* duality), has been identified and thoroughly discussed by Rubinov and Glover (1998), Matveenko (1997, 2010) and Matveenko and Matveenko (2015). These studies have also extended this case into n dimensions. For completeness, here we also present the case where the technology menu or the global function is specified as a maximum function.

Leontief local function. The case where the local function is Leontief is very closely related to our Theorems 2–5 but, strictly speaking, cannot be considered as their special case. The reason is that, contrary to our assumptions, the minimum (Leontief) function:

$$F(BK, AL) = \min\{BK, AL\}, \qquad f(bk) = \min\{bk, 1\}, \tag{43}$$

is not differentiable at the point where $BK = AL$. Nevertheless, the results obtained here can still be conveniently characterized as a limiting case of our setup, where the curvature of the local function tends to infinity at the "kink" (i.e., at the ray from the origin satisfying $BK = AL$). Second order conditions are then automatically verified.

Assuming that the curvature of the technology menu is finite, the first order condition for the primal problem implies $bk = 1$ (and thus $b^*(k) = 1/k$) as well as $f(bk) = bk = 1$. Inserting this choice into the local function for all $(K, L) \in \mathbb{R}^2_+$ we obtain:

$$\phi(k) = \frac{1}{g(1/k)}, \tag{44}$$

which is fully in line with (26). The solution of the dual problem is fully analogous and implies $k^*(b) = 1/b$ and a technology menu satisfying $g(b) = \frac{1}{\phi(1/b)}$.

Furthermore, when the local function is Leontief, a Cobb–Douglas technology menu is dual to a Cobb–Douglas global function (and their exponents coincide); and a CES technology menu is also dual to a CES global function (and their elasticity parameters are mutually inverse, $\alpha = -\xi$).

Technology menu specified as a maximum function. It is also interesting to consider the primal problem under the extreme assumption that the technology menu is given by a maximum function,

$$G(B, A) = \max\{B, A\}, \qquad g(b) = \max\{b, 1\}. \tag{45}$$

This function, not differentiable at $b = 1$, represents a case where the overall level of technology in the economy is pinned down by the *best* of the available factor-specific technologies. It represents a technology menu of a traditional society where goods, factors, or their characteristics are always used in strictly definite proportions. It is also the limit of a sequence of cases where the trade-off between the quality (unit productivity) of the respective factors, very small for highly convex technology menus, gradually disappears.

The current case can be conveniently characterized as a limiting case of our general setup, where the curvature of the technology menu tends to minus infinity at the "kink" (i.e., the ray from the origin where $B = A$). Second order conditions are then automatically verified.

Assuming that the curvature of the local function is finite, the first order condition for the primal problem implies $b^* = 1$ for all k as well as $f(bk) = f(k)$. Inserting this choice into the local function for all $(K, L) \in \mathbb{R}^2_+$ we obtain:

$$\phi(k) = \frac{f(k)}{g(1)} = f(k). \tag{46}$$

Global function specified as a maximum function. The solution to the dual problem with a maximum global function is fully analogous and implies $k^* = 1$ for all b and a technology menu satisfying $g(b) = \frac{f(b)}{\phi(1)} = f(b)$.

Comments. First, the maximum function may look a bit strange as a technology menu and very strange as a global production or utility function. Indeed, we typically expect these functions to be concave and the maximum function represents extreme convexity. Therefore the economic applications of the above examples, and especially the dual problem, are likely to be limited. They may nevertheless be useful as "cautionary" examples indicating the consequences of assuming that the global function or the technology menu have the same functional form as the local function. Namely, the local and global functions can have the same (non-Cobb–Douglas) form only if the technology menu is a maximum function, i.e., there is no trade-off between the qualities of the respective factors. Analogously, the local function can have the same (non-Cobb–Douglas) form as the technology menu only if the global function is specified as a maximum function.

Second, the maximum case is *pathological* in the same sense as is the case with a Cobb–Douglas local function — namely that the technology menu and the global function are not mutually dual because the technology choice is always constant and thus not invertible. Indeed, trying to solve the dual problem with $F = \Phi$,

$$\max_{(K,L)\in\mathbb{R}^2_+} F(BK, AL) \qquad s.t. \qquad F(K, L) = 1, \tag{47}$$

leads to a first order condition of form $\Pi_F(bk) = \Pi_F(k)$ for any given b. This holds either if F is a Cobb–Douglas function, or otherwise only if $b = 1$. The former case has been discussed previously (and flagged as pathological), whereas the latter implies that for $b = 1$ the optimal factor choice is indeterminate, and for $b \neq 1$ there is no interior stationary point. A similar problem is encountered when solving the primal problem for $F = G$. This pathological outcome is a direct consequence of violation of the curvature assumption in Theorem 2 (when solving the primal problem while assuming that F and G have exactly the same functional form) or in Theorem 3 (when making this assumption for F and Φ in the dual problem).

6. The homothetic case

As a generalization of the primal and dual optimization problems (1) and (2), we will now replace homogeneous functions F, G and Φ with their homothetic counterparts, respectively $F_h = f_h \circ F$, $G_h = g_h \circ G$ and $\Phi_h = \phi_h \circ \Phi$, where $f_h, g_h, \phi_h : \mathbb{R}_+ \to \mathbb{R}$ are monotone (typically increasing) and twice continuously differentiable transformations. This additional degree of freedom is particularly useful in the analysis of utility which is often viewed as an ordinal rather than cardinal concept.

We find that when the local function, the technology menu and the global function are not homogeneous but only homothetic then the respective solutions to (1) and (2) still exist and are still mutually dual – as long as the optimal technology choice is invertible – but they are specified only up to a monotone transformation and thus are no longer unique.

Optimal technology choice. Theorems 2–3 can be straightforwardly generalized to the case of homothetic functions, yielding exactly the same outcomes. Intuitively, this is due to the fact that level curves of any function have exactly the same shape whether or not it has been subjected to a monotone transformation.

Theorem 6. *Let $F_h, G_h : \mathbb{R}^2_+ \to \mathbb{R}_+$ be increasing, twice continuously differentiable homothetic functions such that $F_h = f_h \circ F$ and $G_h = g_h \circ G$ where $f_h, g_h : \mathbb{R}_+ \to \mathbb{R}$ are increasing, twice continuously differentiable functions, and F and G are as in Theorem 2. Then the problem*

$$\Phi_h(K,L) = \max_{(B,A)\in\Omega_G} F_h(BK, AL) \qquad s.t.$$
$$\Omega_G = \{(B,A) \in \mathbb{R}^2_+ : G_h(B,A) = g_h(1)\}. \tag{48}$$

allows a unique interior maximum satisfying (20), (21) and (22).

Proof. First, we observe that $G_h(B,A) = g_h(G(B,A)) = g_h(1) \iff G(B,A) = 1$. Thus the technology menu is exactly the same as in (1). We then repeat all the steps of proof of Theorem 2 and observe that all terms related to $f_h'(\cdot)$ and $f_h''(\cdot)$ cancel out in the first and second order conditions, respectively. ∎

Theorem 7. *Let $F_h, \Phi_h : \mathbb{R}^2_+ \to \mathbb{R}_+$ be increasing, twice continuously differentiable homothetic functions such that $F_h = f_h \circ F$ and $\Phi_h = \phi_h \circ \Phi$ where $f_h, \phi_h : \mathbb{R}_+ \to \mathbb{R}$ are increasing, twice continuously differentiable functions, and F and Φ are as in Theorem 3. Then the problem*

$$G_h(B,A) = \max_{(K,L)\in\Omega_\Phi} F_h(BK, AL) \qquad s.t.$$
$$\Omega_\Phi = \{(K,L) \in \mathbb{R}^2_+ : \Phi_h(K,L) = \phi_h(1)\}. \tag{49}$$

allows a unique interior maximum satisfying (23), (24) and (25).

Proof. Fully analogous to the proof of Theorem 6. ∎

Propositions analogous to Theorems 6–7 can be formulated also for the case where some of the functions f_h, g_h, ϕ_h are decreasing, with exactly the same outcomes. The only caveat is that when f_h is decreasing, maximization in (48) and (49) should be replaced with minimization.

Construction of the envelopes. From Theorems 6 and 7 we know that maxima of the primal and dual technology choice problem are invariant under monotone transformations. Building on this result, we shall now extend Theorems 4–5 to homothetic functions. We find that the flipside of allowing for arbitrary monotone transformations is that the resultant envelopes are no longer unique.

More precisely, for every homothetic function $F_h = f_h \circ F$, where $f_h : \mathbb{R}_+ \to \mathbb{R}$ is monotone and twice continuously differentiable and $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing, twice continuously differentiable and homogeneous, from (26) we obtain that

$$\tilde{\Phi}_h(K,L) = F_h(B^*(k)K, A^*(k)L) = f_h(f(b^*(k)k)A^*(k)L)$$
$$= f_h\left(\frac{f(b^*(k)k)}{g(b^*(k))}L\right) = f_h(\Phi(K,L)), \tag{50}$$

$$\tilde{G}_h(B,A) = F_h(BK^*(b), AL^*(b)) = f_h(f(bk^*(b))AL^*(b))$$
$$= f_h\left(\frac{f(bk^*(b))}{\phi(k^*(b))}A\right) = f_h(G(B,A)). \tag{51}$$

This leads to the construction of $\tilde{\Phi}_h = f_h \circ \Phi$ from problem (1) and of $\tilde{G}_h = f_h \circ G$ from problem (2). Clearly, both functions are homothetic. They are also dual to one another in the sense that maximizing $F_h(BK, AL)$ subject to $g_h(G(B,A)) = g_h(1)$ leads to the construction of $\tilde{\Phi}_h(K,L)$ for any monotone function $g_h : \mathbb{R}_+ \to \mathbb{R}$, and maximizing $F_h(BK, AL)$ subject to $\phi_h(\Phi(K,L)) = \phi_h(1)$ leads to the construction of $\tilde{G}_h(B,A)$ for any monotone function $\phi_h : \mathbb{R}_+ \to \mathbb{R}$.

However, inclusion of g_h and ϕ_h in the above formulas underscores that allowing for monotone transformations of the homogeneous functions F, G, Φ compromises uniqueness of the resulting functions. Indeed, the results are unchanged also when we replace $\tilde{\Phi}_h$ with Φ_h (i.e., f_h with an arbitrary ϕ_h) or $\tilde{G}_h$ with G_h (i.e., f_h with an arbitrary g_h). Therefore the slope and curvature properties of the dual objects – the technology menu and the global function – are best characterized when they are expressed in their homogeneous form as in Theorems 4–5.

Example: additively separable log preferences. Extending the Cobb–Douglas example discussed in Section 5, we shall now consider a homothetic local function:

$$F_h(BK, AL) = \ln\left((BK)^{\pi_{0F}}(AL)^{1-\pi_{0F}}\right)$$
$$= \pi_{0F}\ln(BK) + (1-\pi_{0F})\ln(AL). \tag{52}$$

This monotone transformation of a Cobb–Douglas local function is particularly often used in the modeling of consumer choices, where it represents additively separable logarithmic preferences. While analytically convenient, such a specification is also very restrictive and in fact represents a *pathological* case where the optimal technology choice is independent of factor endowments and thus the technology menu and the global function are not mutually dual.

Example: additively separable CRRA preferences. Extending the CES example discussed in Section 5, we may consider a homothetic local function:

$$F_h(BK, AL) = \pi_{0F}\left(\frac{(BK)^\rho - 1}{\rho}\right) + (1-\pi_{0F})\left(\frac{(AL)^\rho - 1}{\rho}\right). \tag{53}$$

This monotone transformation of a CES local function (which uses the formula $f_h(x) = \frac{x^\rho - 1}{\rho}$) is often used in the modeling of consumer choices, where it represents additively separable CRRA (constant relative risk aversion) preferences.

Role of Bergson's theorem. By Bergson's theorem (Theorem 1), preferences given by (52) and (53) are in fact the only specifications which are both homothetic and additively separable with respect to BK and AL. Yet, both homotheticity and additive separability are often viewed as highly desirable properties of utility functions. Homotheticity of preferences helps avoid income effects and keep the optimization problem independent of units of measurement: the proportion of goods chosen by the consumer is then invariant under proportional expansions of the budget set (i.e., marginal rates of substitution are constant along rays through the origin). In contrast, non-homothetic preferences can be viewed as one of the potential causes of structural change in the course of economic development (Kongsamut et al., 2001; Boppart, 2014).

Additively separable preferences, in turn, are a cornerstone of intertemporal utility maximization, as time separability is a key prerequisite for the use of dynamic programming methods. Moreover, when the utility function with consumption and leisure is embedded in a dynamic model, its additive separability helps avoid long-run trends in the fraction of leisure time and match the fact that consumption is empirically less variable than hours worked. However, in such a case the restriction of constant great ratios requires imposing certain (disputable) values on the intertemporal elasticity of substitution. Non-separable preferences would offer an additional degree of freedom here, albeit potentially compromising the long-run trends and the relative volatility of consumption and hours.[11]

7. The technology menu and distributions of ideas

Let us now comment on one important economic interpretation of the factor-specific technology choice problem. Namely, instead of viewing the technology menu $G(B,A)$ as a primitive concept, several studies (Jones, 2005; Growiec, 2008a, b, 2013) have derived it from a probabilistic model. Following this literature, in this section we view the technology menu as a level curve of a certain two-dimensional cumulative distribution function of (stochastic) factor-specific ideas (i.e., unit factor productivities).

This probabilistic setup has originated from the idea that the technology menu may evolve over time as factor productivities are repeatedly drawn from a distribution (Kortum, 1997; Jones, 2005). This distribution has been, in turn, used to microfound aggregate production functions: "the standard production function that we write down, mapping the entire range of capital–labor ratios into output per worker, is a reduced form. It is not a single technology, but rather represents the substitution possibilities across different production techniques. The elasticity of substitution for this global production function depends on the extent to which new techniques that are appropriate at higher capital–labor ratios have been discovered. That is, it depends on the distribution of ideas". Jones (2005, p. 518). Taking the same approach, the current section focuses specifically on the case where the technology menu – derived from the distribution of ideas – is homothetic, i.e., invariant under radial expansions from the origin. This property implies that Hicks-neutral technical change does not affect the relative degree of factor augmentation in the optimum. In other words, if factor supply is fixed then under homotheticity the direction of R&D augmenting the technology menu coincides with the direction of technical change (Caselli and Coleman, 2006; Growiec, 2008a, Appendix D; Growiec, 2013, Appendix B).

It turns out that the class of idea distributions consistent with the homotheticity restriction is rather narrow. In particular, if the (homothetic) technology menu $G(B,A)$ is constructed from *independent* idea distributions, then because of Bergson's theorem it must be of the Cobb–Douglas or CES form, translating respectively into a requirement of Pareto or Weibull marginal idea distributions (Growiec, 2008b, Proposition 3). We generalize this finding by relaxing the independence assumption, leading to the statement of Bergson's theorem for copulas (Theorem 8). As its important application, we discuss the class of Archimedean copulas.

From Sklar's theorem for complementary cumulative distribution functions (ccdfs, see Nelsen, 1999; McNeil and Nešlehová, 2009) it follows that any bivariate distribution can be written as a composition of marginal distributions and a copula:

$$F(x,y)=\mathbb{P}(X>x,Y>y)=C(F_x(x),F_y(y)), \tag{54}$$

where $F_x,F_y:\mathbb{R}_+\to[0,1]$ represent the marginal complementary cumulative distribution functions (ccdfs),

$$F_x(x)=\mathbb{P}(X>x), \qquad F_y(y)=\mathbb{P}(Y>y), \tag{55}$$

and $C:[0,1]^2\to[0,1]$ is the copula. Given this notation, the theorem is stated as follows.

Theorem 8 (*Bergson's Theorem for Copulas*). *Let $F_h:\mathbb{R}^2_+\to[0,1]$ be a homothetic bivariate complementary cumulative distribution function (ccdf) satisfying $F_h(x,y)=C(F_x(x),F_y(y))$, where $F_x,F_y:\mathbb{R}\to[0,1]$ are differentiable marginal ccdfs and $C:[0,1]^2\to[0,1]$ is a differentiable copula which can be written as additively separable after a monotone transformation:*

$$\exists(f_h:\mathbb{R}_+\to[0,1],\ F:\mathbb{R}^2_+\to\mathbb{R}_+) \quad F_h(x,y)=f_h(F(x,y)), \tag{56}$$

$$\exists(f_s:\mathbb{R}\to[0,1],\ D_u,D_v:[0,1]\to\mathbb{R}) \quad C(u,v)=f_s(D_u(u)+D_v(v)), \tag{57}$$

where f_h,f_s,D_u,D_v are decreasing differentiable functions, and F is an increasing, differentiable and homogeneous function. Then $F(x,y)$ must be represented by

$$F(x,y)=c\cdot x^{\frac{\alpha}{\alpha+\beta}}y^{\frac{\beta}{\alpha+\beta}} \qquad \text{or} \qquad F(x,y)=(\alpha x^{\rho}+\beta y^{\rho})^{\frac{1}{\rho}}, \tag{58}$$

where $\alpha>0,\beta>0$; $c_x,c_y\in\mathbb{R}$ are arbitrary constants, $c=\exp\left(\frac{c_x+c_y}{\alpha+\beta}\right)$, and $\rho\neq 0$. Moreover, marginal distributions $F_x(x)$ and $F_y(y)$ must satisfy:

$$D'_u(F_x(x))F'_x(x)=\alpha x^{\rho-1}, \qquad D'_v(F_y(y))F'_y(y)=\beta y^{\rho-1}. \tag{59}$$

Proof. See Appendix. ∎

In plain English, Theorem 8 implies that if the ccdf of the joint idea distribution is homothetic and the copula is additively separable after a monotone transformation, then the ccdf – whose level curve is the technology menu – must be of a very specific, Cobb–Douglas or CES functional form. This form is then translated into very specific requirements imposed on the marginal distributions (Eq. (59)).

Theorem 8 has quite broad applicability. It covers not only the case where both idea distributions are independent (Growiec, 2008b, Proposition 3), but also the case where they are mutually dependent and their dependence is modeled by some representative of the broad and widely applied Archimedean class of copulas (see Appendix for an elaboration of most important representants of this class).

Independent marginal distributions. The independence copula takes the form $C(u,v)=uv$, so it is additively separable after taking logs. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z)=e^{-z}$, $D_u(u)=-\ln u$, $D_v(v)=-\ln v$. From Eq. (59) we then obtain:

$$F_x(x)=e^{-c_x}x^{-\alpha} \text{ if } \rho=0, \qquad F_y(y)=e^{-c_y}y^{-\beta} \text{ if } \rho=0, \tag{60}$$

$$F_x(x)=e^{-c_x}e^{-\frac{\alpha}{\rho}x^{\rho}} \text{ if } \rho\neq 0, \qquad F_y(y)=e^{-c_y}e^{-\frac{\beta}{\rho}y^{\rho}} \text{ if } \rho\neq 0. \tag{61}$$

This means that if the marginal distributions are independent, homotheticity of the technology menu implies that these distributions must take either the Pareto (60) or the Weibull form ((61) with $\rho>0$). In the latter case, both marginal distributions must have equal exponents (i.e., shape parameters).

Archimedean copulas. The bivariate Archimedean copula is defined as (McNeil and Nešlehová, 2009):

$$C(u,v)=\psi(\psi^{-1}(u)+\psi^{-1}(v)), \tag{62}$$

where $\psi:\mathbb{R}_+\to[0,1]$ is a decreasing, continuous function satisfying $\psi(0)=1$ and $\lim_{x\to\infty}\psi(x)=0$. The function ψ is called the *Archimedean generator*.

Hence, it suffices to take $f_s=\psi$ and $D_u=D_v=\psi^{-1}$ in the assumptions of Theorem 8 to observe that in fact all Archimedean copulas are subject to this theorem. Thus, when we assume homotheticity of the joint idea distribution and model dependence of

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11 I am grateful to an anonymous referee for this point.

its marginal distributions by the means of a specific Archimedean copula,[12] the technology menu must take the Cobb–Douglas or CES form, implying that the shapes of the marginal distributions must satisfy a very specific parametric condition which is unique for the given copula.

More precisely, for Archimedean copulas we obtain from (59):

$$\frac{\partial}{\partial x}(\psi^{-1}(F_x(x))) = \alpha x^{\rho-1}, \qquad \frac{\partial}{\partial y}(\psi^{-1}(F_y(y))) = \beta y^{\rho-1},$$
$$\alpha > 0, \beta > 0, \rho \in \mathbb{R}. \tag{63}$$

Integrating, we obtain that $F_x(x)$ and $F_y(y)$ must necessarily follow the formula:

$$F_x(x) = \psi\left(c_x + \alpha \ln x\right) \text{ if } \rho = 0, \qquad F_y(y) = \psi\left(c_y + \beta \ln y\right) \text{ if } \rho = 0, \tag{64}$$

$$F_x(x) = \psi\left(c_x + \frac{\alpha}{\rho}x^\rho\right) \text{ if } \rho \neq 0, \qquad F_y(y) = \psi\left(c_y + \frac{\beta}{\rho}y^\rho\right) \text{ if } \rho \neq 0, \tag{65}$$

where c_x, c_y are arbitrary constants of integration.

Owing to the properties of ψ, it is easily verified that $F_x(x)$ and $F_y(y)$ are indeed decreasing functions. Moreover, if $\rho \geq 0$ then $\lim_{x\to\infty} F_x(x) = \lim_{y\to\infty} F_y(y) = 0$. Other properties depend on the exact choice of the generator ψ and parameters. In particular, for some parametrizations the supports of random variables X and Y may be limited. In such a case, F_x or F_y should be set identically to zero for arguments exceeding the upper bound of the support and to unity for arguments below the lower bound of the support. Then the technology menu should also be defined only on this particular limited support.

8. Conclusion

This paper has provided a detailed treatment of a static, two-dimensional problem of factor-specific technology choice. At the core of this problem there is a local function F, along which the factors are multiplied by their respective unit productivities, drawn from a certain technology menu G. We have derived the optimal technology choices in such a setup and constructed the global function Φ as an envelope of the local functions. We have also solved a symmetric dual problem where Φ is given, and G — sought.

It turns out that the properties of this optimization problem can be characterized with the use of a generalized notion of duality ("F-duality"). In the optimum, partial elasticities of F, G and Φ are all equal, and there exists a clear-cut and economically interpretable relationship between their curvatures.

Our results are marked by their generality and broad applicability. At the same time, however, they also underscore how restrictive the assumptions of homogeneity (constant returns to scale) and homotheticity can be. Crucially, by the virtue of Bergson's theorem (Bergson{Burk}, 1936) homotheticity, when coupled with additive separability, implies the Cobb–Douglas or CES functional form. As we have demonstrated, this result has most bite when one envisages the technology menu as a level curve of a certain bivariate distribution of ideas (Jones, 2005; Growiec, 2008a).

The current study can be extended in a variety of directions as well as applied in a variety of contexts. The most needed theoretical extensions include accommodating non-homothetic local functions and technology menus as well as increasing the dimensionality of the problem by considering more than two factors. These tasks have already been accomplished for special cases such as Cobb–Douglas, CES or Leontief functions. To be addressed in their generality, however, they require the modeler to give up additive separability – a particular inconvenience in higher dimensions – and to make certain decisions with regard to the preferred measures of curvature in higher dimensions, which may just as well mean an opening of Pandora's box.

The scope for applications of the discussed framework is even broader. Firstly, while thus far optimal factor-specific technology choice has been studied predominantly in the context of growth theory, it may just as well be incorporated in models of, e.g., industrial organization, international trade, natural resources, sectoral change, consumption patterns, or social welfare. Secondly, the static technology choice problem studied here could be given a dynamic edge by assuming that the technology is fixed in the short run but not in the long run, and thus the local function represents the short-run technology whereas the global function holds only in the long run. León-Ledesma and Satchi (2018) are the first to formalize this idea, constructing a model where capital and labor are gross complements in the short run but not necessarily so in the long run. In this way they circumvent the Steady State Growth Theorem (Uzawa, 1961) and reconcile the long-run balanced growth requirement with the mounting empirical evidence of gross complementarity of both factors and non-neutral technical change. Their brilliant idea can clearly be taken further, with a wide range of potential extensions and applications.

Appendix. Additional comments and proofs of theorems

Relation to the problem of output/utility maximization subject to a budget constraint. It can be noticed that the primal factor-specific technology choice problem (1) considered in the current study has a similar structure to the classic problem (Shephard, 1953; Diewert, 1974; Fuss and McFadden, 1980) of output/utility maximization subject to a budget constraint (which leads to the construction of an envelope cost function as in (66)), whereas our dual problem (2) resembles the classic dual problem of cost minimization subject to a budget constraint viewed as a function of the prices r and w, (67):

$$C(r, w) = \max_{(K,L)\in\Omega_{B1}} Y(K, L) \qquad s.t. \qquad \Omega_{B1} = \{(K, L) \in \mathbb{R}^2_+ : rK + wL = 1\}, \tag{66}$$

$$F(K, L) = \min_{(r,w)\in\Omega_{B2}} C(r, w) \qquad s.t. \qquad \Omega_{B2} = \{(r, w) \in \mathbb{R}^2_+ : rK + wL = 1\}. \tag{67}$$

There are however differences between both setups: (i) the function linking quantities and prices (the budget constraint) is assumed to be linear here (and not an arbitrary local function F as in our more general setup), (ii) in line with the different economic interpretation but without any impact on the outcomes, the objectives and the constraints have switched places, (iii) to maintain consistency with the economic interpretation, maximization is replaced with minimization in the dual problem.

Although mathematically similar, both problems are "orthogonal" in the sense that the factor-specific technology choice problem abstracts from factor prices and, symmetrically, the standard output/utility maximization problem abstracts from factor quality. This orthogonality property turns out to play a crucial role when we merge both problems into a joint problem of simultaneous factor-specific technology choice and output/utility maximization:

$$C(r, w) = \max_{(K,L)\in\Omega_{B1},(B,A)\in\Omega_G} F(BK, AL) \ \ s.t. \ \ \Omega_{B1} = \{(K, L) \in \mathbb{R}^2_+ : rK + wL = 1\}, \tag{68}$$
$$\Omega_G = \{(B, A) \in \mathbb{R}^2_+ : G(B, A) = 1\}.$$

[12] For example, Growiec (2008a) modeled the dependence of marginal idea distributions with a Clayton copula. His study, however, did not assume homotheticity (apart from a few special cases).

This is a problem where the decision maker is allowed to choose both her favorite technology (subject to the given technology menu) and factor quantities (subject to the given budget constraint) at the same time (cf. León-Ledesma and Satchi, 2018). Inserting these optimal choices for all possible configurations of factor prices permits to construct – instead of the global function taking factor quantities K and L as given – the envelope cost function which depends, in turn, only on the prices r and w.

The associated dual problem can be written as:

$$G(B,A)=\min_{(K,L)\in\Omega_F,(r,w)\in\Omega_C} rK+wL \;\; s.t. \;\; \Omega_F=\{(K,L)\in\mathbb{R}^2_+: F(BK,AL)=1\}, \quad \Omega_C=\{(r,w)\in\mathbb{R}^2_+: C(r,w)=1\}. \tag{69}$$

First order conditions for the joint and the separated optimization problems exactly coincide, underscoring the aforementioned orthogonality property: factor-specific technology choice and output/utility maximization, even when solved simultaneously, are not interdependent. It follows that – as long as factor quality does not enter the budget constraint and factor prices do not enter the technology menu – it is instructive to study the factor-specific technology choice problem separately as we do here. Allowing for interdependence is left for future research.

Additional second order conditions are needed, however, to ensure the existence of an interior solution to the joint problem (see also León-Ledesma and Satchi, 2018, Appendix A.2). The local function F must exhibit sufficient curvature to support an interior maximum with respect to (K,L) in (68). This is verified if the marginal rate of substitution is below $-\frac{w}{r}$ in the limit of $L=0$ and above $-\frac{w}{r}$ in the limit of $K=0$ or, in other words, if the marginal rate of substitution (which is negative and decreasing in K/L as long as F is increasing and concave) crosses the $-\frac{w}{r}$ threshold for some configuration of factors (K,L) along the isoquant/indifference curve. For all Cobb–Douglas and CES functions (even with gross substitutability of factors) this condition is automatically verified (Growiec and Mućk, 2016).

Relation to the literature on factor-augmenting technical change. The discussed setup is static and thus abstracts from technical change which – by definition – happens over time. Moreover, the technological underpinnings of the economy are in fact not only constant but also invisible because in the normalization procedure, the current overall Hicks-neutral technology level of the economy has been conveniently incorporated in F_0, G_0 and Φ_0, whereas the current relative productivity of both factors has been included in π_{0F}, π_{0G} and $\pi_{0\Phi}$. However, the possibility of explicit technical change can be incorporated as an extension of our setup by conditioning at least two of the three functions F, G or Φ on time. In particular, if one wants to consider *factor-augmenting* technical change (which can be decomposed into Hicks-neutral technical change and the bias in technical change, working in favor of one of the factors),[13] one has to replace either:

- $F(BK,AL)$ with $F(\lambda_K BK,\lambda_L AL)=\lambda_L F(\lambda_k BK,AL)$, or
- $G(B,A)$ with $G(\lambda_K B,\lambda_L A)=\lambda_L G(\lambda_k B,A)$, or
- $\Phi(K,L)$ with $\Phi(\lambda_K K,\lambda_L L)=\lambda_L\Phi(\lambda_k K,L)$,

where the variation in $\lambda_K>0$ and $\lambda_L>0$ over time represents capital- and labor-augmenting technical change, respectively. Equivalently, changes in λ_L can be said to represent Hicks-neutral technical change, and then $\lambda_k=\frac{\lambda_K}{\lambda_L}$ measures the capital bias in technical change.[14] Adding a dynamic edge to the considered framework remains an important task which we leave for further research.

Examples of Archimedean copulas. Below we briefly review a few of the most common Archimedean copulas. In each case, we derive the exact functional form that the marginal ccdfs must follow in order to be consistent with homotheticity of the technology menu.

Clayton copula. Clayton copula takes the form $C(u,v)=(\max\{0,u^\delta+v^\delta-1\})^{\frac{1}{\delta}}$, with $\delta\le 1$ and $\delta\neq 0$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z)=\psi(z)=(1-\delta z)^{\frac{1}{\delta}}$ as well as $D_u(u)=\psi^{-1}(u)=-\frac{1}{\delta}\left(u^\delta-1\right)$, $D_v(v)=\psi^{-1}(v)=-\frac{1}{\delta}\left(v^\delta-1\right)$. We then obtain:

$$F_x(x)=(c_x-\alpha\delta\ln x)^{\frac{1}{\delta}} \text{ if } \rho=0, \qquad F_y(y)=\left(c_y-\beta\delta\ln y\right)^{\frac{1}{\delta}} \text{ if } \rho=0, \tag{70}$$

$$F_x(x)=\left(c_x-\frac{\alpha\delta}{\rho}x^\rho\right)^{\frac{1}{\delta}} \text{ if } \rho\neq 0, \qquad F_y(y)=\left(c_y-\frac{\beta\delta}{\rho}y^\rho\right)^{\frac{1}{\delta}} \text{ if } \rho\neq 0. \tag{71}$$

Of particular interest is the case (71) with $c_x=c_y=0$ as well as $\delta\rho<0$. It implies that x and y are Pareto distributed with equal exponents (shape parameters) $\frac{\rho}{\delta}$ (Growiec, 2008a).

Gumbel copula. Gumbel copula takes the form $C(u,v)=\exp\left(-((-\ln u)^\delta+(-\ln v)^\delta)^{\frac{1}{\delta}}\right)$, with $\delta\ge 1$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z)=\psi(z)=e^{-z^{\frac{1}{\delta}}}$ as well as $D_u(u)=\psi^{-1}(u)=(-\ln u)^\delta$, $D_v(v)=\psi^{-1}(v)=(-\ln v)^\delta$. We then obtain:

$$F_x(x)=e^{-(c_x+\alpha\ln x)^{\frac{1}{\delta}}} \text{ if } \rho=0, \qquad F_y(y)=e^{-\left(c_y+\beta\ln y\right)^{\frac{1}{\delta}}} \text{ if } \rho=0, \tag{72}$$

$$F_x(x)=e^{-\left(c_x+\frac{\alpha}{\rho}x^\rho\right)^{\frac{1}{\delta}}} \text{ if } \rho\neq 0, \qquad F_y(y)=e^{-\left(c_y+\frac{\beta}{\rho}y^\rho\right)^{\frac{1}{\delta}}} \text{ if } \rho\neq 0. \tag{73}$$

Of particular interest is the case (73) with $c_x=c_y=0$ as well as $\delta\rho>0$. It implies that x and y are Weibull distributed with equal exponents (shape parameters) $\frac{\rho}{\delta}$.

Ali–Mikhail–Haq copula. Ali–Mikhail–Haq copula takes the form $C(u,v)=\frac{uv}{1-\delta(1-u)(1-v)}$, with $\delta\in[-1,1)$. Hence, in the assumptions of Theorem 8 we should postulate $f_s(z)=\psi(z)=\frac{1-\delta}{e^z-\delta}$ as well as $D_u(u)=\psi^{-1}(u)=\ln\left(\frac{1-\delta(1-u)}{u}\right)$, $D_v(v)=\psi^{-1}(v)=\ln\left(\frac{1-\delta(1-v)}{v}\right)$. We then obtain:

$$F_x(x)=\frac{1-\delta}{x^\alpha e^{c_x}-\delta} \text{ if } \rho=0, \qquad F_y(y)=\frac{1-\delta}{y^\beta e^{c_y}-\delta} \text{ if } \rho=0, \tag{74}$$

$$F_x(x)=\frac{1-\delta}{e^{\frac{\alpha}{\rho}x^\rho}e^{c_x}-\delta} \text{ if } \rho\neq 0, \qquad F_y(y)=\frac{1-\delta}{e^{\frac{\beta}{\rho}y^\rho}e^{c_y}-\delta} \text{ if } \rho\neq 0. \tag{75}$$

13 See, e.g., Acemoglu (2002, 2003), Klump et al. (2007) and León-Ledesma et al. (2010).

14 Growiec (2008a) studies factor-specific technology choice in a dynamic framework with Hicks-neutral technical change. Growiec (2013) allows for biased technical change and discusses the emerging possibility of a difference between the direction of R&D (which only affects the shape of the technology menu G) and the direction of technical change (which also incorporates firms' optimal technology choices). Biased technical change is also allowed within factor-specific technology choice frameworks studied by León-Ledesma and Satchi (2018), an estimated business-cycle model with a short-run CES technology which circumvents the Steady State Growth Theorem (Uzawa, 1961) and reconciles the long-run balanced growth requirement with gross complementarity of both factors and non-neutral technical change; and Growiec et al. (2018), a calibrated model of medium-to-long run swings of the labor share and other macroeconomic variables.

Proof of Theorem 2. Eq. (20) is obtained directly from the two first order conditions for the Lagrangian by eliminating λ. Eq. (21) follows from the fact that along the technology menu, $G(B, A) = g(b)A = 1$.

To ascertain that the found solution is indeed a maximum, we compute the second order conditions, which imply that:

$$\frac{\partial^2 \mathcal{L}_P}{\partial B^2} = \frac{K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{76}$$

$$\frac{\partial^2 \mathcal{L}_P}{\partial A^2} = \frac{b^2K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{77}$$

$$\frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} = -\frac{bK}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right), \tag{78}$$

and thus $\frac{\partial^2 \mathcal{L}_P}{\partial K^2} < 0$ and $\frac{\partial^2 \mathcal{L}_P}{\partial L^2} < 0$ if and only if $\theta_F(b^*(k)k) > \theta_G(b^*(k))$. Though the Hessian is equal to zero because F and G are homogeneous functions (Moysan and Senouci, 2016), concavity is guaranteed along the tangent to the constraint, i.e., along the line

$$\left\{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbb{R}^2 : [g'(b)\ \ g(b) - bg'(b)]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = 0\right\}. \tag{79}$$

Indeed, for all $h_1 \neq 0$ we obtain:

$$\begin{bmatrix} h_1 & -\frac{\Pi_G}{b}h_1 \end{bmatrix}\begin{bmatrix} \frac{\partial^2 \mathcal{L}_P}{\partial B^2} & \frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} \\ \frac{\partial^2 \mathcal{L}_P}{\partial B \partial A} & \frac{\partial^2 \mathcal{L}_P}{\partial A^2} \end{bmatrix}\begin{bmatrix} h_1 \\ -\frac{\Pi_G}{b}h_1 \end{bmatrix} = h_1^2\frac{K}{A}\left(f''(bk)k - \frac{f'(bk)}{g'(b)}g''(b)\right)(1 + \Pi_G^2) < 0. \tag{80}$$

Let us also rewrite (20) as:

$$X_P(b, k) = \Pi_F(bk) - \Pi_G(b) = 0. \tag{81}$$

Using the implicit function theorem and the equality $\pi = \pi_F(bk) = \pi_G(b)$ (which follows from (20)), we obtain:

$$\frac{\partial b^*(k)}{\partial k} = -\frac{\frac{\partial X_P}{\partial k}}{\frac{\partial X_P}{\partial b}} = \frac{\frac{\partial \Pi_F}{\partial(bk)}b}{\frac{\partial \Pi_G}{\partial b} - \frac{\partial \Pi_F}{\partial(bk)}k} = \frac{\frac{f'(bk)b}{f(bk)}\frac{1-\pi_F(bk)-\theta_F(bk)}{(1-\pi_F(bk))^2}}{\frac{g'(b)}{g(b)}\frac{1-\pi_G(b)-\theta_G(b)}{(1-\pi_G(bk))^2} - \frac{f'(bk)k}{f(bk)}\frac{1-\pi_F(bk)-\theta_F(bk)}{(1-\pi_F(bk))^2}} = \frac{b}{k}\left(\frac{1-\pi-\theta_F(bk)}{\theta_F(bk) - \theta_G(b)}\right), \tag{82}$$

or (22). Uniqueness of the optimum $b^*(k)$ follows from the fact that (unless $\frac{\partial \Pi_G}{\partial b} = \frac{\partial \Pi_F}{\partial(bk)} = 0$ which happens only in the excluded case where F and G are Cobb–Douglas functions) the denominator in (82) is positive. ■

Proof of Theorem 4. Existence and uniqueness of Φ solving problem (1) follows from Theorem 2. It also follows that

$$\phi(k) = \frac{\Phi(K, L)}{L} = \frac{F(B^*(k)K, A^*(k)L)}{L} = f(b^*(k)k)A^*(k) = \frac{f(b^*(k)k)}{g(b^*(k))}. \tag{83}$$

Existence and uniqueness of G solving problem (2) follows from Theorem 3. It also follows that

$$g(b) = \frac{G(B, A)}{A} = \frac{F(BK^*(B, A), AL^*(B, A))}{A} = f(bk^*(b))L^*(b) = \frac{f(bk^*(b))}{\phi(k^*(b))}. \tag{84}$$

Both functions are homogeneous by construction. Computing $\phi'(k)$ and $g'(b)$ from (26), using (20), (22), (23), (25) and rearranging we obtain:

$$\frac{\phi'(k)}{\phi(k)} = \frac{f'(bk)}{f(bk)}\left(\frac{\partial b^*(k)}{\partial k}k + b^*(k)\right) - \frac{g'(b)}{g(b)}\frac{\partial b^*(k)}{\partial k} = \frac{\pi}{k} > 0, \tag{85}$$

$$\frac{g'(b)}{g(b)} = \frac{f'(bk)}{f(bk)}\left(b\frac{\partial k^*(b)}{\partial b} + k^*(b)\right) - \frac{\phi'(k)}{\phi(k)}\frac{\partial k^*(b)}{\partial b} = \frac{\pi}{b} > 0, \tag{86}$$

where in each case the positivity of π follows from assumption that the other two functions are increasing. Thus Φ and G are increasing. ■

Proof of Theorem 5. Eq. (29) follows from (20) and (23) in the case where both of them hold at the same time. Moreover, when $1 - \pi_F(bk) - \theta_F(bk) \neq 0$, we can insert (25) into (22), use (29) and obtain:

$$\frac{1 - \pi - \theta_F(bk)}{\theta_F(bk) - \theta_G(b)} = \frac{\theta_F(bk) - \theta_\Phi(k)}{1 - \pi - \theta_F(bk)}, \tag{87}$$

and hence,

$$(1 - \pi - \theta_G(b))(1 - \pi - \theta_\Phi(k)) = (1 - \pi - \theta_F(bk)) \times (1 - \pi - \theta_G(b)) + (1 - \pi - \theta_F(bk))(1 - \pi - \theta_\Phi(k)). \tag{88}$$

It follows that $1 - \pi - \theta_G(b) = 0 \iff 1 - \pi - \theta_\Phi(k) = 0$ and if both terms are nonzero, then we can divide both sides of Eq. (88) by $(1 - \pi_F(bk) - \theta_F(bk))(1 - \pi - \theta_G(b))(1 - \pi - \theta_\Phi(k))$, yielding (30). ■

Proof of Theorem 8. We begin by writing down the marginal rate of substitution of the function F_h. On the one hand we have:

$$MRS = -\frac{\frac{\partial F_h}{\partial y}}{\frac{\partial F_h}{\partial x}} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} \equiv -H\left(\frac{x}{y}\right). \tag{89}$$

The function H depends on the x/y ratio only due to the homogeneity of F. On the other hand, however, using the copula representation,

$$MRS = -\frac{\frac{\partial F_h}{\partial y}}{\frac{\partial F_h}{\partial x}} = -\frac{\frac{\partial C}{\partial v}F_y'(y)}{\frac{\partial C}{\partial u}F_x'(x)} = -\frac{D_v'(F_y(y))F_y'(y)}{D_u'(F_x(x))F_x'(x)} \equiv -\frac{H_y(y)}{H_x(x)}. \tag{90}$$

Therefore $H(\frac{x}{y}) = \frac{H_y(y)}{H_x(x)}$ for all $x, y \in \mathbb{R}_+$. Differentiating both sides of this functional equality with respect to x and y and eliminating $H'(\frac{x}{y})$, we obtain:

$$\frac{H_x'(x)x}{H_x(x)} = \frac{H_y'(y)y}{H_y(y)}, \qquad \text{for all } x, y \in \mathbb{R}. \tag{91}$$

Therefore both sides of (91) must be constant. We denote this constant $\rho - 1$. Integrating, we obtain:

$$H_x(x) = \alpha x^{\rho-1}, \qquad H_y(y) = \beta y^{\rho-1}, \tag{92}$$

for some $\alpha, \beta \in \mathbb{R}$. Substituting for $H_x(x)$ and $H_y(y)$ and observing the signs of respective derivatives yields (59).

This also implies that $H(\frac{x}{y}) = \frac{\beta}{\alpha}\left(\frac{x}{y}\right)^{1-\rho}$, and thus the homogeneous function $F(x, y)$ must take either the Cobb–Douglas or the CES form (58) (cf., Arrow et al., 1961). ■

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