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A microfoundation for normalized CES production functions with factor-augmenting technical change

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ABSTRACT

Based on a tractable "endogeneous technology choice" framework, we provide a microfoundation for aggregate normalized constant elasticity of substitution (CES) production functions with non-neutral, factor-augmenting technical change. In this framework, firms are allowed to choose unit productivities of capital and labor optimally from a technology menu constructed under the assumption that unit factor productivities (UFPs) are independently Weibull-distributed. The Weibull distribution itself is also microfounded here: based on extreme value theory, it is found to be an accurate and robust approximation of the true UFP distribution if technologies consist of a large number of complementary components.

1. Introduction

Aggregate production functions – particularly Cobb–Douglas and CES functions – are used in virtually every article in contemporary theoretical macroeconomics. Surprisingly few models have been put forward so far, however, in which these functions are derived from microfoundations. The purpose of the current contribution is to enrich this sparse literature with an analytically tractable microeconomic framework able to generate these aggregative specifications endogeneously. It is going to be an idea-based "endogeneous technology choice" model where the aggregate production function is derived as a convex hull of local production functions (LPFs), chosen optimally by homogeneous profit-maximizing firms. Each of these local techniques is in turn characterized by a pair of technology-specific unit factor productivities (UFPs) (a,b), which augment labor and capital, respectively.

Frameworks based on similar premises have been studied by Jones (2005) and Growiec (2008a, 2008b).[1] Jones' (2005) model was founded on the assumption that firms producing the final good sample capital- and labor-augmenting UFPs

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[1] The list of related contributions includes also Caselli and Coleman (2006), Nakamura and Nakamura (2008), Nakamura (2009) and León-Ledesma and Satchi (2011) whose "endogeneous technology choice" frameworks are analytically similar to ours. As compared to the current paper, however, they are specified in much less detail in terms of the actual *microfoundations* of the aggregate production function; instead, these papers embed their technology choice frameworks in larger encompassing structures aimed at solving diverse macroeconomic problems. Also, they are endowed with somewhat different

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randomly from a pair of independent Pareto distributions, so that their technology choice is optimal only on average, or in the limit when sufficiently many draws have been made. Growiec (2008a) rewrote Jones' model into a more tractable form that yields equivalent results and put forward its generalization. His key innovation was to allow the firms to pick their preferred technology pair (a,b) deterministically from a given technology menu, and to shift the stochastic technology-search process to the R&D sector, composed of a continuum of researchers who draw the (a,b) technology pairs from a certain pre-defined joint bivariate distribution. This distribution, in turn, was constructed either from a pair of independent Pareto distributions (mirroring the Jones' case), or a pair of marginal Pareto distributions dependent according to the Clayton copula (Growiec, 2008a), or a pair of independent Weibull distributions (Growiec, 2008b). In all versions of this idea-based model, the shape of the resultant aggregate production function, obtained by plugging the optimal technology choices into LPFs, was found to depend, in general, both on the assumed shapes of LPFs and on the assumed joint distribution of UFPs.

Based on the above assumptions, these three papers have succeeded in providing idea-based microfoundations for Cobb–Douglas and CES aggregate production functions. Jones (2005) has shown that if capital- and labor-augmenting ideas are independently Pareto-distributed, then the aggregate production function is Cobb–Douglas; Growiec (2008a, 2008b) has extended this result by proving that if they are independently Weibull-distributed, or Pareto-distributed and dependent according to the Clayton copula, then the aggregate production function is CES.[2] It has also been demonstrated in these papers that endogeneous technology choice allows the aggregate economy to increase the elasticity of substitution between capital and labor beyond the (naturally) low levels characterizing LPFs. A striking omission in these earlier contributions is, however, the lack of any theoretical justification for the assumed Pareto or Weibull forms of UFP distributions. This creates a gap in the literature which ought to be filled.

The contribution of the current paper is to fill this gap by drawing robust conclusions from extreme value theory. Assuming that factor-augmenting technologies are inherently *complex* and consist of a large number of *complementary* components, we find that the Weibull distribution should approximate the true productivity distribution better than any anything else, including the celebrated Pareto distribution (see Kortum, 1997; Gabaix, 1999; Jones, 2005, and references therein). The argument is based on the extreme value property of the Weibull distribution (cf. Kotz and Nadarajah, 2000; de Haan and Ferreira, 2006): if one takes the *minimum* of n independent draws from some (sufficiently well-behaved) distribution, then as $n \to \infty$, this minimum will, after an appropriate normalization, converge in distribution to the standard Weibull distribution with a shape parameter $\alpha > 0$, dependent on the shape of the underlying sampling distribution. Taking the minimum corresponds to the case of complex technologies consisting of complementary components (cf. Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011) whose productivity is determined by the productivity of their "weakest link" (or "bottleneck"). The same complementarity requirement, coupled with the assumption of few substitution possibilities along the LPF, implies also that capital and labor should be gross complements along the aggregate CES production function, i.e., that the aggregate (long-run) elasticity of substitution should be less than unity – in line with empirical evidence (see Chirinko, 2008; León-Ledesma et al., 2010).

We also note that the earlier literature has overlooked a few interesting corollaries from the "endogeneous technology choice" framework, most likely because of their cumbersome parametrizations and a number of unnecessary implicit assumptions. The current paper overcomes these problems by making three modifications in the original model developed by Jones (2005) and Growiec (2008a, 2008b):

1. It features a more detailed specification of the R&D sector, microfounding the assumed functional form of technology menu.
2. It is rewritten in terms of *normalized* CES functions (cf. de La Grandville, 1989; Klump and de La Grandville, 2000). Thanks to this step, we are able to obtain an interpretable link between the parameters of the microfounded aggregate production function, the LPF, and the bivariate UFP distribution. The reason is that under normalization, parameters of the CES production function represent separate concepts which are otherwise deeply intertwined: e.g., the distribution parameter of the un-normalized CES function is itself a function of the elasticity of substitution and the normalized volume units (cf. Klump and Preissler, 2000).[3] We find that normalization can be maintained simultaneously at the local and at the aggregate level, greatly facilitating the interpretation of the aggregate CES production function's parameters.
3. It allows for directed factor-augmenting R&D (Acemoglu, 2003, 2007; Acemoglu and Guerrieri, 2008), able to expand the

(*footnote continued*)
interpretations. Finally, Dupuy (2012) provides an alternative microfoundation for the aggregate CES production function, based on a assignment model with heterogeneous workers and tasks.

[2] Moreover, the assumption of independently Weibull-distributed capital- and labor-augmenting ideas leads to the functional form of the technology menu assumed ad hoc by Caselli and Coleman (2006) and Nakamura (2009).

[3] This in turn greatly obstructs the estimation of parameters and comparative statics exercises. A thorough elaboration of these issues as well as a survey of the related literature can be found in Klump et al. (2012). These authors also request that the parameters of production functions, derived from microfoundations by Jones (2005) and Growiec (2008a, 2008b), should be provided with an interpretation consistent with normalization. Among other accomplishments, the current paper addresses this request.

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technology menu in selected directions more than in others. We thereby relax the assumption implicitly made by Growiec (2008a), that technological progress always augments the technology menu proportionally, as a homothetic transformation from the origin.[4] This step helps understand an important theoretical distinction: in endogeneous technology choice models, the *direction of R&D* (i.e., the direction of expansion of the technology menu) and the *direction of technical change*, actually observed in an economy, are distinct concepts; firms' technology choices respond not only to the augmentation of the technology menu, but also to factor accumulation.[5]

The considered "endogeneous technology choice" framework is readily generalizable to n-factor production functions. Since all the derivations in the n-dimensional case are very similar to the two-dimensional case, formal elaboration of this issue has been delegated to the online appendix.[6] There we also discuss an alternative specification of the R&D process, generating a separate stochastic mechanism which leads to Pareto distributions in R&D productivity, and in consequence to aggregate Cobb–Douglas production functions. It works under the assumption that ideas are simple (consist of a single component only), but in order for new technologies to be useful, their UFPs must exceed a certain high threshold. We argue that the underpinnings of such a mechanism are empirically doubtful, though.

The remainder of the paper is structured as follows. Section 2 provides a microfoundation for the aggregate CES production function given independently Weibull-distributed UFPs. Section 3 justifies the assumption of Weibull distributions. Section 4 discusses the robustness of this justification. Section 5 concludes. Appendix A addresses the theoretical possibility of gross substitutability of inputs along the aggregate CES production function. Appendix B discusses the detailed implications of the framework for the direction of technical change in two important benchmark cases.

2. A microfoundation for the aggregate normalized CES production function

In the current section, we shall derive the aggregate normalized CES production function from idea-based microfoundations, i.e., as a convex hull of LPFs, computed under the restriction that UFPs are chosen from the given (isoelastic) technology menu. The shape of this menu will in turn be computed from a model of directed R&D in Section 3.

2.1. Framework

The discussed "endogeneous technology choice" framework is based on the following assumptions (cf. Jones, 2005; Growiec, 2008a,b).

Assumption 1. The local production function (LPF) takes either the normalized CES or the normalized Leontief form

$$Y = \begin{cases} Y_0\left(\pi_0\left(\dfrac{bK}{b_0K_0}\right)^{\theta} + (1-\pi_0)\left(\dfrac{aL}{a_0L_0}\right)^{\theta}\right)^{1/\theta} & \text{if } \sigma_{LPF}\in(0,1), \\ Y_0 \min\left\{\left(\dfrac{bK}{b_0K_0}\right),\left(\dfrac{aL}{a_0L_0}\right)\right\} & \text{if } \sigma_{LPF}=0, \end{cases} \tag{1}$$

where $\theta\in[-\infty,0)$ is the *substitutability parameter*, related to the elasticity of substitution along the LPF via $\sigma_{LPF}=1/(1-\theta)$. The Leontief LPF, with $\sigma_{LPF}=0$, is obtained as a special case of the more general normalized CES class of LPFs by taking the limit $\theta\to-\infty$ (we denote this case as $\theta=-\infty$ for simplicity). $\pi_0=r_0K_0/Y_0$ is the capital income share at t_0. The LPF exhibits constant returns to scale.

Note that in the normalization procedure of the CES LPFs, benchmark values have been assigned not only to output, capital and labor (Y_0,K_0,L_0), but also to the benchmark technology (b_0,a_0). In the following derivations, this benchmark technology will be identified with the *optimal* technology at time t_0.

By assuming $\theta<0$, or equivalently $\sigma_{LPF}<1$, we concentrate on the likely case where LPFs allow little or no substitutability between inputs. More precisely, capital and labor are assumed to be gross complements along the LPF. Since all results derived in this paper go through also in the limiting case of Leontief LPFs, where inputs are perfectly complementary, the CES specification of the LPF is not strictly necessary for the aggregate CES production function to obtain.

From the economic point of view, the assumption that there is little or no input substitutability along the LPF is consistent with the "recipe" interpretation of particular production techniques (where the LPF is viewed as a list of instructions on how to transform inputs into output, that must be followed as closely as possible, cf. Jones, 2005), and thus it is clearly favorable over the alternative cases $\theta\in(0,1)$ (gross substitutability of inputs along the LPF) and $\theta\to 0$ (Cobb–Douglas LPFs). For this reason, we

[4] This assumption is also maintained by Jones (2005) and León-Ledesma and Satchi (2011), but due to the multiplicative character of the aggregate Cobb–Douglas production function obtained there, lifting this restriction does not change any of their results.

[5] In particular, in the neoclassical growth model with a CES aggregate production function and factor-augmenting technical change, these two concepts are equivalent only along the balanced growth path (BGP) in the unique case when the BGP exists, that is when both R&D and technical change are purely labor-augmenting (cf. Uzawa, 1961). This requires imposing highly specific assumptions on the R&D process. Otherwise, the directions of R&D and technical change diverge.

[6] The framework could also be rather straightforwardly generalized to the case of a *continuum* of inputs (Growiec, 2012).

shall concentrate on this case in the following calculations. From the purely analytical point of view, all these cases may be considered, though. The discussion of the other cases is included in Appendix A.

The CES/Leontief specification of the LPF was used in the earlier contributions of Growiec (2008a, 2008b), but without normalization.

Assumption 2. The technology menu, specified in the (a,b) space, is given by equality:

$$H(a,b)=\left(\frac{a}{\lambda_a}\right)^{\alpha}+\left(\frac{b}{\lambda_b}\right)^{\alpha}=N, \quad \lambda_a,\lambda_b,\alpha,N>0. \tag{2}$$

According to this assumption, the technology menu is thus viewed as a downward-sloping, isoelastic curve (i.e., of CES form) in the (a,b) space, capturing the trade-off between the available UFPs of capital and labor. In Section 3, the technology menu (2) will be derived as a *contour line of the cumulative distribution function* of the joint bivariate distribution of capital- and labor-augmenting ideas ($\tilde{b}$ and $\tilde{a}$, respectively). The key point of Section 3 will be to put forward a tractable model of the R&D sector that will endogeneously generate the technology menu (2) from a general class of individual (marginal) distributions of $\tilde{b}$ and $\tilde{a}$. Under independence of both dimensions (so that marginal distributions of $\tilde{b}$ and $\tilde{a}$ are simply multiplied by one another), Eq. (2) is obtained if and only if the marginal distributions are Weibull with the same shape parameter $\alpha>0$ (Growiec, 2008b)[7]:

$$P(\tilde{a}>a)=e^{-(a/\lambda_a)^{\alpha}}, \quad P(\tilde{b}>b)=e^{-(b/\lambda_b)^{\alpha}}, \tag{3}$$

for $a,b>0$. Under such a parametrization, we have $P(\tilde{a}>a,\tilde{b}>b)=e^{-(a/\lambda_a)^{\alpha}-(b/\lambda_b)^{\alpha}}$, and thus the parameter N in Eq. (2) is interpreted as $N=-\ln P(\tilde{a}>a,\tilde{b}>b)>0$. In what follows, we shall assume N to be constant across time, and λ_a,λ_b to grow as an outcome of factor-augmenting R&D.[8]

From the mathematical point of view, we could also allow the shape parameter α in (2) to be negative. In that case, the technology menu would be reinterpreted as a contour line of a bivariate distribution derived from two independent Fréchet distributions. Since the economic interpretation favors the case with $\alpha>0$ (see Section 3), the derivations for the case $\alpha<0$ have been relegated to Appendix A.

An important caveat is that if $\tilde{a}$ and $\tilde{b}$ are Weibull-distributed but dependent, or independent but following some other distribution than Pareto or Weibull, the resultant aggregate production does not belong to the CES class. It is also vital that both marginal Weibull distributions share the same shape parameter α: if labor- and capital-augmenting ideas are independently Weibull distributed, but with different shape parameters, then the resultant aggregate production function does not belong to the CES class either. Fortunately, as Section 3 shows, under arguably general conditions the model of directed R&D discussed there will yield the same value of $\alpha=1$.

The analytical form of the technology menu postulated in Eq. (2) has been used previously by Caselli and Coleman (2006), Growiec (2008a, 2008b), and Nakamura (2009), but with the unnecessary restriction that λ_a and λ_b are always either constant or increasing proportionately.

Assumption 3. Firms choose the technology pair (a,b) optimally, subject to the current technology menu, such that their profit is maximized:

$$\max_{a,b}\left\{Y_0\left(\pi_0\left(\frac{bK}{b_0K_0}\right)^{\theta}+(1-\pi_0)\left(\frac{aL}{a_0L_0}\right)^{\theta}\right)^{1/\theta}\right\} \quad \text{s.t.} \left(\frac{a}{\lambda_a}\right)^{\alpha}+\left(\frac{b}{\lambda_b}\right)^{\alpha}=N. \tag{4}$$

We note that factor remuneration $rK+wL$, taken into account in the firms' profit maximization problem, does not depend on the chosen technology pair (a,b) so it can be safely omitted from the above optimization problem.[9] The same assumption was made by Jones (2005), Growiec (2008a, 2008b), and Nakamura (2009).

Finally, second order conditions require us to assume that $\alpha>\theta$, so that the interior stationary point of the above optimization problem is a maximum. The proof of this is included in the online appendix (see also Growiec, 2008a). Furthermore, we also need to assume that $\alpha-\theta-\alpha\theta>0$ so that the resultant aggregate production function is concave with respect to K and L. Both these conditions are satisfied automatically in the case $\alpha>0>\theta$, on which we concentrate here. As we shall see shortly, in such a case, capital and labor will always be gross complements along the aggregate production function.

[7] Eq. (2) can also be obtained for Pareto distributions of $\tilde{a}$ and $\tilde{b}$, provided that the pattern of dependence between both marginal distributions is modeled with the Clayton copula (Growiec, 2008a). The alternative case where $\tilde{a}$ and $\tilde{b}$ are independently Pareto-distributed leads to a different specification of the technology menu, discussed in the online appendix.

[8] One could easily reparametrize the technology menu, though, fixing either λ_a or λ_b and allowing N to vary. This would also reparametrize the resultant aggregate production function: the ratio N/N_0 would appear in Eq. (5) and the fixed parameter (λ_a or λ_b) would drop out. One could also (redundantly) vary all three parameters simultaneously. See the discussion in Appendix B.

[9] In the case of Leontief LPFs, optimization implies $bK/b_0K_0=aL/a_0L_0$.

2.2. The aggregation result

Solving the maximization problem set up above yields direct results on the firms' optimal technology choices (Growiec, 2008a). Inserting these optimal choices into the LPF, we obtain the following aggregation result.

Proposition 1. *If Assumptions 1–3 hold, then the aggregate production function takes the normalized CES form*:

$$Y = Y_0 \left(\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0} \right)^{\alpha\theta/(\alpha-\theta)} + (1-\pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0} \right)^{\alpha\theta/(\alpha-\theta)} \right)^{(\alpha-\theta)/\alpha\theta}. \tag{5}$$

Hence, the normalized CES result obtains both in the case of CES and Leontief LPFs.

Proof. In the case of CES LPFs, we form the Lagrangian:

$$\mathcal{L} = Y_0 \left(\pi_0 \left(\frac{bK}{b_0 K_0} \right)^{\theta} + (1-\pi_0) \left(\frac{aL}{a_0 L_0} \right)^{\theta} \right)^{1/\theta} + \Lambda \cdot \left\{ \left(\frac{b}{\lambda_b} \right)^{\alpha} + \left(\frac{a}{\lambda_a} \right)^{\alpha} - N \right\}. \tag{6}$$

Differentiating it with respect to a, b and substituting for Λ yields:

$$\left(\frac{b}{a} \right)^{\alpha-\theta} = \frac{\pi_0}{1-\pi_0} \left(\frac{\lambda_b}{\lambda_a} \right)^{\alpha} \left(\frac{K}{L} \frac{a_0 K_0}{b_0 K_0} \right)^{\theta}. \tag{7}$$

Considering first the reference point of time t_0, when $K = K_0$, $L = L_0$, $\lambda_b = \lambda_{b0}$, $\lambda_a = \lambda_{a0}$, $a = a_0$, $b = b_0$, and using the specification of the technology menu (2) we obtain:

$$b_0^* = (N\pi_0)^{1/\alpha} \lambda_{b0}, \quad a_0^* = (N(1-\pi_0))^{1/\alpha} \lambda_{a0}. \tag{8}$$

For $t \neq t_0$, by plugging (8) into (7), using (2) again and rearranging we obtain that:

$$\left(\frac{b}{b_0} \right)^* = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1-\pi_0) \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{KL_0}{LK_0} \right)^{-\alpha\theta/(\alpha-\theta)} \right)^{-1/\alpha}, \tag{9}$$

$$\left(\frac{a}{a_0} \right)^* = \frac{\lambda_a}{\lambda_{a0}} \left(\pi_0 \left(\frac{\lambda_b}{\lambda_a} \frac{\lambda_{a0}}{\lambda_{b0}} \frac{KL_0}{LK_0} \right)^{\alpha\theta/(\alpha-\theta)} + 1 - \pi_0 \right)^{-1/\alpha}. \tag{10}$$

Plugging this into the LPF and rearranging, we obtain the final result.

In the case of Leontief LPFs, instead of forming the Lagrangian, one should use the equality $bK/b_0K_0 = aL/a_0L_0$, which must hold because of the assumption that the representative firm maximizes profits. Since Eqs. (2) and (8) still hold, plugging these equalities into the LPF yields

$$Y = Y_0 \frac{bK}{b_0 K_0} = Y_0 \left(\pi_0 \left(\frac{\lambda_b}{\lambda_{b0}} \frac{K}{K_0} \right)^{-\alpha} + (1-\pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{L}{L_0} \right)^{-\alpha} \right)^{-1/\alpha}. \tag{11}$$

Note that the same result is obtained by taking the case of CES LPFs and considering the limit $\theta \to -\infty$.

The demonstration that second-order conditions for the maximization of the Lagrangian hold can be found in the online appendix. □

It is worthwhile to comment on each of the parameters of the aggregate normalized CES production function, because they all have sound interpretations:

- the substitutability parameter is $\rho = \alpha\theta/(\alpha-\theta)$ (or $\rho = -\alpha$ in the case of Leontief LPFs). The aggregate elasticity of substitution is thus $\sigma = 1/(1-\rho) = (\alpha-\theta)/(\alpha-\theta-\alpha\theta) > 0$ (or $\sigma = 1/(1+\alpha)$ in the case of Leontief LPFs). It is verified that $\theta < \rho < 0$ and thus $\sigma_{LPF} = 1/(1-\theta) < \sigma < 1$. Hence, endogeneous technology choice *unambiguously increases* the substitutability between production factors as compared to the LPF, but the elasticity of substitution nevertheless remains *bounded from above* by unity, characteristic for the Cobb–Douglas specification,[10]
- the distribution parameter is $\pi_0 = r_0 K_0 / Y_0$, whereas the multiplicative constant term is Y_0. Hence, thanks to normalization, both these parameters are equal to the respective parameters of the LPF,
- the constant parameter N does not appear in the aggregate production function,
- the capital-augmenting factor b present in the LPF is replaced by the capital-augmenting parameter of the technology menu λ_b in the aggregate production function, both at time t_0 and at any time $t \neq t_0$. The same applies to the labor-augmenting factor a and the respective parameter λ_a.[11]

[10] Capital and labor could be gross substitutes in the aggregate production function only if $\theta > 0$ or $\alpha < 0$ (and a few other auxiliary conditions are met). For a discussion of these cases, refer to Appendix A.

Hence, all growth in λ_a and λ_b (with N kept intact), obtained thanks to directed R&D, will ultimately appear as a multiplicative term in front of the respective factor of production in the aggregate production function. Growth in λ_a or λ_b ought not to be confused with the actual factor-augmenting technical change, i.e., growth in a and b: these two types of entities are generally not proportional to one another, unless additional (very unlikely) conditions are met.

We also note the following straightforward corollary.

Corollary 1. *Assuming that factors are priced at their marginal product, the capital and labor income shares are equal to, respectively*:

$$\pi=\frac{rK}{Y}=\frac{\pi_0\left(\frac{\lambda_b}{\lambda_{b0}}\frac{K}{K_0}\right)^{\alpha\theta/(\alpha-\theta)}}{\pi_0\left(\frac{\lambda_b}{\lambda_{b0}}\frac{K}{K_0}\right)^{\alpha\theta/(\alpha-\theta)}+(1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}}\frac{L}{L_0}\right)^{\alpha\theta/(\alpha-\theta)}}, \tag{12}$$

$$1-\pi=\frac{wL}{Y}=\frac{(1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}}\frac{L}{L_0}\right)^{\alpha\theta/(\alpha-\theta)}}{\pi_0\left(\frac{\lambda_b}{\lambda_{b0}}\frac{K}{K_0}\right)^{\alpha\theta/(\alpha-\theta)}+(1-\pi_0)\left(\frac{\lambda_a}{\lambda_{a0}}\frac{L}{L_0}\right)^{\alpha\theta/(\alpha-\theta)}}. \tag{13}$$

The above factor share formulas are instructive as regards the expected direction of their change over time: this direction is strictly determined by the growth rate of $\lambda_b K$ relative to $\lambda_a L$. If both growth rates are equal, the capital income share will remain constant at π_0. If $\lambda_b K$ grows faster, then due to gross complementarity of inputs along the aggregate production function ($\alpha\theta/(\alpha-\theta)<0$), capital's income share will gradually fall to zero over time; conversely, it will gradually rise toward unity if $\lambda_a L$ grows faster. Hence, endogeneous technology choice does not overturn any of the standard results identified in earlier literature.[12]

We may also note that if technology adoption required time, so that there were a nonzero time lag between any exogeneous shift in the firm's factor endowments and the adoption of the new optimal technology by this firm (see León-Ledesma and Satchi, 2011), then the elasticity of substitution along the LPF, σ_{LPF}, could be reinterpreted as the *short-run* elasticity of substitution, whereas σ would then be understood as the *long-run* elasticity of substitution. The implication that $\sigma_{LPF}<\sigma$ is consistent with economic interpretations provided by, e.g., Jones (2005) and León-Ledesma and Satchi (2011). The conclusion that $\sigma_{LPF}<1$ as well as $\sigma<1$ is, in turn, strongly supported by empirical evidence, reviewed by Chirinko (2008) and León-Ledesma et al. (2010).

2.3. Direction of technical change vs. direction of R&D

Given the static character of the endogeneous technology choice framework discussed in the previous section, it is straightforward to embed it in dynamic growth models. To keep things simple, one could, for instance, assume additionally that there are no interactions between technology choice and factor demand on the side of firms. In such case, the aggregate production function derived in Eq. (5) would enter the dynamic model directly. Such a growth model could be closed, e.g., by allowing directed R&D to increase λ_a and λ_b endogeneously (cf. Acemoglu, 2003). Then, depending on the particular assumptions of the embedding growth model, generally any direction of R&D could be obtained in equilibrium. Despite that negative result, clear general conclusions can nevertheless be drawn on the distinction between the *direction of R&D*, captured by the relative growth rates of λ_a and λ_b, and the actual *direction of technical change*, measured by the relative growth rates of a and b.

In particular, very specific interpretable implications follow from the cases of Harrod-neutral R&D ($\hat{\lambda}_a>0, \hat{\lambda}_b=0$), and Hicks-neutral R&D ($\hat{\lambda}_a=\hat{\lambda}_b>0$).[13] Detailed discussion of these cases can be found in Appendix B. Outside of these two special cases, however, we obtain the following generic asymptotical results:

- If $\lambda_b K$ grows asymptotically faster than $\lambda_a L$, then $\hat{y}(t)\to\hat{\lambda}_a(t)$, $\hat{a}(t)\to\hat{\lambda}_a(t)$ and $\hat{b}(t)\to\hat{\lambda}_b(t)+(\theta/(\alpha-\theta))\hat{k}(t)$ as $t\to\infty$.[14] Due to gross complementarity, the pace of capital-augmenting technical change will eventually lag behind the pace of capital-augmenting R&D. The capital income share falls toward zero over time.
- If $\lambda_b K$ grows asymptotically slower than $\lambda_a L$, then $\hat{y}(t)\to\hat{\lambda}_b(t)+\hat{k}(t)$, $\hat{a}(t)\to\hat{\lambda}_a(t)-(\theta/(\alpha-\theta))\hat{k}(t)$ and $\hat{b}(t)\to\hat{\lambda}_b(t)$ as $t\to\infty$.[15]

[11] The last two findings are a corollary from the fact that the technology menu defined in Assumption 2 is a curve in a two-dimensional space, parametrized by three parameters λ_a, λ_b, N, and N is kept constant by assumption.

[12] An obvious remark here is that the above factor share formulas are invalidated once one allows for imperfect competition. However, if it is introduced via the Dixit–Stiglitz model of monopolistic competition, implying constant markups over marginal costs, then this change would merely re-scale factor income shares, without altering any of the results on their dynamics (Growiec, 2012).

[13] We use the notation: $y\equiv Y/L, k\equiv K/L$, and $\hat{x}\equiv\dot{x}/x$ for any variable $x>0$.

[14] Given that $\hat{k}(t)\le\hat{y}(t)$ for sufficiently large t, this case can only be obtained if $\hat{\lambda}_b(t)>0$.

[15] If $\hat{k}(t)=\hat{y}(t)$ in the long run and $\hat{\lambda}_b>0$, then such a model would imply explosive dynamics, potentially achieving infinite output in finite time.

Due togross complementarity, the pace of labor-augmenting technical change will eventually lag behind the pace of labor-augmenting R&D. The capital income share increases toward unity over time.

- If $\lambda_b K$ grows asymptotically at the same pace as $\lambda_a L$, then $\hat{y}(t) \to \hat{\lambda}_a(t) = \hat{\lambda}_b(t) + \hat{k}(t)$, $\hat{a}(t) \to \hat{\lambda}_a(t)$ and $\hat{b}(t) \to \hat{\lambda}_b(t)$ as $t \to \infty$.[16] In the long run, the pace of factor-augmenting technical change coincides with the pace of factor-augmenting R&D. The capital income share tends to aconstant.

Hence, we see that under endogeneous technology choice, the direction of R&D and the direction of technical change are generally different from one another. Moreover, the temporal evolution of optimal technology choices and factor income shares is determined by comparing the growth rates of $\lambda_b K$ and $\lambda_a L$, i.e., of capital and labor expressed in efficient units evaluated along the aggregate production function.

3. A microfoundation for Weibull UFP distributions

Let us now justify the functional form of the technology menu, taken for granted in Assumption 2. This will be done using a novel, analytically tractable model of two independent R&D sectors, producing capital- and labor-augmenting innovations, respectively. It bears some similarity with the framework discussed in Appendix D of Growiec (2008a), but has a few unique distinguishing features, thanks to which it can be viewed as the primary contribution of the current paper to the related literature.

3.1. Distributions of complex ideas

The point of departure of the current model is the assumption that ideas are inherently *complex* and consists of a large number of *complementary* components. Formally, this can bin Appendix B.e written down in the following way.

Assumption 4. The (capital- or labor-augmenting) R&D sector consists of an infinity of researchers located along the unit interval $I=[0,1]$. At each instant t, every researcher $i \in I$ determines the quality of her innovation ($\tilde{b}_i$ or $\tilde{a}_i$, respectively) by taking the minimum over $n \in \mathbb{N}$ independent draws from the elementary idea distribution with cdf $\mathcal{F}$. The distribution $\mathcal{F}$ has positive density on $[w, v)$, where v can be infinite, and zero density otherwise, and satisfies the condition of a regularly varying left tail

$$\lim_{p \to 0_+} \frac{\mathcal{F}(w+px)}{\mathcal{F}(w+p)} = x^{\alpha} \tag{14}$$

for all $x>0$ and a certain $\alpha>0$.

The parameter n in the above assumption captures the number of constituent components of any given (composite) idea, and thus measures the complexity of any state-of-the-art technology. Allowing for such complexity puts the current framework in stark contrast to earlier studies (such as Jones, 2005, or Growiec, 2008a) where the quality of ideas was determined via a *single* draw from the elementary idea distribution $\mathcal{F}$.[17]

Moreover, the assumption that the quality of an innovation is the *minimum* (a Leontief function) of a range of n independent draws from the distribution $\mathcal{F}$ reflects the view that the components of an idea are complementary to one another (Kremer, 1993; Blanchard and Kremer, 1997; Jones, 2011). More precisely, we consider the extreme case here, where they are *perfectly* complementary, and thus the actual productivity of a complex idea is determined by the productivity of its "weakest link" (or "bottleneck"). Clearly, this need not hold exactly in reality, since certain deficiencies of design can often be covered by advantages in different respects. However, the example of the explosion of the space shuttle *Challenger* due to a failure of an inexpensive O-ring, put forward by Kremer (1993), is perhaps the best possible illustration of the potentially complementary character of components of complex ideas.

Letting the technology complexity n be arbitrarily large, we obtain the following result:

Proposition 2. *If Assumption 4 holds, then as $n \to \infty$, the minimum of n independent random draws from the distribution $\mathcal{F}$, after appropriate normalization, converges in distribution to the Weibull distribution with the shape parameter α:*

$$[1-\mathcal{F}(xp_n+w)]^n \xrightarrow{d} e^{-(x/\lambda)^{\alpha}}, \tag{15}$$

where $w=\inf\{x \in \mathbb{R} : \mathcal{F}(x)>0\}$, $p_n=(1/\lambda)(\mathcal{F}^{-1}(1/n)-w)$ and the free parameter $\lambda>0$ is assumed to be proportional to the mean of the underlying distribution $\mathcal{F}$.

[16] If $\hat{k}(t)=\hat{y}(t)$ in the long run, then this case can appear only if $\hat{\lambda}_b(t) \to 0$, which boils down to the balanced growth path case discussed in Appendix B.

[17] Jones (2005) viewed the technology menu as a convex hull of a finite number of ideas, say M. In line with the findings of extreme value theory, in the limit $M \to \infty$, this menu took the form of a contour line of a Fréchet distribution, which is the limiting distribution of the *maximum* of M independent draws from a distribution $\mathcal{F}$ that has a regularly varying right tail. This assumption has been later replaced, both in Growiec (2008a) and in the current paper, with Assumption 5. At this point, one should note however that Jones (2005) was preoccupied with the distribution of the maximum *across ideas* and here we are considering the minimum across *components of each idea*. Across ideas, it is still the best ones that matter.

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Table 1
Selected distributions $\mathcal{F}$ such that for $X_1, \dots, X_n \sim \mathcal{F}$, $\min\{X_1, \dots, X_n\}$ converges in distribution to the Weibull distribution as $n \to \infty$.

Distribution	Cdf for $x \in [w, v)$	Lower bound	Postulated p_n	Implied λ	Implied α
Pareto(ϕ)	$\mathcal{F}(x) = 1 - \left(\frac{\gamma_x}{x}\right)^{\phi}$	$w = \gamma_x$	$p_n = \left(1 - \frac{1}{n}\right)^{-1/\phi}$	$\lambda = \gamma_x$	$\alpha = 1$
Uniform $U([w, v])$	$\mathcal{F}(x) = \frac{x-w}{v-w}$	Given w	$p_n = \frac{1}{n}$	$\lambda = v - w$	$\alpha = 1$
Truncated $N(\mu, \sigma)$	$\mathcal{F}(x) = \frac{\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}$	Given w	$p_n = \overline{p}_n$	$\lambda = \mu$	$\alpha = 1$
Weibull(α, λ)	$\mathcal{F}(x) = 1 - e^{-(x/\lambda)^{\alpha}}$	$w = 0$	$p_n = -\frac{1}{\alpha} \ln\left(1 - \frac{1}{n}\right)$	Given λ	Given α

Note: (i) to obtain convergence to the Weibull distribution, one may equivalently take $p_n = 1/n$ in the Pareto case, and $p_n = n^{-1/\alpha}$ in the Weibull case; (ii) we used the notation

$$\overline{p}_n = 1 - \frac{w}{\mu} + \frac{\sigma}{\mu}\Phi^{-1}\left(\left[1 - \Phi\left(\frac{w-\mu}{\sigma}\right)\right]\frac{1}{n} + \Phi\left(\frac{w-\mu}{\sigma}\right)\right).$$

The mean of a random variable drawn from the truncated Gaussian distribution increases both with the mean of the original distribution μ and the truncation point w, according to the formula

$$EX = \mu + \frac{\sigma\varphi\left(\frac{w-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{w-\mu}{\sigma}\right)}.$$

We have chosen technological change in λ to affect μ, but we might have alternatively chosen it to affect w, or some mixture of both.

Proof. The proposition follows directly from the Fisher–Tippett–Gnedenko extreme value theorem, applied to the distribution $\mathcal{F}$ (Theorem 1.1.3 in de Haan and Ferreira, 2006, rephrased so that it captures the minimum instead of maximum). From the theorem specifying the domain of attraction of the Weibull distribution (Theorem 1.2.1 in de Haan and Ferreira, 2006; Section 1.3 in Kotz and Nadarajah, 2000), we obtain the necessary and sufficient conditions for the complementarity mechanism to work. □

From the mathematical point of view, the parameter λ is superfluous and can be normalized to unity without loss of generality, by a simple re-normalization of the sequence p_n as $\tilde{p}_n = p_n \cdot \lambda = \mathcal{F}^{-1}(1/n) - w$. In the case of the currently discussed R&D model, the distinction between p_n and λ is important, however, because it allows R&D activity to influence the means of the distributions of $\tilde{a}$ and $\tilde{b}$ at any moment in time: it is precisely λ which pins down the mean of the limiting Weibull distribution.

Hence, turning to our original distinction between capital- and labor-augmenting R&D, let us now distinguish between λ_a determining the mean of $\tilde{a}$, and λ_b pinning down the mean of $\tilde{b}$. In the limit of $n \to \infty$, we obtain the following generic results:

$$E\tilde{a} = \lambda_a \Gamma\left(1 + \frac{1}{\alpha}\right), \quad E\tilde{b} = \lambda_b \Gamma\left(1 + \frac{1}{\alpha}\right), \tag{16}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ is Euler's Gamma function.

A few meaningful applications of Proposition 2 have been summarized in Table 1. This table indicates that the Weibull result can be obtained for a wide range of distributions, including the ones frequently discussed in the related literature. It also provides a clear explanation of the relationship between λ and the characteristics of the underlying productivity distribution $\mathcal{F}$, and announces the resulting values of the parameter α.

The main message drawn from the results presented in Table 1 is that we can model directed R&D, affecting the mean of the underlying distribution $\mathcal{F}$ and thus λ_a and λ_b, in an arbitrary way; if only the conditions specified in Assumption 4 hold, then the Weibull result in the limit of $n \to \infty$ will always go through. If, on top of that, the shape parameters α of capital- and labor-augmenting developments happen to be equal to one another, then the aggregate normalized CES production function result will always follow, too.

Fortunately for the last statement, the implied parameter α is found to be unitary for a wide range of distributions $\mathcal{F}$, and therefore the condition that α is equal for both capital- and labor-augmenting ideas is indeed quite plausible. Furthermore, if $\alpha = 1$, then also $E\tilde{a} = \lambda_a$ and $E\tilde{b} = \lambda_b$, which makes the link between the underlying distribution $\mathcal{F}$ and the limiting Weibull distribution (which in such a case specializes to the exponential distribution) especially apparent. We note the following corollary.

Corollary 2. *If the underlying idea distributions $\mathcal{F}$ are Pareto, uniform or truncated Gaussian, then $\alpha = 1$ and thus the limiting idea distribution is exponential. In such a case, the elasticity of substitution of the aggregate CES production function is equal to $\sigma = (1-\theta)/(1-2\theta) \in [1/2, 1)$, increasing from 1/2 in the case of Leontief LPFs to unity in the limiting case of Cobb–Douglas LPFs.*

Also note that a number of well-known classes of distributions have not been included in Table 1 because they do not satisfy Assumption 4. First of all, the support of the distribution must be bounded from below, which rules out distributions defined on the whole $\mathbb{R}$ such as the Gaussian. Also, the pdf of such a distribution cannot increase smoothly from zero at w; there must be a jump. This rules a few more candidate distributions such as the lognormal or the Fréchet. Furthermore, the lowest possible value of the random variable cannot be an isolated atom, which rules out all discrete distributions such as the two-point distribution, the binomial, negative binomial, Poisson, etc.

Ultimately, it must be remembered that the aforementioned derivation of the Weibull distribution requires that the technologies consist of a large number of *complementary* components. If this were not the case, i.e., if technologies were rather simple (consisting of a few components only) or if their components were mutually substitutable, then the current Weibull result would not follow. One particularly instructive case of such a departure is the following. Namely, if one substituted the minimum of n draws from distribution $\mathcal{F}$ (with $n\to\infty$) with the *maximum* of these draws, then the limiting distribution would be Fréchet instead of Weibull (cf. Kortum, 1997; Kotz and Nadarajah, 2000; Jones, 2005). In consequence, one would obtain the same functional form of the technology menu (2), but with $\alpha<0$; consequently, the resulting aggregate production function would still be (normalized) CES, but with gross substitutability between inputs ($\sigma>1$) instead of gross complementarity (see Appendix A).

The limiting case where we take the maximum of n draws from distribution $\mathcal{F}$, with $n\to\infty$ can be interpreted as the case where technologies are simple – consist of a single component – but the R&D is a repeated "trial-and-error" process where only the best try is considered useful. Hence, this case would correspond to a world where researchers work on simple, narrowly focused projects and are allowed to redo these projects over and over again in the search for improvements. In the light of this interpretation, the Fréchet case seems to be much less plausible empirically than the Weibull case: the current times are marked by increasing technological complexity of new inventions, skill-biased technical change, mounting burden of knowledge, and increasing R&D collaboration (Caselli and Coleman, 2006; Pintea and Thompson, 2007; Jones, 2009). Last but not least, it is also the Weibull case which better corresponds to the empirical findings reviewed by Chirinko (2008), suggesting that the aggregate elasticity of substitution between capital and labor is below unity, both in the short and in the long run.

3.2. Derivation of the technology menu

Let us finally show how the individual draws of (complex, factor-augmenting) technologies $\tilde{a}$ and $\tilde{b}$ are combined, yielding the functional form of the technology menu postulated in Assumption 2. We shall close the model of the two independent R&D sectors by making the following assumption.

Assumption 5. Every capital- or labor-augmenting technology draw is allowed to enter the technology menu if it has been confirmed by at least a pre-defined fraction of researchers in I (z_b or z_a, respectively).

In the parlance of the above assumption, a "confirmed" technology is such that the given fraction of researchers has simultaneously obtained the same or a higher technology draw ($\tilde{b}$ or $\tilde{a}$, respectively).

Formally, given the above assumption and the Law of Large Numbers, a labor-augmenting technology a will be included in the technology menu at time t if and only if $P(\tilde{a}>a)\geq z_a$, and a capital-augmenting technology b, if and only if $P(\tilde{b}>b)\geq z_b$. Since both R&D sectors are independent from one another, it follows that a technology pair (a,b) is included in the technology menu if $P(\tilde{a}>a,\tilde{b}>b)=P(\tilde{a}>a)P(\tilde{b}>b)\geq z_a z_b$.[18] Furthermore, since no profit-maximizing firm would ever choose a dominated technology, we may as well replace the above "≥" inequality with equality in the formulation of the technology menu. This brings us directly to Assumption 2.

4. Robustness of the Weibull result: non-iid components of ideas

The point made in the previous section can be substantially generalized because, in fact, the result of a limiting Weibull UFP distribution is robust to numerous changes in the specification of the underlying R&D process.[19] Assessing the robustness of this result is in turn an important question because the restriction which we have maintained so far, that components of ideas should be independent and identically distributed (iid), is admittedly arbitrary and probably overly restrictive. In reality, some components of ideas might be easy to invent, some others might be straightforward but tedious, some further ones might be very difficult, etc. Hence, the distributions of components of ideas $\mathcal{F}$ might vary in terms of their location, slope, and skewness. Also, these distributions could be correlated, as e.g. solving some problems could greatly help solve other ones.

Fortunately, it turns out that allowing for such extensions will generally not overturn the Weibull result announced in the previous section. In fact, the assumption that the components of ideas are iid with a predefined distribution $\mathcal{F}$ can be

[18] Note that the last inequality could also be assumed directly, leading to a generalization of our framework (see Appendix D in Growiec, 2008a). In such a case, different pairs of z_a and z_b (such that their product $z_a z_b$ is given) would be allowed at the technology menu simultaneously, whereas currently we fix the values of z_a and z_b separately. This extension does not bring about any significant change in results, however, and thus we leave it aside.

[19] Moreover, the technology menu given in Eq. (2) can also be microfounded without referring to Weibull distributions at all: e.g., Growiec (2008a) has achieved this goal by considering Pareto-distributed ideas dependent according to the Clayton copula.

relaxed to a great extent without affecting the shape of the limiting distribution (Leadbetter et al., 1983; de Haan and Ferreira, 2006). The possibility to obtain the limiting Weibull distribution of sample minima under very general circumstances is an important strength of extreme value theory. Let us see how this mathematical argument applies to our framework.

4.1. Differently distributed components of ideas

The first way to relax the iid assumption is to allow the components of ideas $i=1,\dots,n$ to follow different distributions, captured by cdfs $\mathcal{F}_i$ (in this case we retain the assumption of their independence). This corresponds to a case where some components of ideas – viewed as problems to be solved – vary in terms of their difficulty and the ingenuity required to find the solution. In turn, the distributions $\mathcal{F}_i$ might vary in terms of their location, slope, and skewness.

An important result for such a case is that the limiting Weibull (α) distribution is still obtained as long as the distributions $\mathcal{F}_i$ have positive density on $[w_i, v_i)$, where v_i can be infinite, and zero density otherwise, and satisfy the condition of a regularly varying lower tail

$$\lim_{p\to 0_+} \frac{\mathcal{F}_i(w_i+px)}{\mathcal{F}_i(w_i+p)} = x^\alpha \tag{17}$$

for all $x>0$ and a certain $\alpha>0$, provided that $w_i \in [w_{\min}, w_{\max}]$ for all $i=1,2,\dots,n$ with $w_{\min}, w_{\max} \in \mathbb{R}$. Mathematically, this generalization requires only a modification of the sequence p_n and replacing w with component-specific w_n in Eq. (15), and the same result is maintained.

One could also allow the shape parameters α to be component-specific (α_i), and the Weibull limiting distribution would still follow – in such a case, the shape parameter α of the resultant Weibull distribution would depend on all underlying α_i's, though. Furthermore, in relation to Table 1 we note that whenever the underlying distributions $\mathcal{F}_i$ are Pareto, uniform, or truncated Gaussian, then $\alpha=1$ (and thus an exponential limiting distribution of factor-augmenting ideas) should be expected even if all other underlying parameters are component-specific.

4.2. Correlated components of ideas

The second way to relax the iid assumption is to allow the components of ideas $i=1,\dots,n$ to be dependent on one another. This corresponds to the case where solving some problems is helpful in solving other ones. In relation to this case, we learn from extreme value theory that as long as the stochastic process $(X_i)_{i=1}^\infty$ is stationary (in particular, this requires the components of ideas X_i, $i=1,\dots,n$, to follow the same distribution $\mathcal{F}$), and the assumption of asymptotic independence of minima (AIM) is satisfied, then the Weibull result still goes through (see Leadbetter et al., 1983). The AIM condition requires that separated groups of extreme points (subsample minima) become asymptotically independent as their separation increases and their level decreases, at appropriate rates. The proof of this result is rather involved mathematically, but some corollaries are straightforward. First, the Weibull distribution of sample minima is obtained in the case where the variables X_i are correlated with variables $X_{i-q},\dots,X_{i+q}$, for any finite q, and independent from others. Second, it also follows if the correlation between X_i and X_{i-q} or X_{i+q} decreases sufficiently fast (e.g., exponentially) with q.

One could also think of considering components of ideas which are both dependent and differently distributed. We leave this for further research.

4.3. Weighted components of ideas

The Weibull distribution of ideas can also be generated in various other ways. One of them involves assigning components of any composite idea with different degrees of importance. For example,[20] one could think of a situation where the components of ideas X_i are drawn from independent Pareto distributions $\mathcal{F}_i(x) = 1-(\gamma_i/x)^{\phi_i}$. In such a case, transformed random variables $Y_i = \gamma(X_i/\gamma_i)^{\phi_i/\phi}$ have the same cdf $\mathcal{F}(y) = 1-(\gamma/y)^\phi$ for all i. Clearly, $\min\{Y_1,\dots,Y_n\}$ – after appropriate normalization – will then converge in distribution to the exponential distribution (Weibull distribution with $\alpha=1$). In a sense, this transformation ("pre-processing") of variables according to a power function implies that components of the composite idea are weighted, i.e., associated with different importance (some are amplified while others are muted). Note that the direct approach, consisting in computing $\min\{X_1,\dots,X_n\}$ where the variables are not transformed, leads asymptotically to the same Weibull outcome as $n\to\infty$.

5. Conclusion

The current paper has provided robust idea-based microfoundations for the aggregate normalized CES production function, based on an "endogeneous technology choice" framework where this function is obtained as a convex hull of local

[20] I am grateful to an anonymous Referee for providing this example.

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production techniques. This result follows under the assumption that unit productivities (UFPs) of capital and labor are independently Weibull-distributed (Growiec, 2008a, 2008b).

The central contribution of the current paper to the literature has been to develop a novel, tractable model of directed R&D which microfounds the postulated Weibull UFP distributions. Drawing from established results in extreme value theory, we have demonstrated why the Weibull distribution should approximate real-world UFP distributions very well. Our argument is based on the identifying assumption that ideas (technologies, production techniques) are not simple, as it was implicitly assumed in earlier literature, but inherently complex, consisting of a large number of complementary components. Under such circumstances, the efficiency of a given technology closely follows the efficiency of its "weakest link", i.e., the least efficient component. The Weibull distribution is, in turn, the extreme value distribution characterizing the minimum of n random draws from the same underlying distribution (which is bounded from below and has a regularly varying lower tail), in the limit of $n \to \infty$. Consequently, we have shown that if technologies consist of a wide range of complementary components, and they are then optimally chosen by firms, then the aggregate production function should take the CES form.

The discussed framework has a number of other interesting features which have been overlooked so far. First, thanks to normalization of CES functions, all parameters of the derived aggregate production function can be provided with a sound interpretation in terms of the parameters of local production functions and the underlying unit factor productivity (UFP) distributions. Second, normalization can be maintained simultaneously at the local and aggregate level. Third, the equilibrium direction of (factor-augmenting) technical change is typically not equal to the direction of underlying R&D.

There is a wide range of issues, closely related to the current paper, that should be studied in further research. Let us just name a few. First, it would be worthwhile to investigate the real-world productivity distributions underlying our setup, attempting to discriminate econometrically between the Weibull specification and the Pareto one (or perhaps some further distributions, too). Another challenge would be to develop an empirical approach able to identify jointly the parameters of the aggregate production function and the technology menu. Third, it would be interesting to see the consequences of allowing for dependence between the marginal Weibull distributions. Fourth, the current framework could also be potentially applied in the modeling of multi-stage production processes, or used as means to motivate or endogenize selected dimensions of firm heterogeneity.

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Appendix A. Aggregate CES production function with gross substitutability of inputs

The "endogeneous technology choice" framework discussed here implies that if (i) LPFs are Leontief or CES with $\theta < 0$, so that inputs are gross complements along the LPF and (ii) the technology menu is generated as a contour line of a bivariate distribution whose marginals are independent Weibull distributions (and thus $\alpha > 0$), then the aggregate production function is CES with an elasticity of substitution bounded from above by unity ($\sigma < 1$). Hence, capital and labor are necessarily gross complements not only along the LPF, but also along the aggregate production function. We note that the Cobb–Douglas limit of $\sigma \to 1_-$ is approached when $\theta \to 0_-$ (so that the LPFs tend to Cobb–Douglas functions themselves) or when $\alpha \to 0_+$ (so that the Weibull distributions of UFPs get ever fatter tails).

As quoted in the main text, the implication that $\sigma < 1$ is in good agreement with empirical evidence surveyed by Chirinko (2008) and León-Ledesma et al. (2010). Despite that, our theoretical procedure can also be easily generalized to accommodate the case with $\sigma > 1$ as well. This can be done in two alternative ways, discussed in the following subsections. Both of them are analytically straightforward.

As far as the dynamic implications of gross substitutability of inputs along the aggregate production function are concerned, one must remember that in such a case, factor income shares move in the opposite direction than in the case of gross complementarity. The direction of their change is still strictly determined by the growth rate of $\lambda_b K$ relative to $\lambda_a L$, and if both growth rates are equal, then the capital income share will still remain constant at π_0. If $\lambda_b K$ grows faster, however, then in the gross substitutability case, the capital income share will gradually increase to unity over time; conversely, it will gradually fall to zero if $\lambda_a L$ grows faster.

Another caveat is that gross substitutability of inputs along the aggregate production function dramatically changes the conclusions regarding the long-run direction of technical change, due to two reasons: first, gross substitutability acts in favor of increasing the UFP of the relatively abundant factor instead of the relatively scarce factor. Since capital is the accumulable factor here, this implies that technical change should be largely capital-augmenting, rather than labor-augmenting as in the case of gross complementarity, especially in the long run (cf. Klump and de La Grandville, 2000; Acemoglu, 2003; Growiec, 2008a). Second, gross substitutability can act as an engine of perpetual growth due to capital

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accumulation alone (de La Grandville, 1989; Klump and Preissler, 2000; Palivos and Karagiannis, 2010). A precise discussion of these issues is beyond the scope of the current paper.

A.1. The case with $\theta \geq 0$

The first of the two possibilities allowing for gross substitutability of inputs along the aggregate production function is to assume directly that inputs are gross substitutes already along the LPF (so that $\theta > 0$). In such a case, the additional restriction $\alpha > \theta$ is required for second-order optimality conditions to hold. We also need to assume that $\alpha - \theta - \alpha\theta > 0$ (or $\theta < \alpha/(1+\alpha)$) to ensure that the aggregate production function is concave with respect to its arguments:

$$\sigma = \frac{\alpha-\theta}{\alpha-\theta-\alpha\theta} \in (1, +\infty). \tag{A.1}$$

It is easily verified that if $\theta \in (0, \alpha/(1+\alpha))$, then all our derivations still go through, and Proposition 1 still holds, but this time with $\theta > 0$. Furthermore, as $\theta \to (\alpha/(1+\alpha))_-$ then the elasticity of substitution of the aggregate production function tends to infinity and the function becomes linear in the limit. Conversely, when $\theta \to 0_+$ then $\sigma \to 1_+$ and the aggregate production function converges to the Cobb–Douglas form.

If the LPFs are Cobb–Douglas (which happens in the limiting case $\theta = 0$), then the aggregate production function is also necessarily Cobb–Douglas (cf. Growiec, 2008a). If

$$Y = Y_0 \left(\frac{bK}{b_0 K_0}\right)^{\psi} \left(\frac{aL}{a_0 L_0}\right)^{1-\psi}, \tag{A.2}$$

with $\psi = \pi_0$ being the initial capital income share (which will soon turn out to be constant for all t), then at any moment in time t the optimal technology choice satisfies:

$$a^* = \lambda_a (N(1-\psi))^{1/\alpha}, \quad b^* = \lambda_b (N\psi)^{1/\alpha}, \tag{A.3}$$

and thus

$$\left(\frac{a}{a_0}\right)^* = \frac{\lambda_a}{\lambda_{a0}}, \quad \left(\frac{b}{b_0}\right)^* = \frac{\lambda_b}{\lambda_{b0}}, \tag{A.4}$$

implying that the aggregate production function takes the Cobb–Douglas form with the same partial capital elasticity ψ:

$$Y = Y_0 \left(\frac{\lambda_b K}{\lambda_{b0} K_0}\right)^{\psi} \left(\frac{\lambda_a L}{\lambda_{a0} L_0}\right)^{1-\psi}. \tag{A.5}$$

Despite the analytical simplicity of the above derivations, it must be remembered that the assumption that inputs are gross substitutes along the LPF is widely at odds with the "recipe" interpretation of the LPF, which views it as a set of strictly specified instructions, indicating how to turn inputs into output (Jones, 2005). Given this interpretation, one should rather expect the LPFs to be approximately Leontief, than to have more than unitary elasticity of substitution.

A.2. The case with $\alpha < 0$

The second of the two possibilities allowing for gross substitutability of inputs along the aggregate production function is to assume that the parameter α in the specification of the technology menu (2) is negative. Again, the additional restriction $\alpha > \theta$ is required for second-order optimality conditions to hold, and one needs to assume that $\alpha - \theta - \alpha\theta > 0$ (or $\alpha > \theta/(1-\theta)$) to ensure that the aggregate production function is concave with respect to its arguments.

It is easily verified that if $\alpha \in (\theta/(1-\theta), 0)$ then all our derivations still go through, and Proposition 1 still holds, but this time with $\alpha < 0$. We note that as $\alpha \to (\theta/(1-\theta))_+$ then the elasticity of substitution of the aggregate production function tends to infinity and the function becomes linear in the limit. Conversely, when $\alpha \to 0_-$ then $\sigma \to 1_+$ and the aggregate production function converges to the Cobb–Douglas form, just like it does when $\alpha \to 0_+$.

Even if $\alpha < 0$, the technology menu (2) can still be derived as a contour line of the cumulative distribution function of the joint bivariate distribution of capital-augmenting ideas $\tilde{b}$ and labor-augmenting ideas $\tilde{a}$. We find that under independence of both dimensions (so that marginal distributions are multiplied by one another), Eq. (2) with $\alpha < 0$ obtains if and only if the marginal distributions are Fréchet with the same shape parameter $\alpha < 0$:

$$P(\tilde{a} \leq a) = e^{-(a/\lambda_a)^{\alpha}}, \quad P(\tilde{b} \leq b) = e^{-(b/\lambda_b)^{\alpha}}, \tag{A.6}$$

for $a, b > 0$. Under such a parametrization, we have $P(\tilde{a} \leq a, \tilde{b} \leq b) = e^{-(a/\lambda_a)^{\alpha} - (b/\lambda_b)^{\alpha}}$, and thus the parameter N in Eq. (2) is interpreted as $N = -\ln P(\tilde{a} \leq a, \tilde{b} \leq b) > 0$. As in the main text, we may assume N to be constant across time, and λ_a, λ_b to increase over time as an outcome of factor-augmenting R&D.

The Fréchet distribution is also an extreme value distribution, just like the Weibull distribution is (cf. de Haan and Ferreira, 2006); however, while the Weibull is min-stable, the Fréchet is max-stable. It is the limiting distribution of the maximum of n independent random draws from a given distribution $\mathcal{F}$ bounded from below, approached when $n \to \infty$. Once $\mathcal{F}$ additionally satisfies the condition of a regularly varying upper tail, then the maximum of these draws, after appropriate normalization, will converge in distribution to the Fréchet distribution (for mathematical reference, see Kotz and Nadarajah, 2000; for economic applications, see Kortum, 1993; Jones, 2005).

Appendix B. Implications for the direction of technical change

As mentioned in the main text, our static "endogeneous technology choice" framework can yield interesting implications when embedded in a dynamic growth model. The general finding is that the direction of technical change and the direction of R&D are distinct concepts. The key insights in this regard are obtained by considering the asymptotic behavior of the growth rate of $\lambda_b K$ relative to $\lambda_a L$. However, even more lessons can be drawn from the discussion of the benchmark cases of Harrod-neutral R&D ($\hat{\lambda}_a > 0, \hat{\lambda}_b = 0$), and Hicks-neutral R&D ($\hat{\lambda}_a = \hat{\lambda}_b > 0$). We shall discuss these two cases below.

B.1. Balanced growth path, Harrod-neutral R&D, and purely augmenting technical change

It is frequently postulated in the economic growth literature – partly due to the plentiful established "stylized facts" and partly due to analytical convenience – that the aggregate economy should follow a balanced growth path (BGP), or at least converge to it in the long run. It should be emphasized, however, that the existence of a BGP is a very restrictive, *knife-edge* assumption (cf. Growiec, 2007). In particular, in the case of neoclassical growth models, it requires that either the aggregate production function is Cobb–Douglas or technical change is purely labor-augmenting (Uzawa, 1961).

A seminal example of a growth model with (endogeneous) directed R&D, that is able to generate a BGP despite being based upon an aggregate CES production function, is due to Acemoglu (2003). This model can be straightforwardly used as an embedding structure over our endogeneous technology choice framework. It requires the equations of motion of factor-augmenting shifts in the technology menu to be captured by linear equations of form

$$\dot{\lambda}_a = f_a(\ell_a)\lambda_a, \tag{B.1}$$

$$\dot{\lambda}_b = f_b(\ell_b)\lambda_b, \tag{B.2}$$

where ℓ_a and ℓ_b are the fractions of population (total employment or hours worked) engaged in labor- and capital-augmenting R&D, respectively, and f_a, f_b are some smooth increasing functions. Such a model is scale-free. We also assume the usual capital's equation of motion

$$\dot{K} = sY - \delta K, \quad \delta > 0, \tag{B.3}$$

where s is the (potentially endogeneous) savings rate of the aggregate economy.

Analyzing the model's implications, both in the social planner allocation and the decentralized equilibrium à la Acemoglu, reveals that in the long run, the economy will converge to a balanced growth path where

$$\hat{y} = \hat{k} = \hat{a} = \hat{\lambda}_a = f(\ell_a^*) > 0, \tag{B.4}$$

$$\hat{b} = \hat{\lambda}_b = 0, \tag{B.5}$$

and thus in the long-run equilibrium:

(i) R&D is Harrod-neutral, i.e., directed toward labor-augmenting developments only ($\hat{\lambda}_b = 0$),
(ii) technical change is also Harrod-neutral, i.e., purely labor-augmenting ($\hat{b} = 0$),
(iii) factor income shares are constant at π_0 and $1-\pi_0$, respectively, despite the fact that the aggregate production function is not Cobb–Douglas.

Hence, in this very specific framework – with linear technology equations in both R&D sectors and no spillovers between both sectors – technical change must follow the direction of R&D on one-to-one basis.

Further analysis reveals that the equations of motion of capital- and labor-augmenting developments λ_a and λ_b, (B.1) and (B.2), could be made slightly more general without altering any of the aforementioned predictions. This would happen if one allowed for mutual spillovers between both R&D sectors, yet also imposed a particular knife-edge condition on their strength (measured by partial elasticities). This result has been obtained by Li (2000), for a somewhat different two-R&D-sector model, which is however isomorphic to the current one in its reduced form (that is, after stripping its solution to the form of a system of dynamic equations governing the dynamics of its state variables).

What is much more important here, however, is that for every other reduced-form specification of the encompassing growth model, the above BGP result will fail. Hence, in the *typical* (non-knife-edge) case, technical change will *not* be purely labor-augmenting, it will *not* reflect the direction of R&D, and factor income shares will *not* be constant across time (Growiec, 2007).

B.2. Hicks-neutral R&D

The second benchmark case is the model of Hicks-neutral R&D, where R&D always expands the technology menu *proportionally*, without any bias toward any of the production factors. On the one hand, such a model is clearly just as parsimonious as the Harrod-neutral R&D model discussed above: arriving at this particular case also requires one to make a certain knife-edge assumption. On the other hand, however, it provides us with a second reasonable benchmark to which we can compare the general results for all non-neutral cases.

Hicks-neutral R&D has been considered in very similar endogeneous technology choice frameworks by Caselli and Coleman (2006) and Growiec (2008a). These authors have made the implicit assumption that factor-augmenting idea distributions $\tilde{a}$ and $\tilde{b}$ evolve proportionally, so that the ratio λ_a/λ_b is constant and thus $\hat{\lambda}_a = \hat{\lambda}_b$ for all times t.[21] An example of a model where R&D is Hicks-neutral for all t can be written as follows:

$$\dot{\lambda}_a = f(\ell_a, \ell_b)\lambda_a^{\alpha+1}\lambda_b^{\beta}, \tag{B.6}$$

$$\dot{\lambda}_b = f(\ell_a, \ell_b)\lambda_a^{\alpha}\lambda_b^{\beta+1}, \tag{B.7}$$

and hence by assumption $\hat{\lambda}_a = \hat{\lambda}_b$.[22] We shall also use the usual capital's law of motion (B.3) again.

As argued above, Hicks-neutral R&D precludes the existence of a balanced growth path. More surprisingly, however, it also implies that technical change in the aggregate economy, determined jointly by the direction of R&D and firms' endogeneous technology choices, is *not* Hicks-neutral. In particular, under the assumptions that (i) Hicks-neutral R&D improves both UFPs at the same constant rate, so that $\hat{\lambda}_a = \hat{\lambda}_b \equiv g > 0$, (ii) the economy is able to maintain positive growth rates of physical capital per worker k until infinite time, with $\lim_{t\to\infty} k(t) = +\infty$, technical change will augment both factors of production in the long run, according to

$$\lim_{t\to\infty} \hat{y}(t) = g, \tag{B.8}$$

$$\lim_{t\to\infty} \hat{a}(t) = g, \tag{B.9}$$

$$\lim_{t\to\infty} \hat{b}(t) = g + \frac{\theta}{\alpha-\theta} \lim_{t\to\infty} \hat{k}(t) \Rightarrow \lim_{t\to\infty} \hat{b}(t) \in \left[\left(\frac{\alpha}{\alpha-\theta}\right)g, g\right). \tag{B.10}$$

These results have been obtained by taking limits of optimal technology choices (9) and (10) under the assumption that $k(t) \to +\infty$. We have also used the inequality $\lim_{t\to\infty} \hat{k}(t) \le \lim_{t\to\infty} \hat{y}(t)$, because in the opposite case, $y(t)/k(t)$ would be falling toward zero, ultimately violating the capital's equation of motion.[23]

Hence, in the case of Hicks-neutral R&D, technological change augments both factors of production in the long run. Capital-augmenting technical change remains positive forever, too, in contrast to the findings based on the Cobb–Douglas production function (Jones, 2005).

On the other hand, even if R&D is Hicks-neutral, endogeneous technology choice still introduces a bias in the direction of technical change in favor of labor, the scarce non-accumulable input. Firms decide optimally to increase the UFP of labor faster than that of capital in order to adjust to the ongoing changes in factor proportions, which are in favor of capital. This is natural given gross complementarity of both inputs.

Furthermore, under Hicks-neutral R&D, the capital income share π is bound to fall gradually toward zero, provided that the rate of capital accumulation remains positive in the long run.

Appendix C. Supplementary data

Supplementary data associated with this article can be found in the online version at (http://dx.doi.org.10.1016/j.jedc.2013.06.006).

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[21] Growiec (2008a) assumed that λ_a and λ_b were fixed and proxied technological progress by growth in N instead. This is equivalent, however, in terms of the evolution of the technology menu over time, keeping N fixed and varying λ_a and λ_b proportionately. An analogous assumption was made by Caselli and Coleman (2006) in the cross-sectional context: they allowed only N to vary across countries, but λ_a and λ_b were kept fixed.

[22] In a somewhat larger (yet, still very specific) class of models, R&D will be Hicks-neutral in the limit of $t\to\infty$. The long-run results obtained within this section hold for such models as well.

[23] Assuming furthermore that $\lim_{t\to\infty} \hat{k}(t) = \lim_{t\to\infty} \hat{y}(t) = g$, it follows that $\lim_{t\to\infty} \hat{b}(t) = (\alpha/(\alpha-\theta))g > 0$.

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