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ISOELASTIC ELASTICITY OF SUBSTITUTION PRODUCTION FUNCTIONS

Jakub Growiec and Jakub Mućk
SGH Warsaw School of Economics and Narodowy Bank Polski

We generalize the normalized constant elasticity of substitution (CES) production function by allowing the elasticity of substitution to vary isoelastically with (i) the relative factor share, (ii) the marginal rate of substitution, (iii) the capital–labor ratio, (iv) the capital share, (v) the capital's rate of return, or (vi) the capital–output ratio. Ensuing isoelastic elasticity of substitution (IEES) functions have intuitively and analytically desirable properties, for example, self-duality. Empirically, for the post-war US economy we robustly reject the CES specification in favor of the IEES alternative. Assuming the IEES production structure we find that the capital–labor elasticity of substitution has remained around 0.8–0.9 from 1948 to the 1980s, followed by a period of secular decline.

Keywords: Production Function, Factor Share, Elasticity of Substitution, Factor-Augmenting Technical Change

1. INTRODUCTION

The constant elasticity of substitution (CES) production function, first introduced to economics by Arrow et al. (1961), is a very popular framework which allows factor shares to be affected by factor endowments. But what if in reality its key parameter, the elasticity of substitution σ, is not constant after all? What if σ is not deep and structural, but rather just a function of the factor ratio K/L (k; either in raw or technology-adjusted units) or of the input–output ratio K/Y? And crucially, what if σ is systematically above unity for some configurations of factor endowments and below unity for others?

This is an important question because contemporary theoretical models of economic growth, business cycles, international trade, industrial organization, resource and energy economics, etc. frequently assume CES production. Relaxing

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Table 1. How IEES production functions generalize Cobb–Douglas and CES functions

		Cobb–Douglas	CES	IEES
Output per worker	y	Isoelastic		
Relative factor share	$\frac{\pi}{1-\pi}$	Constant	Isoelastic	
Elasticity of substitution	σ	1	Constant	Isoelastic
Elasticity of elasticity of substitution	ψ	0	0	Constant

this assumption may have far-reaching consequences. For example, in macroeconomics we know that the properties of an economy with CES production depend critically on the value of the elasticity of substitution σ. Whether the factors of production are gross complements ($\sigma < 1$) or substitutes ($\sigma > 1$) is crucial for both long-run growth perspectives and short-run fluctuations of the economy. First, above-unity elasticity of substitution can be perceived as an engine of long-run endogenous growth [Solow (1956); Jones and Manuelli (1990), and Palivos and Karagiannis (2010)]. If capital and labor are gross substitutes, then neither of them is essential for production, and thus physical capital accumulation alone can, under otherwise favorable circumstances, drive perpetual growth. Otherwise, the scarce factor limits the scope for economic development and output is bounded. Second, the magnitude of the elasticity of substitution is also vital for the immediate impact of factor accumulation (and factor-augmenting technical change) on factor shares. Under gross substitutes, accumulation of capital relative to labor increases the capital's share of output; under gross complements the opposite effect is observed. This has a bearing on the comovement of variables both at business cycle frequencies [Cantore et al. (2015)] and over the medium run. In particular, labor share declines observed across the world since the 1970s–1980s [Karabarbounis and Neiman (2014), and Piketty (2014)] can be directly explained by capital deepening or capital-augmenting technological progress under CES production only if $\sigma > 1$.[1]

Theoretical contribution. In this paper we put forward and thoroughly characterize a novel, tractable, and empirically useful class of isoelastic elasticity of substitution (IEES)[2] production functions. Our basic idea is simple. We design IEES functions so that they generalize the CES function in the same way as the CES function generalizes the Cobb–Douglas (Table 1): the Cobb–Douglas is isoelastic and implies constant factor shares, the CES function implies isoelastic factor shares and has a constant elasticity of substitution, whereas IEES functions have an isoelastic elasticity of substitution and a constant *elasticity of elasticity of substitution*. Moreover, just like both their predecessors, IEES functions are consistent with factor-augmenting technical change and exhibit globally constant returns to scale (CRS).

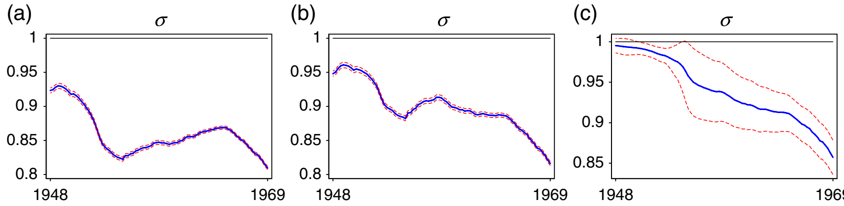
We consider six alternative variants of IEES functions by allowing the elasticity of substitution to vary isoelastically with (i) the relative factor share, (ii) the marginal rate of substitution (MRS), (iii) the capital–labor ratio, (iv) the capital share, (v) the capital's rate of return, or (vi) the capital–output ratio. Considering each of the six possibilities signifies that the IEES class is quite versatile and underscores that we remain agnostic in our choice of exact functional specification, at least in the space of two-input, CRS production functions. This ought to be an empirical question.

All our calculations have been carried out in normalized units [de La Grandville (1989) and Klump and de La Grandville (2000)]. In consequence, not only is our basic idea simple, but also our analytical results remain sharp and are not cluttered with unnecessary algebra. Thanks to production function normalization the role of each parameter of IEES functions has been precisely disentangled from all others, facilitating theoretical discussions and parameter estimation [see Klump et al. (2012)].

Empirical relevance. Being a multi-purpose tool, IEES functions can be used in many contexts. In macroeconomics, to take one important example, their empirical relevance builds on the fact that they can help reconcile the disagreement between various estimates of σ between aggregate capital and labor, which still persists in the literature. Indeed, empirical identification of this elasticity of substitution has appeared a notoriously difficult task: on the one hand, a voluminous literature exploiting time-series and cross-firm variation for the USA [Antràs (2004); Chirinko (2008); Klump et al. (2007, 2012); Young (2013); Oberfield and Raval (2014); Herrendorf et al. (2015), and Chirinko and Mallick (2017)] finds that the elasticity of substitution between capital and labor is below unity ($\sigma \approx 0.6 - 0.8$), and thus both factors of production are gross complements; on the other hand, numerous studies exploiting the cross-country variation in factor shares [Duffy and Papageorgiou (2000); Karabarbounis and Neiman (2014); Piketty (2014), and Piketty and Zucman (2014)] tend to imply gross substitutability, with $\sigma \approx 1.2$–1.3. Moreover, studies allowing for cross-country heterogeneity in σ find that it can be quite substantial [Duffy and Papageorgiou (2000); Mallick (2012), and Villacorta (2016)], with gross complementarity in some countries and gross substitutability in others.

In fact, caveats apply already when looking at the US data alone. We have estimated the elasticity of substitution σ under a normalized CES specification [following Klump et al. (2007)] in rolling windows, obtaining clear evidence of a downward trend in this parameter across the consecutive windows (Figure 1). Our results imply that while capital and labor have consistently been gross complements in the USA ($\sigma < 1$), the degree of their mutual complementarity has significantly increased.[3]

These empirical findings constitute a puzzle which can be resolved only by conditioning σ on additional variables or relaxing the assumption of its constancy. IEES production functions provide a framework in which the latter can be done in a systematic, tractable way. Our estimation of IEES functions (with three alternative estimation strategies, see Section 8) confirms that σ between capital and



Notes: σ has been estimated within a normalized supply-side system with CES production, in 45-year rolling windows. The horizontal axis marks the initial year of a given window. Dashed lines represent 95% confidence intervals. Specifications: (a) Baseline; (b) Baseline + Quality-adjusted labor input; and (c) Baseline + Quality-adjusted labor input + Exponential capital-augmenting technical change. See Section 8 for details of the empirical approach.

FIGURE 1. Rolling window estimates of the elasticity of substitution σ.

labor in the US has indeed been consistently below unity in the postwar period, first fluctuating around 0.8–0.9 until the 1980s and then embarking on a secular downward trend. As opposed to rolling window CES estimation, however, the IEES structure allows us to back out σ for each year and link its changes to the underlying shifts in factor endowments. We find that σ has been systematically positively related to the capital–labor ratio in effective units, $\bar{k}$ (i.e., after accounting for factor-augmenting technical change) as well as the capital–output ratio k/y. The null hypothesis of the CES specification is robustly rejected.

Finally, from the econometric point of view, IEES functions can be used as a tool to eliminate or at least reduce the extent of inconsistency and bias in parameter estimates, incurred under the Cobb–Douglas and CES specifications if σ is actually time varying. Indeed, function misspecification is among chief sources of bias in econometric studies based on macroeconomic data, beside measurement error and omitted variables.

Comparison with background literature. Obviously, we are not the first to generalize the CES function. Many theoretical articles, proposing various production functions with a variable elasticity of substitution (VES), were published in the late 1960s and early 1970s. Next, after a three-decade-long break, the topic re-emerged around 2000, with a much more empirical focus, fueled by the progress associated with production function normalization. Still, in our opinion, the literature has not managed so far to design a satisfactory framework for modeling endowment-specific elasticities of substitution.

IEES production functions have a few notable advantages compared to functions with a VES which have already been analyzed in the literature. First, the class of IEES functions is sufficiently general to nest some of them directly, such as the Revankar's VES (1971), the Stone–Geary production function [Geary (1950) and Stone (1954)], or the Jones–Manuelli function [Jones and Manuelli (1990)]. In contrast to Revankar's VES, most IEES functions allow σ to cross unity. This is crucial because it makes IEES functions useful in analyzing poverty traps and growth reversals: physical capital accumulation alone can become an engine of unbounded endogenous growth only if the elasticity of substitution $\sigma(k)$ exceeds unity, which in the IEES case may be true only for k sufficiently large. Second, as opposed to the empirically popular translog function [Christensen et al. (1973) and Kim (1992)], the empirically motivated VES function due to Lu (1967), or the flexible variety aggregator proposed by Kimball (1995), IEES functions are not a local approximation of an arbitrary function but are well behaved and have economically interpretable properties globally. Third, alike the Cobb–Douglas and CES function but in contrast to the translog one, IEES functions are *self-dual*: cost functions associated with IEES production functions are also of the IEES form. Fourth, as opposed to a recent idea to view the production function as an arbitrary spline of CES functions with different σ's [Antony (2010)], it implies that $\sigma(k)$ is a smooth function of k. Fifth, alike the translog function but unlike VES production functions discussed in a wave of articles around 1970 [Lu (1967); Sato and Hoffman (1968), and Kadiyala (1972)],[4] it naturally lends itself to further generalizations. For example, mirroring the extension from the Cobb–Douglas to the CES and from the CES to the IEES, the elasticity of elasticity of substitution could be made isoelastic instead of constant. One could thus eliminate one of the potential limitations of IEES functions: that $\sigma(k)$ is monotone in k.[5]

Our research is also tangent to the papers which endogenize the elasticity of substitution within various general equilibrium frameworks. First, following the lead of Miyagiwa and Papageorgiou (2007), some authors have studied growth models with two-level CES production structures [Papageorgiou and Saam (2008); Saam (2008), and Xue and Yip (2013)]. This literature implies that the aggregate elasticity of substitution is a linear combination of elasticities of substitution between capital and labor in intermediate goods sectors as well as the elasticity of substitution between intermediate goods in final goods production. In equilibrium, $\sigma(k)$ can be either monotone, hump-shaped, or U-shaped in k [Xue and Yip (2013)].[6] Second, following the lead of Jones (2005), some authors have considered frameworks with an aggregate CES production function arising from optimal technology choices at the level of firms [Growiec (2008a,b, 2013), and Matveenko and Matveenko (2015)]. In these models, the aggregate elasticity of substitution σ is different from the local one but does not depend on k in equilibrium. Third, Irmen (2011) and Leon-Ledesma and Satchi (2018) have put forward dynamic models with endogenous technology choice, demonstrating how the equilibrium value of σ can evolve over time, driven by factor accumulation and technical change. Fourth, looking from an industrial organization perspective, Zhelobodko et al. (2012) have also demonstrated how σ could be endogenously determined by market forces under monopolistic competition. In contrast to all these papers, our contribution posits that the linkage between σ and k is technological, not economic.

Structure. The remainder of the article is structured as follows. Section 2 defines IEES production functions and presents the way they can be analytically constructed. Section 3 derives the properties of three cases of IEES functions: where the elasticity of substitution is isoelastic with respect to the relative factor share, the MRS, and the factor ratio k. Section 4 complements the analysis with the *capital deepening* production function representation [Klenow and Rodriguez-Clare (1997), and Madsen (2010)]. Section 5 derives the properties of IEES functions where the elasticity of substitution is isoelastic with respect to the capital share, the capital's rate of return, and the degree of capital deepening, k/y. Section 6 demonstrates that IEES functions are self-dual. Section 7 discusses the role of factor-augmenting technical change with IEES production. Section 8 illustrates the usefulness of IEES production functions in empirical applications by applying the framework to post-war US data. Section 9 concludes. Analytical details, the description of our data set, and robustness checks of the empirical exercise are relegated to the Supplementary Appendix.

2. DEFINITIONS AND CONSTRUCTION

Let F be a CRS production function of two inputs, K and L. In its intensive form, $Y = F(K, L)$ is written as $y = f(k)$, where $y = Y/L$ and $k = K/L$. We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is three times continuously differentiable, increasing, and concave in its whole domain.[7,8] We assume that factor markets are perfectly competitive.

We carry out our analysis in *normalized units*. The reason is that production function normalization, while redundant for the Cobb–Douglas production function due to its multiplicative character, has been shown [de La Grandville (1989) and Klump and de La Grandville (2000)] to be crucial for obtaining clean identification of the role of each parameter of the CES function. We observe that this argument applies equally forcefully to the proposed class of IEES functions.

The natural objects of reference in the current study are the Cobb–Douglas and the CES production function. The normalized Cobb–Douglas function is written as:

$$y = f(k) = y_0 \left(\frac{k}{k_0}\right)^{\pi_0}, \qquad k_0, y_0 > 0, \pi_0 \in (0, 1). \tag{1}$$

The normalized CES production function is, in turn:

$$y = f(k) = y_0 \left(\pi_0 \left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}} + (1 - \pi_0)\right)^{\frac{\sigma}{\sigma-1}}, \quad k_0, y_0 > 0, \pi_0 \in (0, 1), \sigma > 0, \tag{2}$$

converging to the Cobb–Douglas function as the elasticity of substitution $\sigma \to 1$, to a linear function as $\sigma \to +\infty$, and to a Leontief (minimum) function as $\sigma \to 0_+$.

The following elementary concepts are central to our analysis:

- *Factor shares.* The partial elasticity of output Y with respect to K is defined as $\pi = \pi(k) = \frac{kf'(k)}{f(k)} \in [0, 1]$. Given that factor markets are perfectly competitive, this elasticity is also equal to the capital's share of output, $\frac{rk}{y}$. By CRS, implying that the labor share is $1 - \pi$, it is also easily obtained that the ratio of factor shares (and of partial elasticities) Π, strictly increasing in π, is equal to $\Pi = \Pi(k) = \frac{\pi(k)}{1-\pi(k)} = \frac{kf'(k)}{f(k)-kf'(k)} \geq 0$.

The Cobb–Douglas production function is characterized by constant factor shares, with $\pi(k) \equiv \pi_0$ for all $k \geq 0$. For the CES production function, the relative factor share $\Pi(k) = \frac{\pi(k)}{1-\pi(k)} = \frac{\pi_0}{1-\pi_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$ increases with k, from 0 when $k=0$ to $+\infty$ as $k \to \infty$, if $\sigma > 1$. Conversely, if $\sigma < 1$, then the ratio gradually declines, from $+\infty$ toward 0.

- *MRS*. For CRS functions of two inputs, the MRS—capturing the slope of the isoquant—is computed as $\text{MRS} = \varphi(k) = -\frac{1-\pi(k)}{\pi(k)}k = -\frac{f(k)}{f'(k)} + k \leq 0$. Given that factor markets are perfectly competitive, the MRS is also equal to minus the relative price of labor as compared to capital, $\frac{w}{r} = \frac{1-\pi(k)}{\pi(k)}k = -\varphi(k)$. Monotonicity and concavity of the production function f imply that the MRS is negative and (at least weakly) declines with k.

  The Cobb–Douglas function has a linearly declining MRS $\varphi(k) = \varphi_0\left(\frac{k}{k_0}\right)$.[9] The CES function, in turn, has an isoelastic MRS $\varphi(k) = \varphi_0\left(\frac{k}{k_0}\right)^{1/\sigma}$. In both cases, the MRS unambiguously declines from 0 when $k=0$ to $-\infty$ when $k \to \infty$.
- *Elasticity of substitution.* The elasticity of substitution—measuring the curvature of the isoquant, that is, the elasticity of changes in the factor ratio k in reaction to changes in the MRS—is computed as $\sigma = \sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} = -\frac{f'(k)(f(k)-kf'(k))}{kf(k)f''(k)} \geq 0$. Concavity of the production function f implies that the elasticity of substitution is non-negative.

  The Cobb–Douglas function implies $\sigma(k) \equiv 1$ for all $k \geq 0$. For CES functions, the elasticity of substitution $\sigma > 0$ is a constant parameter.

We also adopt the following definitions.

DEFINITION 1. *The* elasticity of elasticity of substitution with respect to *x*, $\text{EES}(x)$*, where* $x \in \left\{\Pi, \varphi, k, \pi, r, \frac{k}{y}\right\}$*, is the elasticity with which the elasticity of substitution* σ *reacts to changes in x:*

$$\text{EES}(x) = \frac{\partial \sigma}{\partial x}\frac{x}{\sigma} = \frac{\sigma'(k)}{\sigma(k)}\frac{x(k)}{x'(k)}. \quad (3)$$

DEFINITION 2. *The* isoelastic elasticity of substitution production function $\text{IEES}(x)$*, where* $x \in \left\{\Pi, \varphi, k, \pi, r, \frac{k}{y}\right\}$*, is a CRS production function* $F : \mathcal{D} \to \mathbb{R}_+$*, with a domain* $\mathcal{D} \subset \mathbb{R}^2_+$*, such that* $\text{EES}(x) \equiv$ *const for all* $(K, L) \in \mathcal{D}$*.*

Inserting the six alternative arguments into the EES formula we obtain, respectively:

$$\text{EES}(\Pi) = \frac{\pi(k)(1-\pi(k))}{\pi'(k)}\frac{\sigma'(k)}{\sigma(k)}, \quad (4)$$

$$\text{EES}(\varphi) = \frac{\varphi(k)}{\varphi'(k)}\frac{\sigma'(k)}{\sigma(k)} = k\sigma'(k), \quad (5)$$

$$\text{EES}(k) = \frac{k\sigma'(k)}{\sigma(k)}, \tag{6}$$

$$\text{EES}(\pi) = \frac{\pi(k)}{\pi'(k)} \frac{\sigma'(k)}{\sigma(k)}, \tag{7}$$

$$\text{EES}(r) = \frac{f'(k)}{f''(k)} \frac{\sigma'(k)}{\sigma(k)} = \frac{kf(k)\sigma'(k)}{kf'(k) - f(k)}, \tag{8}$$

$$\text{EES}\left(\frac{k}{y}\right) = \frac{k}{1-\pi(k)} \frac{\sigma'(k)}{\sigma(k)}. \tag{9}$$

In what follows, we characterize the six respective IEES functions.[10] Each of the considered variants is a different functional representation of the idea that the elasticity of substitution should vary systematically with factor endowments.

Please observe that for every CES or Cobb–Douglas function, due to the constancy of σ it is obtained that $\text{EES}(x) = 0$ for all x, and thus they naturally belong to the wider IEES class as well. Another observation is that the EES is a *third-order characteristic* of any function f: existence of $\sigma'(k)$ for all k requires that f is at least three times differentiable in its domain. Standard axioms of production functions do not place any sign restrictions on $f^{(3)}(k)$ and thus on EES, a degree of freedom that we shall exploit.

Construction. The construction of a function f whose elasticity of substitution $\sigma(k)$ is of a given form will be obtained in two steps: in the first step, $\sigma(k)$ is integrated up to yield the MRS $\varphi(k)$; in the second step $\varphi(k)$ is integrated up to yield the function $f(k)$ itself.[11] Formally,

$$\sigma(k) = \frac{\varphi(k)}{k\varphi'(k)} \Rightarrow \varphi(k) = -\exp\left(\int \frac{dk}{k\sigma(k)}\right), \tag{10}$$

$$\varphi(k) = -\frac{f(k)}{f'(k)} + k \Rightarrow f(k) = \exp\left(\int \frac{dk}{k - \varphi(k)}\right). \tag{11}$$

Both constants of integration have to be picked specifically to maintain production function normalization. For IEES production functions, integration (10) can be executed analytically, yielding closed, economically interpretable formulas for the MRS as a function of k. In contrast, integration (11) generally cannot be performed in elementary functions—but for a few notable exceptions, some of which have already been discussed in the literature.

3. PROPERTIES OF IEES FUNCTIONS

The logic behind the construction of first three IEES functions is best followed by looking at the identity:

$$\frac{\pi}{1-\pi} = \frac{r}{w}\frac{K}{L} \Rightarrow \qquad \Pi = -\frac{1}{\varphi} \cdot k. \tag{12}$$

The identity follows directly from firms' first-order conditions of optimization under perfect competition. For each IEES function we posit that the elasticity of substitution is isoelastic with the respective element in this identity.

3.1. The IEES(Π) Function

The IEES(Π) production function, defined as a function for which EES(Π) $= \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{1-\pi} \frac{1-\pi_0}{\pi_0} \right)^{\psi}. \tag{13}$$

In this case, integration (10) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 \left(\frac{1}{\sigma_0} \left(\frac{k}{k_0} \right)^{-\psi} + \left(1 - \frac{1}{\sigma_0} \right) \right)^{-\frac{1}{\psi}}, \tag{14}$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0} \right) k_0$.

Hence, the relative factor share satisfies

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0} \left(\frac{1}{\sigma_0} + \left(1 - \frac{1}{\sigma_0} \right) \left(\frac{k}{k_0} \right)^{\psi} \right)^{\frac{1}{\psi}}. \tag{15}$$

The functional form in equation (15) reveals that the IEES(Π) function is an equally natural generalization of the CES as the CES is a generalization of the Cobb–Douglas (isoelastic) production function. For the CES function, the MRS and the relative factor share are Cobb–Douglas (isoelastic) functions of k and the elasticity of substitution is constant. For the IEES(Π) function, the MRS and the relative factor share are CES functions of k and the elasticity of substitution is Cobb–Douglas (isoelastic) in the relative factor share.

Inserting equation (15) back into equation (13), we find that the elasticity of substitution is the following function of k:

$$\sigma(k) = 1 + (\sigma_0 - 1) \left(\frac{k}{k_0} \right)^{\psi}, \tag{16}$$

and hence $\sigma(k) > 1$ for all k if $\sigma_0 > 1$, irrespective of the value of ψ, and conversely, $\sigma(k) < 1$ for all k if $\sigma_0 < 1$. Hence, in the IEES(Π) case the factors of production are either always gross substitutes or always gross complements. Due to the strict monotonicity of the relative factor share with respect to k [equation (15)], the elasticity of substitution $\sigma(k)$ cannot cross unity. Moreover, the case $\sigma_0 = 1$ automatically reduces the IEES(Π) function directly to the Cobb–Douglas specification.

A more detailed discussion of properties of IEES(Π) functions is relegated to Appendix A.1 of the Supplementary Material, but we shall now discuss the special cases with $\psi = \pm 1$ for which integration (11) yields known closed-form formulas.[12] The case $\psi = 1$ corresponds to the "VES" production function due to Revankar (1971), whereas the case $\psi = -1$ captures the Stone–Geary production function.

Revankar's VES production function. Assuming that $\psi = 1$, following Revankar (1971), allows us to find the antiderivative in equation (11) in elementary functions. The normalized "variable elasticity of substitution" (Revankar's VES) production function with CRS reads

$$y = f(k) = y_0 \left(\frac{k}{k_0}\right)^{\frac{\pi_0}{\pi_0+\sigma_0(1-\pi_0)}} \left(\pi_0 \left(\frac{\sigma_0 - 1}{\sigma_0}\right)\left(\frac{k}{k_0}\right) + \frac{\pi_0 + \sigma_0(1-\pi_0)}{\sigma_0}\right)^{\frac{\sigma_0(1-\pi_0)}{\pi_0+\sigma_0(1-\pi_0)}}, \tag{17}$$

or in non-normalized notation, $f(k) = Ak^{\alpha}(Bk + 1)^{1-\alpha}$, with $\alpha \in (0, 1), A > 0$, and $B \in \mathbb{R}$. Please observe the domain restriction $k \le -1/B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

We note that while most production functions derived around 1970, which typically do not belong to the class of IEES functions, have remained something of a theoretical curiosity, the Revankar's VES function has been sometimes used in empirical studies, such as Karagiannis et al. (2005). However, its applicability is limited by the fact that it does not offer the type of curvature which is requested by the data: our estimates of IEES(Π) functions for US data (see Section 8) locate ψ far away from unity and rather between -5 and -8.

Stone–Geary production function. Assuming that $\psi = -1$ also allows us to find the antiderivative in equation (11) in elementary functions. The normalized Stone–Geary production function (i.e., Cobb–Douglas production function of a shifted input) is

$$y = f(k) = y_0 \left(\left(\frac{k}{k_0}\right)\left(\frac{\sigma_0\pi_0 + (1-\pi_0)}{\sigma_0}\right) + (1-\pi_0)\frac{\sigma_0 - 1}{\sigma_0}\right)^{\frac{\sigma_0\pi_0}{\sigma_0\pi_0+(1-\pi_0)}}, \tag{18}$$

or in non-normalized notation, $f(k) = A(k + B)^{\alpha}$, with $\alpha \in (0, 1), A > 0$, and $B \in \mathbb{R}$. Please observe the domain restriction $k \ge -B$ which is in force if $B < 0$ (i.e., $\sigma_0 < 1$).

3.2. The IEES(MRS) Function

The IEES(MRS) production function, defined as a function for which EES(φ) $= \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varphi}{\varphi_0}\right)^{\psi}, \tag{19}$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right) k_0$.

In this case, integration (10) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 \left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{\frac{1}{\psi}}. \quad (20)$$

Hence, the relative factor share satisfies

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\frac{k}{k_0}\left(1 + \frac{\psi}{\sigma_0} \ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{\psi}}. \quad (21)$$

Inspection of the above formulas reveals that the MRS is a logarithmic function of k. The relative factor share is, on the other hand, a product of a logarithmic and a linear function of k. As opposed to the cases of the Cobb–Douglas, CES, and IEES(Π) functions, relative factor shares are no longer a monotonic function of k. There exists a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0 e^{-\frac{\sigma_0 - 1}{\psi}}$ with $\sigma(\tilde{k}) = 1$.

Inserting equation (21) back into equation (19), we find that the elasticity of substitution is the following function of k:

$$\sigma(k) = \sigma_0 + \psi \ln\left(\frac{k}{k_0}\right). \quad (22)$$

A more detailed discussion of properties of IEES(MRS) functions is relegated to Appendix A.1 of the Supplementary Material.

3.3. The IEES(k) Function

The IEES(k) production function, defined as a function for which EES(k) $= \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{k}{k_0}\right)^{\psi}. \quad (23)$$

In this case, integration (10) yields the following formula for the MRS:

$$\varphi(k) = \varphi_0 e^{\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}, \quad (24)$$

where $\varphi_0 = -\left(\frac{1-\pi_0}{\pi_0}\right) k_0$.

Hence, the relative factor share satisfies

$$\frac{\pi}{1-\pi} = \frac{\pi_0}{1-\pi_0}\frac{k}{k_0} e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}. \quad (25)$$

Inspection of the above formulas reveals that the MRS is an exponential function of k. The relative factor share is, on the other hand, a product of an exponential and a linear function of k. As opposed to the cases of the Cobb–Douglas, CES, and IEES(Π) functions, and alike the IEES(MRS) function, the relative factor share is a non-monotonic function of k. There is a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{k} = k_0 \sigma_0^{-1/\psi}$ with $\sigma(\tilde{k}) = 1$.

A more detailed discussion of properties of IEES(k) functions is relegated to Appendix A.1 of the Supplementary Material.

4. THE CAPITAL DEEPENING PRODUCTION FUNCTION REPRESENTATION

In order to define further types of IEES functions, we have to rewrite the aggregate production function so that it takes the capital–output ratio $\kappa \equiv K/Y = k/y$ instead of k as its input. Increases in κ are then identified with *capital deepening*. This transformation is frequently used in the growth and development accounting literature [see, e.g., Klenow and Rodriguez-Clare (1997) and Madsen (2010)] because unlike k, the capital–output ratio κ typically does not exhibit a strong upward trend in the data, and dealing with variables without discernible trends has its documented statistical advantages.[13]

To maintain a useful point of reference, we first rewrite the normalized Cobb–Douglas and CES functions with CRS in the capital deepening form:

$$y = y_0 \left(\frac{\kappa}{\kappa_0}\right)^{\frac{\pi_0}{1-\pi_0}}, \qquad \kappa_0, y_0 > 0, \pi_0 \in (0,1), \tag{26}$$

$$y = y_0 \left(\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma}{\sigma-1}}, \ \kappa_0, y_0 > 0, \pi_0 \in (0,1), \sigma > 0. \tag{27}$$

The implied relative factor shares Π are, respectively, equal to $\Pi = \frac{\pi_0}{1-\pi_0}$ (a constant) in the Cobb–Douglas case, and

$$\Pi = \Pi(\kappa) = \frac{\frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}}{\frac{1}{1-\pi_0} - \frac{\pi_0}{1-\pi_0}\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}} \qquad \Rightarrow \qquad \pi = \pi(\kappa) = \pi_0 \left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}} \tag{28}$$

in the CES case. Hence, in the latter case the capital share is isoelastic in the capital–output ratio κ and increases with κ if and only if $\sigma > 1$, that is, if the factors are gross substitutes. Finally, observe that the functional form of equation (28) does not by itself preclude cases with $\pi(\kappa) > 1$. These cases are made impossible only by the range of the CES function which restricts the support of $\kappa = k/y$ appropriately.

More generally, any function $F(K,L)$ satisfying our assumptions can be rewritten in terms of κ. Let us now recall some known relevant results.

Existence. First, any increasing, concave, CRS production function of two inputs, $Y = F(K,L)$, can be rewritten as $F\left(\frac{K}{Y}, \frac{L}{Y}\right) = 1$. Then, by the implicit function theorem,[14] there exists a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\frac{L}{Y} = \frac{1}{h(K/Y)}$ and thus $y = h(\kappa)$. Note that due to the concavity of F, the capital–output ratio κ is always increasing in k. At this point it can be observed that the relative factor share can be computed directly as the elasticity of $h(\kappa)$ with respect to κ:

$$\Pi = \Pi(\kappa) = \frac{\pi(\kappa)}{1-\pi(\kappa)} = \frac{h'(\kappa)\kappa}{h(\kappa)}. \tag{29}$$

The existence of an *explicit* form of the function $h(\kappa)$, however, hinges on the requirement that $F(\kappa, 1/y) = 1$ can be solved for y explicitly, which need not be the case even if the functional form of F is given. Notably, it cannot be done for IEES functions whose explicit form is not known.[15]

Construction. Using this notation, the proposed two-step method for finding functions whose elasticity of substitution is given as a predefined function of the capital–output ratio κ is as follows:

$$\sigma(\kappa) = \frac{1}{1-\frac{\pi'(\kappa)\kappa}{\pi(\kappa)}} = \Rightarrow \pi(\kappa) = \exp\left(\int \frac{\sigma(\kappa)-1}{\kappa\sigma(\kappa)} d\kappa\right), \tag{30}$$

$$\Pi(\kappa) = \frac{h'(\kappa)\kappa}{h(\kappa)} \Rightarrow h(\kappa) = \exp\left(\int \frac{\Pi(\kappa)}{\kappa} d\kappa\right). \tag{31}$$

For IEES production functions, integration (30) can be executed analytically, yielding closed, economically interpretable formulas for the capital share as a function of κ. In contrast, the integral (31) can be computed in elementary functions only for a very narrow set of functional specifications of $\sigma(\kappa)$. This apparatus enables us to define and characterize the IEES(π), IEES(r), and IEES(κ) production functions.

5. PROPERTIES OF FURTHER IEES PRODUCTION FUNCTIONS

The logic behind the construction of the next three IEES functions is best followed by looking at the identity:

$$\pi = r \cdot \frac{K}{Y} \Rightarrow \pi = r \cdot \kappa. \tag{32}$$

In each case we posit that the elasticity of substitution is isoelastic with the respective element in this identity. We also note that as $1-\pi = w \cdot \frac{L}{Y}$, one could also symmetrically define IEES($1-\pi$), IEES(w), and IEES(L/Y) production functions. By perfect competition and CRS, however, they would have fully analogous properties to the ones derived below.

5.1. The IEES(π) Function

The IEES(π) production function, defined as a function for which EES(π) = ψ, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\pi}{\pi_0}\right)^{\psi}. \tag{33}$$

In this case, solving equation (30) as a Bernoulli equation yields the following formula for the capital share as a function of the capital–output ratio κ:

$$\frac{\pi}{\pi_0} = \left(\frac{1}{\sigma_0} + \left(1 - \frac{1}{\sigma_0}\right) \left(\frac{\kappa}{\kappa_0}\right)^{\psi} \right)^{\frac{1}{\psi}}. \tag{34}$$

Owing to the similarity of equation (34) with equation (15), the IEES(π) function shares many properties with the IEES(Π) one. Crucially, inserting equation (34) back into equation (33), we find that the elasticity of substitution is the following function of κ:

$$\sigma(\kappa) = 1 + (\sigma_0 - 1)\left(\frac{\kappa}{\kappa_0}\right)^{\psi}, \tag{35}$$

and hence $\sigma(\kappa) > 1$ for all κ if $\sigma_0 > 1$, irrespective of the value of ψ, and conversely, $\sigma(\kappa) < 1$ for all κ if $\sigma_0 < 1$. Hence, both factors are either always gross substitutes or always gross complements. Due to the strict monotonicity of the capital share with respect to κ [equation (33)], the elasticity of substitution $\sigma(\kappa)$ cannot cross unity. Moreover, the case $\sigma_0 = 1$ automatically reduces the IEES(π) function directly to the Cobb–Douglas specification.

A more detailed discussion of properties of IEES(π) functions is relegated to Appendix A.1 of the Supplementary Material. Here, we shall only discuss the special cases with $\psi = \pm 1$ for which integration (31) yields known closed-form formulas. The case $\psi = 1$ corresponds to the production function due to Jones and Manuelli (1990), whereas the case $\psi = -1$ captures the capital deepening Stone–Geary production function.

Jones–Manuelli production function. Assuming that $\psi = 1$ allows us to find the antiderivative in equation (31) in elementary functions. The capital deepening representation of the resulting production function reads

$$y = h(\kappa) = y_0 \left(\frac{\kappa}{\kappa_0}\right)^{\frac{\pi_0}{\sigma_0 - \pi_0}} \left(\frac{\pi_0}{1-\pi_0}\frac{1-\sigma_0}{\sigma_0}\left(\frac{\kappa}{\kappa_0}\right) + \frac{\sigma_0 - \pi_0}{\sigma_0(1-\pi_0)} \right)^{\frac{-\sigma_0}{\sigma_0 - \pi_0}}. \tag{36}$$

Inserting $\frac{\kappa}{\kappa_0} = \frac{k}{k_0}\frac{y_0}{y}$ and solving for $y = f(k)$ yields

$$y = f(k) = y_0 \left(\frac{(\sigma_0 - 1)\pi_0}{\sigma_0 - \pi_0}\left(\frac{k}{k_0}\right) + \frac{\sigma_0(1-\pi_0)}{\sigma_0 - \pi_0}\left(\frac{k}{k_0}\right)^{\frac{\pi_0}{\sigma_0}} \right), \tag{37}$$

or in non-normalized notation, $f(k) = Ak + Bk^{\alpha}$, with either $\alpha \in (0,1), A > 0, B > 0$ if $\sigma_0 > 1$, or $\alpha \in (0,1), A < 0, B > 0$ if $1 > \sigma_0 > \pi_0$, or $\alpha > 1, A > 0, B < 0$ if $\sigma_0 < \pi_0$. Please observe the domain restrictions if $A < 0$ or $B < 0$ (which are in force if $\sigma_0 < 1$). In all cases, however, the function f is increasing and concave in its domain.[16]

Observe that in the case where $\sigma_0 > 1$, this is precisely the production function considered by Jones and Manuelli (1990), that is, a sum of a linear (AK) and a Cobb–Douglas function.

Capital deepening Stone–Geary production function. Assuming that $\psi = -1$ also allows us to find the antiderivative in equation (31) in elementary functions. The capital deepening representation of the resultant production function takes the form of a Stone–Geary function (i.e., Cobb–Douglas production function of a shifted input):

$$y = h(\kappa) = y_0 \left(\frac{1-\sigma_0\pi_0}{\sigma_0(1-\pi_0)} \left(\frac{\kappa}{\kappa_0} \right) + \frac{\sigma_0 - 1}{\sigma_0(1-\pi_0)} \right)^{\frac{\sigma_0\pi_0}{1-\sigma_0\pi_0}}, \qquad (38)$$

or in non-normalized notation, $h(\kappa) = (A\kappa + B)^{\alpha}$, with either $\alpha < 0, A < 0, B > 0$ if $\sigma_0 > \frac{1}{\pi_0} > 1$, or $\alpha > 0, A > 0, B > 0$ if $\frac{1}{\pi_0} > \sigma_0 > 1$, or $\alpha > 0, A > 0, B < 0$ if $\sigma_0 < 1$. Please observe the domain restrictions if $A < 0$ or $B < 0$ (which are in force if $\sigma_0 > \frac{1}{\pi_0}$ or if $\sigma_0 < 1$). Unfortunately, this function cannot be solved explicitly as $y = f(k)$.[17]

5.2. The IEES(r) Function

The IEES(r) production function is defined as a function for which EES(r) $= -\psi$, where $\psi \in \mathbb{R}$ is a constant. The minus is optional; we include it to maintain symmetry with the IEES(MRS) function, with which the current one shares many properties. Under this specification, the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{r}{r_0} \right)^{-\psi} = \left(\frac{\kappa}{\kappa_0} \frac{\pi_0}{\pi} \right)^{\psi}, \qquad (39)$$

where $r_0 = \frac{\pi_0}{\kappa_0}$.

Integration (30) yields the following formula for the capital share:

$$\frac{\pi}{\pi_0} = \frac{\kappa}{\kappa_0} \left(1 + \frac{\psi}{\sigma_0} \ln \left(\frac{\kappa}{\kappa_0} \right) \right)^{-\frac{1}{\psi}}. \qquad (40)$$

Thus, factor shares are not a monotonic function of k or κ. There exists a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{\kappa} = \kappa_0 e^{-\frac{\sigma_0 - 1}{\psi}}$ with $\sigma(\tilde{\kappa}) = 1$.

Inserting equation (40) back into equation (39), we find that the elasticity of substitution is the following function of κ:

$$\sigma(\kappa) = \sigma_0 + \psi \ln \left(\frac{\kappa}{\kappa_0} \right). \qquad (41)$$

A more detailed discussion of properties of IEES(r) functions is relegated to Appendix A.1 of the Supplementary Material.

5.3. The IEES(k/y) Function

The IEES(κ) production function, defined as a function for which EES(κ) $= \psi$, where $\psi \in \mathbb{R}$ is a constant, implies (upon normalization) that the elasticity of substitution follows:

$$\frac{\sigma}{\sigma_0} = \left(\frac{\kappa}{\kappa_0}\right)^{\psi} = \left(\frac{k}{y}\frac{y_0}{k_0}\right)^{\psi}. \qquad (42)$$

In this case, integration (30) yields the following formula for the capital share:

$$\frac{\pi}{\pi_0} = \left(\frac{\kappa}{\kappa_0}\right) e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}. \qquad (43)$$

As opposed to the cases of the Cobb–Douglas, CES, IEES(Π), and IEES(π) functions, and alike the IEES(MRS), IEES(k), and IEES(r) functions, relative factor shares are a non-monotonic function of κ here (and thus, owing to the concavity of $F(K, L)$, of k as well). There exists a unique point of reversal, coinciding with the point where the elasticity of substitution crosses unity, $\tilde{\kappa} = \kappa_0\sigma_0^{-1/\psi}$ with $\sigma(\tilde{\kappa}) = 1$.

A more detailed discussion of properties of IEES(κ) functions is relegated to Appendix A.1 of the Supplementary Material.

5.4. Summary of Analytical Results

A summary of key analytical results obtained in the current paper is provided in Table 2.

6. SELF-DUALITY

Having discussed the main properties of IEES functions, let us now demonstrate one of their important theoretical advantages, their *self-duality*: we find that cost functions associated with IEES production functions are also of the IEES form. Self-duality is, in turn, a desirable property of production functions because it facilitates estimation of their parameters based on empirical data. Parameter estimates can then also be recovered from the dual cost function representation.[18] Moreover, while self-duality also characterizes Cobb–Douglas and CES functions [see Sato (1981)], it is crucial to observe that this property is not shared by other specifications often used in the literature, such as the translog function [Chambers (1988)].

To derive dual cost functions associated with IEES production functions under perfect competition, we apply Shephard's lemma [Shephard (1953)] and use CRS of the total cost function $TC(r, w)$ to write down the relative factor share Π as a function of the ratio of capital to labor remuneration $\eta = \frac{r}{w}$ only. The details of the

TABLE 2. Summary of analytical results

Main results

	$\sigma=1$ C–D	$\frac{\sigma}{\sigma_0}=1$ CES	$\frac{\sigma}{\sigma_0}=\left(\frac{\pi}{1-\pi}\frac{1-\pi_0}{\pi_0}\right)^{\psi}$ IEES(Π)	$\frac{\sigma}{\sigma_0}=\left(\frac{\varphi(k)}{\varphi_0}\right)^{\psi}$ IEES(MRS)	$\frac{\sigma}{\sigma_0}=\left(\frac{k}{k_0}\right)^{\psi}$ IEES(k)
$\Pi(k)=\frac{\pi(k)}{1-\pi(k)}$	$\frac{\pi_0}{1-\pi_0}$	$\frac{\pi_0}{1-\pi_0}\left(\frac{k}{k_0}\right)^{\frac{\sigma-1}{\sigma}}$	$\frac{\pi_0}{1-\pi_0}\left(\frac{1}{\sigma_0}+\left(1-\frac{1}{\sigma_0}\right)\left(\frac{k}{k_0}\right)^{\psi}\right)^{\frac{1}{\psi}}$	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}\left(1+\frac{\psi}{\sigma_0}\ln\left(\frac{k}{k_0}\right)\right)^{-\frac{1}{\psi}}$	$\frac{\pi_0}{1-\pi_0}\frac{k}{k_0}e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{k}{k_0}\right)^{-\psi}\right)}$
(relative factor share)	Constant	$\sigma>1$: increasing $\sigma<1$: decreasing	$\sigma_0>1$: increasing $\sigma_0<1$: decreasing	$\psi>0$: ∪-shaped $\psi<0$: ∩-shaped	$\psi>0$: ∪-shaped $\psi<0$: ∩-shaped
$\sigma(k)$	1	σ_0	$1+(\sigma_0-1)\left(\frac{k}{k_0}\right)^{\psi}$	$\sigma_0+\psi\ln\left(\frac{k}{k_0}\right)$	$\sigma_0\left(\frac{k}{k_0}\right)^{\psi}$
(elasticity of substitution)	Constant	Constant	$\sigma<1(>1)\Leftrightarrow\sigma_0<1(>1)$ $\psi(\sigma_0-1)>0$: increasing $\psi(\sigma_0-1)<0$: decreasing	$\sigma=1$ if $\frac{k}{k_0}=e^{-\frac{\sigma_0-1}{\psi}}$ $\psi>0$: increasing $\psi<0$: decreasing	$\sigma=1$ if $\frac{k}{k_0}=\sigma_0^{-\frac{1}{\psi}}$ $\psi>0$: increasing $\psi<0$: decreasing

Results for the capital deepening representation (with $\kappa=k/y$)

	$\sigma=1$ C–D	$\frac{\sigma}{\sigma_0}=1$ CES	$\frac{\sigma}{\sigma_0}=\left(\frac{\pi}{\pi_0}\right)^{\psi}$ IEES(π)	$\frac{\sigma}{\sigma_0}=\left(\frac{r}{r_0}\right)^{-\psi}$ IEES(r)	$\frac{\sigma}{\sigma_0}=\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$ IEES(κ)
$\pi(\kappa)$	π_0	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)^{\frac{\sigma-1}{\sigma}}$	$\pi_0\left(\frac{1}{\sigma_0}+\left(1-\frac{1}{\sigma_0}\right)\left(\frac{\kappa}{\kappa_0}\right)^{\psi}\right)^{\frac{1}{\psi}}$	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)\left(1+\frac{\psi}{\sigma_0}\ln\left(\frac{\kappa}{\kappa_0}\right)\right)^{-\frac{1}{\psi}}$	$\pi_0\left(\frac{\kappa}{\kappa_0}\right)e^{-\frac{1}{\psi\sigma_0}\left(1-\left(\frac{\kappa}{\kappa_0}\right)^{-\psi}\right)}$
(capital share)	Constant	$\sigma>1$: increasing $\sigma<1$: decreasing	$\sigma_0>1$: increasing $\sigma_0<1$: decreasing	$\psi>0$: ∪-shaped $\psi<0$: ∩-shaped	$\psi>0$: ∪-shaped $\psi<0$: ∩-shaped
$\sigma(\kappa)$	1	σ_0	$1+(\sigma_0-1)\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$	$\sigma_0+\psi\ln\left(\frac{\kappa}{\kappa_0}\right)$	$\sigma_0\left(\frac{\kappa}{\kappa_0}\right)^{\psi}$
(elasticity of substitution)	Constant	Constant	$\sigma<1(>1)\Leftrightarrow\sigma_0<1(>1)$ $\psi(\sigma_0-1)>0$: increasing $\psi(\sigma_0-1)<0$: decreasing	$\sigma=1$ if $\frac{\kappa}{\kappa_0}=e^{-\frac{\sigma_0-1}{\psi}}$ $\psi>0$: increasing $\psi<0$: decreasing	$\sigma=1$ if $\frac{\kappa}{\kappa_0}=\sigma_0^{-\frac{1}{\psi}}$ $\psi>0$: increasing $\psi<0$: decreasing

TABLE 3. Overview of primal and dual IEES function representations

	Primal		Dual
Cobb–Douglas	$\text{CD}(\pi_0)$	—	$\text{CD}(\pi_0)$
CES	$\text{CES}(\pi_0, \sigma)$	—	$\text{CES}\left(\pi_0, \frac{1}{\sigma}\right)$
	$\text{IEES}(\Pi)(\pi_0, \sigma_0, \psi)$	—	$\text{IEES}(\Pi)\left(\pi_0, \frac{1}{\sigma_0}, -\psi\right)$
	$\text{IEES}(\text{MRS})(\pi_0, \sigma_0, \psi)$	╳	$\text{IEES}(k)\left(\pi_0, \frac{1}{\sigma_0}, \psi\right)$
IEES	$\text{IEES}(k)(\pi_0, \sigma_0, \psi)$		$\text{IEES}(\text{MRS})\left(\pi_0, \frac{1}{\sigma_0}, \psi\right)$
	$\text{IEES}(\pi)(\pi_0, \sigma_0, \psi)$	—	$\text{IEES}(\pi)\left(\pi_0, \frac{1}{\sigma_0}, -\psi\right)$
	$\text{IEES}(r)(\pi_0, \sigma_0, \psi)$	╳	$\text{IEES}(k/y)\left(\pi_0, \frac{1}{\sigma_0}, \psi\right)$
	$\text{IEES}(k/y)(\pi_0, \sigma_0, \psi)$		$\text{IEES}(r)\left(\pi_0, \frac{1}{\sigma_0}, \psi\right)$

derivation have been relegated to Appendix A.2 of the Supplementary Material. The key results are, however, summarized in Table 3 signifying that:

(i) the IEES(Π) function is dual to itself, with the baseline elasticity of substitution of the dual cost function being the reciprocal of σ_0 from the primal production function, and the elasticity of elasticity of substitution of the cost function being equal to $-\psi$, where ψ is the elasticity of elasticity of substitution of the primal production function;
(ii) the IEES(MRS) and IEES(k) functions are dual to one another, with equal ψ's and reciprocal σ_0's;
(iii) the IEES(π) function is dual to itself, with the baseline elasticity of substitution of the dual cost function being the reciprocal of σ_0 from the primal production function, and the elasticity of elasticity of substitution of the cost function being equal to $-\psi$, where ψ is the elasticity of elasticity of substitution of the primal production function; and
(iv) the IEES(r) and IEES(k/y) functions are dual to one another, with equal ψ's and reciprocal σ_0's.

As a side remark, we observe that thanks to production function normalization, the role of each parameter is precisely delineated and there is a very transparent relationship between the primal and dual function parameterization.

7. FACTOR-AUGMENTING TECHNICAL CHANGE

A big advantage of IEES functions is that they are useful for empirical applications. Their usefulness follows from the fact that they provide testable predictions for the functional relationships between two observables: the relative factor share and the factor ratio k (or capital–output ratio, κ). Each of the nonlinear equations (15), (21), (25), (34), (40), and (43) can be estimated, either separately or in a larger system, based on country-level, sectoral-level, or even firm-level data.

However, a typical caveat is that before any production function could be meaningfully taken to the data, it ought to be augmented with factor-augmenting technical change. This point is particularly important when dealing with aggregate data (such as the data we employ in Section 8). Fortunately, such form of technical change can be straightforwardly incorporated in any CRS production function by replacing $Y = F(K, L)$ with $Y = F(\Gamma^{\mathcal{K}} K, \Gamma^{\mathcal{L}} L)$, or—in the intensive form—by replacing $y = f(k)$ with $\bar{y} = f(\bar{k})$, where $\bar{y} = \frac{Y}{\Gamma^{\mathcal{L}} L}$ and $\bar{k} = \frac{\Gamma^{\mathcal{K}} K}{\Gamma^{\mathcal{L}} L}$. Crucially, owing to CRS, the functional form of f remains unchanged. And if one is ultimately interested in y instead of $\bar{y}$, then one may simply compute $y = \Gamma^{\mathcal{L}} \bar{y} = \Gamma^{\mathcal{L}} f(\bar{k}) = F(\Gamma^{\mathcal{K}} k, \Gamma^{\mathcal{L}})$ after all the necessary derivations.

This last step implicitly separates the Hicks-neutral component of technical change from the *capital bias* in technical change [cf., e.g., Leon-Ledesma et al. (2010)]. This is the key insight for the current study because it allows us to define the capital share $\pi(\bar{k})$, the MRS $\varphi(\bar{k})$ and, crucially, the elasticity of substitution $\sigma(\bar{k})$, as a function of the capital–labor ratio *in effective units*. Hence, any capital-biased technical change (i.e., increase in $\Gamma^{\mathcal{K}} / \Gamma^{\mathcal{L}}$) acts just like physical capital accumulation, whereas labor-biased technical change (decline in $\Gamma^{\mathcal{K}} / \Gamma^{\mathcal{L}}$) affects factor shares, MRS and σ alike a decline in the capital–labor ratio k. All functional forms remain unchanged.

Factor-augmenting technical change can be studied in the capital deepening production function representation as well. With the notation $\bar{\kappa} = \frac{\bar{k}}{\bar{y}} = \frac{\Gamma^{\mathcal{K}} k}{y}$, one can easily replace $y = h(\kappa)$ with $\bar{y} = h(\bar{\kappa})$. Again, all functional forms remain unchanged. At the same time, using this notation underscores that capital-augmenting technical change adds to capital deepening just like capital accumulation, whereas labor-augmenting technical change is neutral for capital deepening.

Clearly, both theory and data suggest that, at least at the macroeconomic scale, labor-augmenting technical change is likely to be dominant over the long run [Acemoglu (2003) and Klump et al. (2012)], and therefore, the capital–labor ratio in effective units $\bar{k}$ will likely grow slower (if at all) than the raw capital–labor ratio k. Indeed, the US data include, apart from periods of growth, also prolonged periods of decline in $\bar{k}$ (see Figure A.1 of the Supplementary Appendix). Hence, for empirical applications of IEES functions (and CES ones as well), it is important whether one considers the capital–labor ratio in effective units ($\bar{k}$) or just as a raw variable, measured in dollars per worker (k).

8. APPLICATION: THE US AGGREGATE PRODUCTION FUNCTION

In this section we consider an empirical application of the proposed class of IEES production functions to the aggregate US data. Our data set covers the non-residential business sector of the US economy [see Rognlie (2015), for discussion] and consists of quarterly time series spanning the period 1948q1–2013q4. All variables are normalized around their respective geometric sample means. Detailed description of the data set and construction of variables is included in Appendix A.3 in the Supplementary Material.

In our estimations of IEES functions, we will determine simultaneously the magnitude of elasticity of elasticity of substitution ψ and the (geometric) average elasticity of substitution in the sample, σ_0. As for the latter parameter, elasticity of substitution σ estimated under the CES specification will work as a natural benchmark for comparisons.

Having estimated the parameters of IEES functions, we shall study the implied trajectory of the elasticity of substitution between capital and labor in the USA, σ_t. Thus far, the magnitude of this deep technological parameter has been consistently estimated for the USA only when assuming its constancy over time [see Klump et al. (2012), for a review]. In contrast, our empirical investigation reveals that σ_t has exhibited substantial variation over time, and since the 1980s it has also systematically followed a downward trend.

We shall also provide a framework for discriminating empirically between the CES production function specification and the six alternative IEES variants. First, as each IEES function nests the CES as its special case with $\psi = 0$, we test the null hypothesis of CES against the alternative of IEES. To this end we perform parameter significance tests ($\mathcal{H}_0 : \psi = 0$ against $\mathcal{H}_1 : \psi \neq 0$) as well as likelihood ratio tests. We find that the CES specification is systematically rejected. Second, as there is no nesting structure among the alternative IEES functions, we discriminate between them using goodness-of-fit characteristics such as the Bayesian information criterion (BIC). Our results slightly favor the IEES(k/y) specification; however, differences in goodness of fit are small and sensitive to the choice of estimation strategy and treatment of technical change. One cannot, in fact, confidently claim that any of the six specification is *strongly* or *robustly* preferred over others for post-war US macro data.

8.1. Estimation Strategies

The parameters of IEES functions will be estimated under the baseline assumption that technological progress is exponential and purely labor augmenting.[19] Hence, we shall assume that $\Gamma_t^{\mathcal{K}} \equiv 1$ and $\Gamma_t^{\mathcal{L}} = e^{\gamma_l(t-\bar{t})}$ where γ_l is the constant and exogenous rate of labor-augmenting technical change. The capital–labor ratio in effective units is then equal to

$$\bar{k}_t = \frac{\Gamma_t^{\mathcal{K}} K_t}{\Gamma_t^{\mathcal{L}} L_t} = \frac{K_t}{L_t} e^{-\gamma_l(t-\bar{t})}. \quad (44)$$

The parameter $\bar{t}$ is set such that the sample average of $\ln \bar{k}_t$ is zero. It is also observed that under purely labor-augmenting technical change, $\bar{\kappa}_t = \kappa_t$.

We consider three alternative estimation strategies.

Single-equation nonlinear least squares estimation. This strategy consists in estimating the parameters of each of the nonlinear equations (15), (21), (25), (34), (40), and (43) with nonlinear least squares (NLS), after taking logs. The pace of labor-augmenting technical change, γ_l, is set at 0.0045 per quarter (about 0.018 per annum)[20] and not estimated.

The advantage of single-equation NLS estimation is that it is simple to execute and requires data solely on the relative factor share, $\Pi_t = \frac{\pi_t}{1-\pi_t}$, and the capital–labor ratio expressed in effective units, $\bar{k}_t$, or alternatively the capital–output ratio, κ_t. The problem with this estimation method is, however, that the identification of the estimated parameters—the average elasticity of substitution σ_0 and the elasticity of elasticity of substitution ψ—based on such a scarce data set is hard because they are deep technological constants. This argument is analogous to the one put forward in the CES literature which has identified the advantages of using normalized supply-side system estimation over the single-equation approach in estimating the (constant) elasticity of substitution based on time-series data [Klump et al. (2007) and Leon-Ledesma et al. (2010)]. Hence, we shall also seek to estimate σ_0 and ψ jointly with π_0 and γ_l in a three-equation system, using additional data on output and relative prices.

Two-step estimation. This strategy consists in, first, estimating the parameters of a CES production function with factor-augmenting technical change $(\pi_0, \sigma_0, \gamma_l)$ following the three-equation system strategy due to Klump et al. (2007) and, next, assuming that the elasticity of substitution $\sigma(k)$ is given by an IEES specification and thus estimating ψ.

In the first step, the normalized supply-side system with CES production is jointly estimated:

$$\ln\left(\frac{r_t K_t}{P_t Y_t}\right) = \ln(\pi_0) + \frac{1-\sigma_0}{\sigma_0}\left(\ln\left(\frac{Y_t}{K_t}\frac{K_0}{Y_0}\right) - \ln(\xi) - \ln\left(\Gamma_t^{\mathcal{K}}\right)\right), \tag{45}$$

$$\ln\left(\frac{w_t L_t}{P_t Y_t}\right) = \ln(1-\pi_0) + \frac{1-\sigma_0}{\sigma_0}\left(\ln\left(\frac{Y_t}{L_t}\frac{L_0}{Y_0}\right) - \ln(\xi) - \ln\left(\Gamma_t^{\mathcal{L}}\right)\right), \tag{46}$$

$$\ln\left(\frac{Y_t}{Y_0}\right) = \ln(\xi) + \frac{\sigma_0}{\sigma_0 - 1}\ln\left(\pi_0\left(\frac{K_t}{K_0}\Gamma_t^{\mathcal{K}}\right)^{\frac{\sigma_0-1}{\sigma_0}} + (1-\pi_0)\left(\frac{L_t}{L_0}\Gamma_t^{\mathcal{L}}\right)^{\frac{\sigma_0-1}{\sigma_0}}\right), \tag{47}$$

where $\frac{r_t K_t}{P_t Y_t} = \pi_t$ and $\frac{w_t L_t}{P_t Y_t} = 1 - \pi_t$ stand for the capital and labor share, respectively.[21] The first two equations are first-order conditions of profit maximization under perfect competition,[22] for capital (45) and labor (46). The third equation captures the log of a CES production function.[23] Residuals are allowed to be correlated across equations, and therefore, we use a generalized nonlinear least squares (GNLS) estimator.[24]

In the second step, the estimate of σ_0 and the parameters describing deterministic factor-augmenting technical progress, obtained in the previous stage, are taken as given. It allows us to estimate the second deep parameter ψ with NLS, based on equations (15), (21), (25), (34), (40), and (43), with much more precision.

The advantages of two-step estimation are that (i) the properties of the first step have been thoroughly characterized in the CES literature, and that (ii) the second step is less demanding of data than in the single-equation approach. On the other hand, the disadvantage of this approach is that it inconsistently assumes the production function to be CES in the first step and IEES in the second step. This may lead to systematic errors in the case of a substantial discrepancy between both specifications (i.e., if ψ is far away from zero). Therefore, we also seek to estimate ψ jointly with π_0, σ_0, and γ_l in a single system.

Joint estimation of the supply-side system with IEES production. This strategy consists in estimating all parameters of the supply-side system (45)–(47) jointly with ψ while allowing the elasticity of substitution to be time varying. Thus σ_0 in equations (45)–(47) is replaced with σ_t, which follows the considered IEES production function:

$$\text{IEES}(\Pi): \qquad \sigma_t = 1 + (\sigma_0 - 1)\left(\frac{\bar{k}_t}{\bar{k}_0}\right)^{\psi}, \tag{48}$$

$$\text{IEES(MRS)}: \qquad \sigma_t = \sigma_0 + \psi \ln\left(\frac{\bar{k}_t}{\bar{k}_0}\right), \tag{49}$$

$$\text{IEES}(k): \qquad \sigma_t = \sigma_0 \left(\frac{\bar{k}_t}{\bar{k}_0}\right)^{\psi}, \tag{50}$$

$$\text{IEES}(\pi): \qquad \sigma_t = 1 + (\sigma_0 - 1)\left(\frac{\kappa_t}{\kappa_0}\right)^{\psi}, \tag{51}$$

$$\text{IEES}(r): \qquad \sigma_t = \sigma_0 + \psi \ln\left(\frac{\kappa_t}{\kappa_0}\right), \tag{52}$$

$$\text{IEES}(\kappa): \qquad \sigma_t = \sigma_0 \left(\frac{\kappa_t}{\kappa_0}\right)^{\psi}. \tag{53}$$

There are two key advantages of joint estimation of all parameters of this system: it is consistent with the theoretical formulation of IEES production functions, and it uses a richer data set than the single-equation NLS approach. On the other hand, it must be emphasized that after substituting any of the equations (48)–(53) into the systems (45)–(47), a highly nonlinear system of equations is obtained. In particular, note, in the first three IEES specifications, factor-augmenting technical change is embodied in the formula for the elasticity of substitution. Empirical identification of parameters of such a complex system is generally possible but sometimes challenging.[25]

8.2. Estimation Results: The IEES(k/y) Function

The discussion of our empirical results is organized as follows. First, in the current subsection we present and discuss estimation results for the IEES(k/y) production function specification, which we view as mildly preferred. Next, in Subsection 8.3 we motivate this choice of specification based on goodness-of-fit characteristics. We also argue, however, that the key empirical findings are in fact similar

TABLE 4. Summary of baseline estimates of IEES (k/y) production function

	Single-equation			System approach	
		IEES (k/y)			
	CES	NLS	Two-step	CES	IEES (k/y)
π_0	0.326***	0.324***	0.324***	0.331***	0.326***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.720***	0.745***	0.757	0.757***	0.795***
	(0.026)	(0.029)	(not est.)	(0.001)	(0.014)
ψ		2.676***	2.867***		1.032***
		(0.726)	(0.588)		(0.243)
ξ				0.999***	1.003***
				(0.002)	(0.002)
γ_l	0.004	0.004	0.004	0.004***	0.004***
	(not est.)	(not est.)	(not est.)	(0.000)	(0.000)
$\mathcal{H}_0 : \mathcal{CES}$		[0.001]	[0.000]		[0.000]
implied σ_t		↘	↘		↘
ADF$_K$	−2.427**	−2.688***	−3.004***	−2.715***	−2.903***
ADF$_L$				−3.646***	−3.601***
ADF$_Y$				−2.494**	−2.49**

Notes: ***, **, and * denote rejection of the null about parameters' insignificance at the 1%, 5%, and 10% significance levels, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. The ***, **, and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5%, and 10% significance levels. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively. In the single-equation approach, CES estimates are based on the FOC for the (logged) capital share equation (45).

across all six IEES function variants, at least under system estimation. Finally, in Subsection 8.4 we discuss additional robustness checks, regarding in particular the treatment of technical change.

The choice of the IEES(k/y) specification as our baseline is motivated by three facts. First, this specification exhibits relatively good fit to the data (see Table 6 and Table A.9 of the Supplementary Appendix), under each estimation strategy. Second, estimates obtained under this specification are relatively robust to the choice of estimation strategy and the treatment of technical change, in particular unlike the IEES(Π), IEES(MRS), and IEES(k) variants they consistently imply a downward tendency in the elasticity of substitution, σ_t. Third, it has the theoretical advantage of allowing the elasticity of substitution to cross unity within the sample, unlike IEES(Π) and IEES(π). Yet, it must be understood that differences in goodness of fit among the six IEES variants are small, often inconclusive, and sensitive to other modeling choices.

Results of estimation of the IEES(k/y) function are presented in Table 4. From left to right, we present single-equation NLS estimates, two-step estimates, and results of joint estimation of the supply-side system with IEES production. CES estimates are included in the first and fourth columns for comparison. The results for two-step estimation of the CES function are not provided because in such a case, the first step corresponds exactly to the system approach (presented in the fourth column) whereas the second step is void.

These numbers deliver three key messages. First, capital and labor are, on average, gross complements. The estimated value of σ_0 is comprised between 0.7 and 0.8, fully corroborating the CES results summarized by Klump et al. (2012). Second, the elasticity of substitution is *not* constant over time. The estimates of ψ are statistically significantly different from zero at the 1% significance level. Likelihood ratio tests confirm that the null hypothesis of a CES production function specification is confidently rejected. Third, the relationship between the elasticity of substitution $\sigma(\kappa)$ and the capital–output ratio κ is positive, $\psi > 0$: the greater is the capital–output ratio, the higher the elasticity of substitution σ. Hence, a clear downward trend in κ recorded over the last decades (negative capital deepening, see Figure A.1 of the Supplementary Appendix) translates to an implied secular decline in the elasticity of substitution in the USA since 1980s. This result has been indicated in the row "implied σ_t."

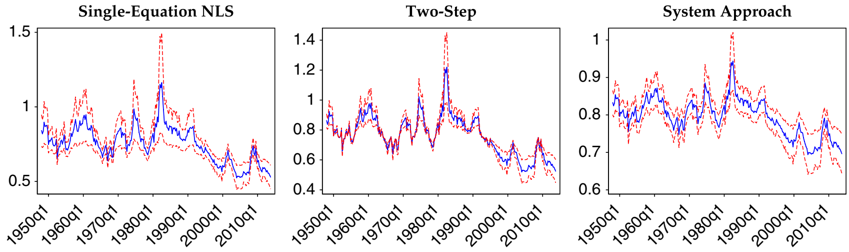
More specifically, our point estimate of the elasticity of substitution based on the normalized supply-side system with CES production equals 0.757 (fourth column) whereas under IEES production the estimated value of σ_0 (average elasticity of substitution) is only slightly higher, equalling 0.795 (fifth column). Reassuringly, both numbers are close to the literature consensus of $\sigma_0 \approx 0.6-0.7$ and allow to confidently reject the null of $\sigma_0 = 1$ (the Cobb–Douglas form). Our estimate of the pace of labor-augmenting technical change, $\gamma_l = 0.0045$ per quarter (0.018 per annum), is equally well aligned with the literature.

ADF statistics indicate that residuals from all estimated equations are stationary at all conventional significance levels. It is particularly important because, on the other hand, we do not find evidence for stationarity of relative factor shares.[26] Viewed from the cointegration perspective, this ensures that there is no problem of spurious regression.

Plugging the actual time series of the capital–output ratio κ_t (see Figure A.1 of the Supplementary Appendix) into equation (53) allows us to infer the exact time path of σ_t under the IEES(k/y) specification. Three estimates of this path, along with 95% confidence intervals computed with the delta method, are presented in Figure 2. We find that the elasticity of substitution between capital and labor in the United States has exhibited substantial variability over time. The time pattern of σ_t is such that first it fluctuated around a constant mean from 1948 to the early 1980s, then it briefly spiked in 1982, after which it entered a period of secular decline, bottoming from 2004 onwards.

8.3. Comparison of IEES Function Estimates

Let us now compare the above IEES(k/y) specification against the alternative forms. Table 5 demonstrates that key empirical observations following from Table 4 are remarkably robust to the choice of IEES production function



Notes: Dashed lines represent 95% confidence intervals computed with the delta method. In the case of two-step estimates (middle panels), the assessment of variance of estimates may be downward biased because the delta method has been applied to the second step only, taking the estimates from the first step as fixed numbers

FIGURE 2. Implied time paths of the elasticity of substitution σ_t, IEES(k/y) specification.

specification. The top panel summarizes single-equation NLS estimates, the middle one—two-step estimates, and the bottom one—results of joint estimation of the supply-side system with IEES production. Consecutive columns pertain to the respective variants of IEES functions. CES estimates are included in the first column for comparison.[27]

We find that capital and labor are always gross complements: the estimated value of σ_0 ranges from 0.590 to 0.866, depending on the production function specification and estimation method. Second, the CES is always rejected: across all IEES specifications, the estimates of ψ are statistically significantly different from zero at least at the 10% significance level (and typically also at the 1% level).[28] In likelihood ratio tests, the null hypothesis of a CES production function is also rejected against the alternative of IEES in all its variants at least at the 10% significance level. Third, apart from three single-equation NLS estimates of ψ which are likely biased (see below), we find that the relationship between the elasticity of substitution $\sigma(\bar{k})$ and the effective capital–labor ratio $\bar{k}$ as well as the capital–output ratio κ is consistently positive, that is, $\psi > 0$ in the cases of IEES(MRS), IEES(k), IEES(r), and IEES(κ), and $(\sigma_0 - 1)\psi > 0$ with $\sigma_0 < 1$ and $\psi < 0$ in the cases of IEES(Π) and IEES(π). In other words, we find that the more capital is accumulated per worker (in effective units, $\bar{k}$) or the greater is the capital–output ratio κ, the higher is the elasticity of substitution σ. This translates to an implied secular decline in the elasticity of substitution in the United States, where both $\bar{k}$ and κ have recorded clear downward trends over the last decades (see Figure A.1 of the Supplementary Appendix). This has been indicated in table rows "implied σ_t." ADF statistics indicate that residuals from all estimated equations are stationary at all conventional significance levels.

Let us now clarify why we believe our single-equation NLS estimates are likely less reliable than the other ones and probably biased. The reason is that simultaneous identification of two deep parameters of the production function, σ_0 and ψ, based on a single, highly nonlinear equation and two time series only (the

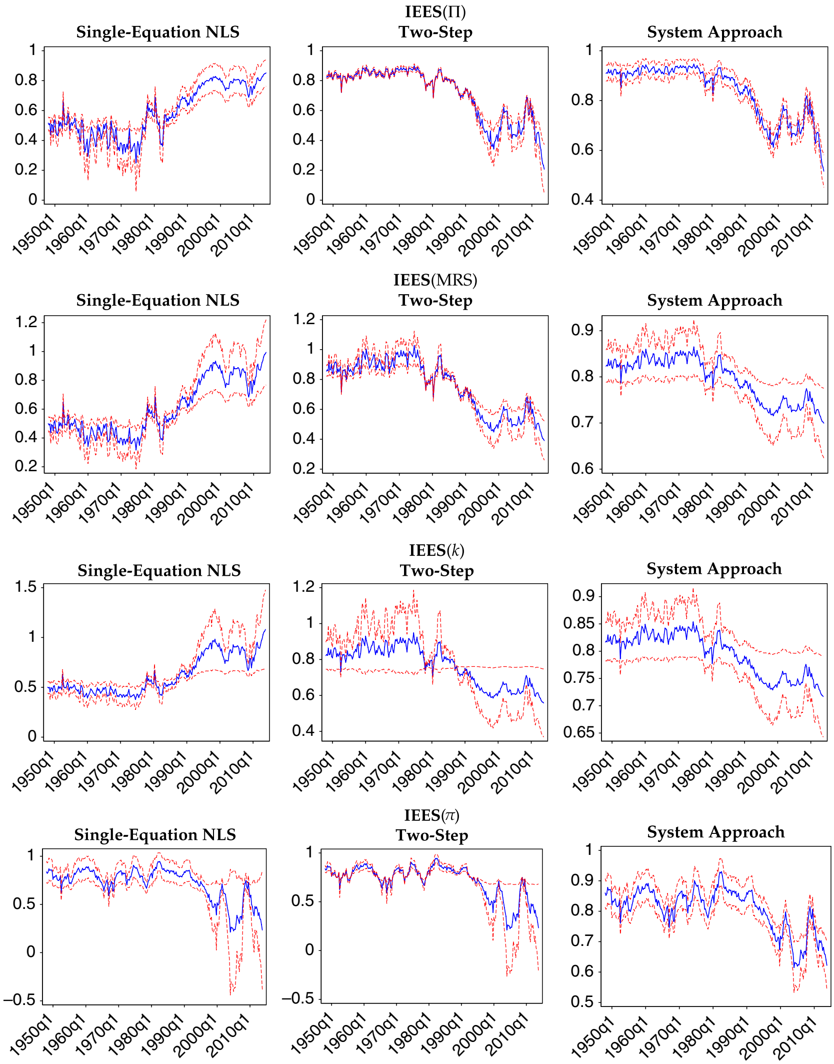
TABLE 5. Summary of baseline estimates of IEES production functions

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
				Single-equation NLS ($\gamma_l = 0.0045$)			
π_0	0.327***	0.331***	0.331***	0.331***	0.363***	0.324***	0.324***
	(0.001)	(0.001)	(0.001)	(0.002)	(0.02)	(0.001)	(0.001)
σ_0	0.613***	0.623***	0.605***	0.59***	0.757***	0.751***	0.745***
	(0.019)	(0.027)	(0.021)	(0.019)	(0.041)	(0.034)	(0.029)
ψ		6.064***	−2.535***	−3.936***	−8.958**	1.879**	2.676***
		(1.586)	(0.671)	(1.201)	(4.225)	(0.744)	(0.726)
Implied σ_t		↗	↗	↗	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.001]	[0.034]	[0.012]	[0.001]
ADF	−3.37***	−3.738***	−3.697***	−3.619***	−2.648***	−2.674***	−2.688***
				Two-step ($\sigma_0 = 0.757$ and $\gamma_l = 0.0045$)			
π_0		0.324***	0.324***	0.325***	0.363***	0.324***	0.324***
		(0.001)	(0.001)	(0.001)	(0.011)	(0.001)	(0.001)
ψ		−7.687***	2.381***	1.985*	−8.986***	1.96***	2.867***
		(0.688)	(0.434)	(1.127)	(2.341)	(0.547)	(0.588)
Implied σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.000]	[0.078]	[0.000]	[0.000]	[0.000]
ADF		−3.562***	−3.418***	−3.292***	−2.965***	−2.99***	−3.004***

TABLE 5. Continued

	CES	IEES(Π)	IEES(MRS)	IEES($\bar{k}$)	IEES(π)	IEES(r)	IEES(κ)
				System approach			
π_0	0.331***	0.324***	0.326***	0.326***	0.326***	0.326***	0.326***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
σ_0	0.757***	0.866***	0.796***	0.793***	0.819***	0.798***	0.795***
	(0.001)	(0.018)	(0.014)	(0.015)	(0.022)	(0.019)	(0.014)
ξ	0.999***	1.002***	1.001***	1.001***	1.004***	1.003***	1.003***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
γ_l	0.004***	0.005***	0.004***	0.004***	0.004***	0.004***	0.004***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ψ		−7.956***	0.608***	0.644**	−5.737***	0.89***	1.032***
		(1.048)	(0.228)	(0.301)	(1.349)	(0.279)	(0.243)
Implied σ_t		↘	↘	↘	↘	↘	↘
$\mathcal{H}_0 : \mathcal{CES}$		[0.000]	[0.008]	[0.033]	[0.000]	[0.001]	[0.000]
ADF_K	−2.715***	−3.132***	−2.856***	−2.83***	−2.968***	−2.916***	−2.903***
ADF_L	−3.646***	−3.578***	−3.519***	−3.52***	−3.672***	−3.613***	−3.601***
ADF_Y	−2.494**	−2.47**	−2.487**	−2.489**	−2.474**	−2.488**	−2.49**

Notes: ***, **, and * denote rejection of the null about parameters' insignificance at the 1%, 5%, and 10% significance level, respectively. In the case of σ_0, the null hypothesis is that $\sigma_0 = 1$ (Cobb–Douglas production). ADF stands for the Augmented Dickey-Fuller test without a constant term. ***, **, and * in the ADF test denote rejection of the null about a unit root of the respective residuals at the 1%, 5%, and 10% significance level. The number of lags for the ADF test has been determined by the BIC criterion. The numbers in round and squared parentheses denote robust standard errors and probability values, respectively. In the single-equation approach, CES estimates are based on the FOC for the (logged) relative factor share equation (see Table 2).



Notes: Dashed lines represent 95% confidence intervals computed with the delta method. In the case of two-step estimates (middle panels), the assessment of variance of estimates may be downward biased because the delta method has been applied to the second step only, taking the estimates from the first step as fixed numbers. Observe that σ_t is a nonlinear function of ψ, as in equations (48)–(53). This implies that (i) around the normalization point, $\bar{k}_t \approx \bar{k}_0$ and $\kappa_t \approx \kappa_0$, $\sigma_t \approx \sigma_0$ regardless of the value of ψ. Then the upper and lower bounds converge to the point estimate of σ_t; (ii) in some cases the p-value of the ψ parameter estimate is only slightly below 0.05. Then at the bound of the 95% confidence interval, $\psi \approx 0$ and thus $\sigma_t \approx \sigma_0$ implying that the given bound is almost flat (even if the point estimate of σ_t and the other bound is not).

FIGURE 3. Implied time paths of the elasticity of substitution σ_t under five alternative IEES specifications.

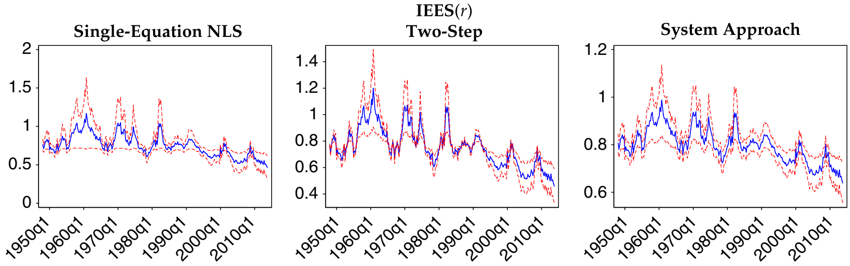


FIGURE 3. (Continued).

capital–labor ratio in effective units and the relative factor share) is very demanding of the data [Leon-Ledesma et al. (2010)]. It is likely that the puzzling result of an opposite sign of ψ estimates in the case of IEES(Π), IEES(MRS), and IEES(k) functions is driven by short-run correlation between $\bar{k}_t$ and Π_t, and thus captures cyclical co-movement rather than the underlying production technology. This conclusion is further strengthened by our robustness checks which tend to agree with our two-step and system estimates. In particular, the signs of ψ estimates under the single-equation strategy are reversed once we use a quality-adjusted labor input.

Plugging the actual time series of the effective capital–labor ratio $\bar{k}_t$ and the capital–output ratio κ_t into equations (48)–(53), we infer the exact time path of σ_t under each IEES specification; see Figure 3. We find that the overall time pattern of σ_t is robust across various IEES function specifications and estimation strategies, except under single-equation estimation of IEES(Π), IEES(MRS), and IEES(k) which is likely biased. First σ_t remained relatively stable, at about $0.8-0.9$, from 1948 to the 1980s, after which it entered a period of secular decline.[29] In its core, this robust finding mirrors the decline in the effective capital–labor ratio and negative capital deepening in the US economy since the 1980s (Figure A.1 of the Supplementary Appendix).

As the six IEES function specifications are not nested, we cannot discriminate among them by parametric hypothesis testing. Therefore, we make our choice based on goodness-of-fit statistics, presented in Table 6. In the first line, we compare the BIC under the single-equation NLS estimation strategy. In the second line, the same is done for the two-step estimation strategy. In the third line, pertaining to system estimation, we extract the residual sum of squares (SSE) from the production function equation and present it relative to the CES benchmark.[30] We refrain from providing the BIC in this case because the system estimation strategy deals with three estimating equations with correlated residuals and our focus is on the goodness of fit of the production function only. Finally, the comparison is carried out separately for the group of IEES(Π), IEES(MRS), and IEES($\bar{k}$) functions, and for the group of functions with a capital deepening representation, IEES(π), IEES(r), and IEES(k/y). This is because the former group is estimated based on data on the relative factor share and the effective capital–labor ratio $\bar{k}$,

TABLE 6. Summary of goodness-of-fit measures

	CES	IEES (Π)	IEES (MRS)	IEES (k)
$\mathcal{BIC}^1$	−659.126	**−664.400**	−663.235	−661.752
$\mathcal{BIC}^2$	−659.126	−640.478	−634.156	**−631.612**
$\text{SSE}^i/\text{SSE}^{\mathcal{CES}}$	1.000	0.991	**0.835**	0.990
	CES	IEES (π)	IEES (r)	IEES (k/y)
$\mathcal{BIC}^1$	−814.720	−815.989	−816.548	**−816.821**
$\mathcal{BIC}^2$	−814.720	−821.565	−822.096	**−822.256**
$\text{SSE}^i/\text{SSE}^{\mathcal{CES}}$	1.000	0.990	**0.835**	0.989

Notes: $\mathcal{BIC}^1$ is the Schwarz criterion for the single-equation NLS estimates, $\mathcal{BIC}^2$ denotes the Schwarz criterion for the two-step estimates, while $\text{SSE}^i/\text{SSE}^{\mathcal{CES}}$ is the relative (to the CES estimates) residual sum of squares (SSE) for the production function equation in system estimation. Lowest values indicated in bold.

whereas in the latter group the capital–output ratio κ is used instead. Naturally, goodness-of-fit statistics are also different for the CES function itself, depending on whether it is estimated in the capital deepening form or not.

Beyond the fact that the CES specification is systematically outperformed by IEES functions, our key finding here is that the differences between alternative IEES specifications are small and often inconclusive. In the first group of functions, there is no clear winner, whereas in the second group the IEES(k/y) specification wins in two cases out of three. Therefore, we view this form as mildly preferred. It should be also remarked that in the case of system estimation, the apparent large advantage of IEES(MRS) and IEES(r) specifications is counterbalanced by markedly worse fit of the other two estimating equations, so that the differences in goodness of fit for the whole three-equation system are again miniscule.

Extensive robustness analysis of the above results is provided in Table A.9 of the Supplementary Appendix, which features goodness-of-fit measures under a range of alternative assumptions regarding technical change. They confirm that (i) the differences between IEES specifications are small and often inconclusive, and but nevertheless (ii) the IEES(k/y) specification is most frequently preferred (even if only slightly).

8.4. Robustness Checks

As a robustness check of the previous empirical results, we have used additional data as well as modified our assumptions on technical progress. Details have been relegated to Appendix A.5 of the Supplementary Material.

First, we have considered an alternative measure of the labor input. Indeed, total employment (employees plus the self-employed) or aggregate hours might in fact be a poor proxy of the actual flow of labor services in the economy because they ignore the ongoing changes in labor composition, or "quality."

Therefore, quality-adjusted aggregate hours due to Fernald (2012) have been used as our measure of the labor input in this robustness check (see Appendix A.3 of the Supplementary Material for a detailed description). Table A.2 of the Supplementary Appendix confirms that all our previous empirical findings are robust to this change. Interestingly, signs of all estimated parameters are now consistent across all estimation strategies (the inconsistency in the single-equation NLS case disappears), whereas the implied trajectory of σ_t (Figure A.10 of the Supplementary Appendix) consistently features a substantial decline in the elasticity of substitution since the early 1980s.

Second, we have relaxed the restriction of exponential labor-augmenting technical change by introducing a more general and flexible Box–Cox technology term [following Klump et al. (2007)]. Table A.3 of the Supplementary Appendix confirms the robustness of our baseline estimates to this change. Puzzlingly, the estimated curvature parameter λ_l is slightly (but statically significantly) above unity, suggesting that labor-augmenting technical change has been *accelerating* throughout the sample. Nevertheless, the estimated parameter ψ has the same sign as in the baseline setting and is statistically significant in all specifications. Apart from the single-equation NLS estimation strategy, this exercise also replicates the secular decline in the elasticity of substitution since the 1980s.

Third, we have combined the above two scenarios, allowing both for (i) quality-adjusted labor input and (ii) Box–Cox labor-augmenting technical progress. Table A.4 of the Supplementary Appendix summarizes the estimates for this case. The estimated curvature parameter of labor-augmenting technical change λ_l is now (statistically significantly) below unity, in line with the earlier CES-based results of Klump et al. (2007). Thus we identify a slight decreasing tendency in the rates of labor-augmenting technical change. More flexibility in the specification of technological progress does not affect our other findings, though. Signs of all estimated parameters remain consistent across all estimation strategies, whereas σ_t, irrespectively of estimation strategy and the IEES function specification, has been roughly constant until the late 1980s and displayed a substantial decline afterwards.

Fourth, we have considered the possibility of a structural break in the pace of labor-augmenting technical change (Table A.5 of the Supplementary Appendix).[31] It is estimated that the break occurred in 2003, when the pace of technical change increased. We have then also coupled this extension with the inclusion of a quality-adjusted labor input (Table A.6 of the Supplementary Appendix). In the latter case, the break appears in 1964 and represents a slowdown in the pace of labor-augmenting technical change. In both cases, however, we strongly confirm the robustness of our baseline estimates. Signs of all estimated parameters are consistent across all estimation strategies, and the implied trajectory of σ_t consistently features a decline in the elasticity of substitution since the early 1980s.

Fifth, we have also allowed technical change to be simultaneously labor- and capital-augmenting (Table A.7 of the Supplementary Appendix). We have then coupled this extension with the inclusion of a quality-adjusted labor input (Table A.8 of the Supplementary Appendix). Although the signs of key estimated coefficients are generally preserved in this series of robustness checks, the ensuing results are somewhat less convincing than the previous ones. Crucially, our estimates of the pace of capital-augmenting technical change are consistently negative. This outcome has a bearing on other findings as well: σ_0 is now found to be visibly closer to unity than in the baseline case (though still in the range of gross complementarity) and the elasticity of elasticity of substitution ψ is now closer to zero and often statistically insignificant. We still find that σ_t has been declining over time, but the predicted paths of σ_t are now less consistent across the six IEES specifications and no longer indicate a qualitative change in the behavior of σ_t after 1980. This is likely because when we agnostically fit a highly nonlinear model featuring both labor- and capital-augmenting technical change to the data, the observed declines in the rate of real investment rate and the capital–output ratio after 1980 (and even more strongly so after the world economic crisis) can potentially be (mis)interpreted as technological regress. Alternatively, however, this result could also be explained by the ongoing *routinization* of production [Acemoglu and Restrepo (2018)] or shifts in the composition of industries [Elsby et al. (2013) and Oberfield and Raval (2014)]. We observe that the puzzling result of negative capital-augmenting technical change appears strongest in the system approach which is relatively most demanding of the data (technological progress terms are nested in the $\sigma(\bar{k})$ and $\sigma(\bar{\kappa})$ formulas) and is likely further amplified by the fact that our estimation strategy does not allow for markups. Addressing these issues in more detail is left for further research.

9. CONCLUSION

In the current paper, we have constructed a novel class of normalized IEES production functions and analyzed its properties. Our analytical results are summarized in Table 2, expanding upon Table 1 provided in the Introduction. We have also shown that IEES functions are self-dual and demonstrated their empirical usefulness. Our empirical results for the aggregate production function in the post-war US economy imply that the elasticity of substitution σ between capital and labor has been systematically positively related to the capital–labor ratio in effective (technology-adjusted) units, $\bar{k}$, as well as to the capital–output ratio κ. We have also observed that σ has been consistently below unity, first fluctuating around 0.8–0.9 until the 1980s and then embarking on a secular downward trend.

The scope for further applications of IEES functions is very broad. First, when understood as macroeconomic production functions with capital and labor, they could improve our understanding of the dynamic behavior of factor shares over time as well as their dispersion across countries, regions, sectors, and firms. They could also turn out useful in growth and levels accounting.

Second, when applied to the substitution possibilities across other pairs of inputs, they could be helpful in analyzing the problems of essentiality of exhaustible resources, skill-biased technical change, capital-skill complementarity, aggregation of intermediate goods, aggregation of domestic and imported goods, preferences over consumption goods with varying degrees of complementarity, and so on.

Third, from the theoretical point of view, IEES functions may become a useful tool for analyzing long-run growth, poverty traps, medium-run swings, and short-run fluctuations in economic activity. In particular, allowing for an endogenous shift between production factors being gross complements and gross substitutes can substantially change long-run predictions of known growth models. We know that physical capital accumulation alone can become an engine of unbounded endogenous growth only if the elasticity of substitution exceeds unity, $\sigma > 1$ [Solow (1956); Jones and Manuelli (1990), and Palivos and Karagiannis (2010)]. Because IEES(MRS), IEES(k), IEES(r), and IEES(κ) functions with $\psi > 0$ imply that $\sigma(k) > 1$ if and only if k is sufficiently large, therefore, they may become a useful tool not only in the modeling of endogenous growth, but also poverty traps and multiple equilibria in growth performance. Endogenous shifts between production factors being gross complements and gross substitutes could also enrich our understanding of short- and medium-run responses of factor shares to technological shocks.

SUPPLEMENTARY MATERIAL

To view supplementary material for this article, please visit https://dx.doi.org/10.1017/S1365100518000950.

NOTES

1. The same caveat applies when dealing with other pairs of inputs. CES functions have been applied to the issue of substitutability between exhaustible natural resources and accumulable physical capital [Dasgupta and Heal (1979) and Bretschger and Smulders (2012)] or human capital [or quality-adjusted labor, see e.g., Smulders and de Nooij (2003)]. No surprise that the central question in this literature is whether these two inputs are gross complements ($\sigma < 1$) or substitutes ($\sigma > 1$), and thus if exhaustible resources are essential for production. CES functions have also been applied to the question of substitution possibilities between skilled and unskilled labor [e.g., Caselli and Coleman (2006)] as well as capital-skill complementarity [Krusell et al. (2000) and Duffy et al. (2004)]. Whether σ is above or below unity determines whether capital accumulation and factor-augmenting technical change increase or decrease the relative demand for skilled versus unskilled workers. The magnitude of σ is also important for issues such as the substitutability among consumption goods in an agent's utility function, between intermediate goods in the production of a final good, in the aggregation of domestic and imported goods by an open economy, etc. See also Chirinko (2008) and Leon-Ledesma and Satchi (2018) on the distinction between the short-run and the long-run elasticity of factor substitution.

2. Best pronounced as "yes." Abbreviation designed to avoid confusion with the intertemporal elasticity of substitution (IES).

3. The results follow from estimating the system of equations (45)–(47) in rolling windows. They are robust across various specifications of technical change as well as across various window lengths. It must be kept in mind that rolling window estimation requires sequential re-normalization of units. For detailed results of our rolling window CES estimations, please consult Appendix A.4 of the

Supplementary Material. We also note that in contrast to our results, Cantore et al. (2017) found σ to increase across the consecutive 30-year windows of the post-war US data, from about 0.2 to about 0.7. Their result, however, concentrates on the short-run variation in the data instead of long-run trends as it is based on estimations of a Real Business Cycle model with CES production and a non-smooth, autoregressive specification of factor-augmenting technical change.

4. See Mishra (2010) for a review of the history of production functions.

5. Another issue which ought to be addressed in the future is how to generalize IEES functions into higher dimensions. Notable early contributions heading in this direction have been due to Gorman (1965) and Hanoch (1971). The task is, however, plagued by the fact that the elasticity of substitution is not a unique concept for functions of more than two inputs [Blackorby and Russell (1989)].

6. Complementarily to these findings, Oberfield and Raval (2014) have shown empirically that in the USA, σ declines with higher levels of aggregation, that is, when moving from industry-level data to sector-level data and then to aggregate production. They explain the observed decline in the US labor share since the 1970s–1980s with changes in the way how intermediate and industry inputs are combined to produce industry or sectoral output. Hence, their study also implies that the relationship between aggregate σ and k may be a consequence of ongoing changes in the micro structure of the economy.

7. Allowing K and L to be expressed in effective, technology-adjusted units is relegated to Section 7. At this point, it suffices to mention that all our results remain unchanged.

8. Note also that whenever we call K "capital" and L "labor," this is only to focus the discussion and keep it close to the associated literature. However, the theory can be readily applied to any other pair of inputs as well.

9. Using the notation $\varphi_0 = -\frac{1-\pi_0}{\pi_0} k_0 < 0$.

10. The latter three cases require also the *capital deepening* production function representation, that is, rewriting $y = f(k)$ in the form of $y = h(k/y)$.

11. Solving it in a single step is also theoretically possible but requires solving a second-order nonlinear differential equation.

12. Symbolic integration reveals that closed-form formulas (albeit huge and generally difficult to interpret) exist also for $\psi = \pm 2, \pm\frac{1}{2}$. They are available from the authors upon request.

13. Indeed, relative stability of the capital–output ratio (one of the "great ratios" in macroeconomics) has been long taken as a stylized fact, together with the relative stability of factor shares. Only quite recently have both postulates been questioned; still, if both y and k exhibit upward trends, by definition k/y *must* be at least growing much slower than k (if at all), underscoring the empirical value of the current representation.

14. Which can be used because F is increasing and concave in its entire domain.

15. It can be done for the special cases of Revankar's VES and Stone–Geary production function, though details are available upon request.

16. Under the special parameterization $\sigma_0 = \pi_0$, solving equation (31) yields the production function $y = h(\kappa) = y_0 \left(\frac{\kappa_0}{\kappa}\right) e^{\left(\frac{\kappa_0}{\kappa} - 1\right)\frac{1}{\sigma_0 - 1}}$, or equivalently, $y = f(k) = y_0 \left(\frac{k}{k_0}\right)\left(1 + (\sigma_0 - 1)\ln\frac{k}{k_0}\right)$.

17. Under the special parameterization $\sigma_0 = \frac{1}{\pi_0}$, solving equation (31) yields the production function $y = h(\kappa) = y_0 e^{\left(\frac{\kappa}{\kappa_0} - 1\right)\frac{1}{\sigma_0 - 1}}$. This function cannot be solved explicitly as $y = f(k)$, either.

18. Without self-duality, there is no direct correspondence between the parameters of the production and cost functions which share the same analytical form.

19. This assumption is consistent with a bulk of empirical literature [see the review by Klump et al. (2012)]. Moreover, extensive robustness analysis confirms that all our key results remain in force also when this assumption is relaxed. See the extensive Supplemental Appendix.

20. This number corresponds to our estimates of the US supply-side system which will be discussed shortly.

21. The additional scaling parameter ξ is expected to be around unity, cf. Klump et al. (2007), and will not play any role in our analysis.

22. Allowing for constant markups would not change our results qualitatively.

23. System estimation is expected to yield superior estimates of σ_0 (and π_0) compared to single-equation CES estimates based uniquely on the difference between equations (45) and (46), cf. Klump et al. (2007) and Leon-Ledesma et al. (2010).

24. However, our findings are robust to alternative choices of the estimator (e.g., multivariate NLS), as well as initial values used in the estimation procedure.

25. See the discussion of our robustness checks with capital-augmenting technical change.

26. The ADF statistics for the log relative factor share, log capital share, and log labor share are above -1.967, so that the null about a unit root cannot be rejected at the 10% significance level. This result aligns with the fact that evidence for covariance stationarity of the US labor share is weak [Mućk et al. (2018)], mirroring the fact that more than 70% of the total variance of the US labor share can be attributed to (highly persistent) medium-run swings and long-run nonlinear trends [Growiec et al. (2018)].

27. Estimation was performed in *Stata* and then replicated in *EViews*, with identical results. We used the default gradient-based optimization algorithm. We tested a wide range of starting points for the estimation. Crucially, we experimented with starting with $\sigma > 1$ and $\sigma < 1$ for each estimated variant. The results are robust to all such manipulations, which makes us confident that we have indeed found the global minimum.

28. Recall that magnitudes of ψ estimates are not comparable across IEES specifications because they pertain to *different* elasticity parameters, that is, they are estimates of elasticity with which the elasticity of substitution σ reacts to changes in *different* variables, EES(x) where $x \in \{\Pi, \varphi, k, \pi, r, \kappa\}$, see Definition 1.

29. In fact, this decline is even more pronounced when we use quality-adjusted aggregate hours as our labor input (see Figure A.10 of the Supplementary Appendix).

30. As expected, reported values are below unity, confirming that each of the more general IEES specifications fits the data better than the CES function.

31. The break point has been selected in the following way. First, following a standard procedure, we have trimmed the first 10% and the last 10% of the sample. Second, we have identified the break point among the remaining observations via likelihood maximization. The parameter $\gamma_{l,\mathcal{B}}$ in Tables A.5 and A.6 of the Supplementary Appendix captures the increase in the rate of labor-augmenting technical change after the identified break point.

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