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Lecture 10: Generalised Method of Moments

Econometric Methods

Andrzej Torój

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	GMM estimator	Example: Gali&Gertler (1999)
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3 Application: Gali&Gertler's hybrid Phillips curve (1999)

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2 GMM estimator

3 Application: Gali&Gertler's hybrid Phillips curve (1999)

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Example: Gali&Gertler (1999) 0 000

Instrumental variables: general idea

- OLS estimaton based on the general underlying assumption that $E\left(\mathbf{X}_{T \times k}^{T} \boldsymbol{\varepsilon}_{T \times 1}\right) = \mathbf{0}_{k \times 1}$ (by Gauss-Markov).
- It may be broken i.a. for the following reasons:
 - just by construction of the economic model;
 - two-way causality between y_t and a subset of x_t ;
 - non-random sample selection.
- Solution: find variables that are truly orthogonal to $\varepsilon_{T \times 1}$ ("instrumental variables").

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Introduction	GMM estimator	Example: Gali&Gertler (1999)
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OLS and IV: alternative approach

• OLS: residuals uncorrelated with regressors, $E(\mathbf{X}_{T \times k}^{T} \boldsymbol{\varepsilon}_{T \times 1}) = \mathbf{0}_{k \times 1}$

• these are k "moment conditions" from which we infer the estimates • $\mathbf{X}^{\mathsf{T}}\mathbf{y} = \mathbf{X}^{\mathsf{T}}\mathbf{X}\boldsymbol{\beta} + \underbrace{\mathbf{X}^{\mathsf{T}}\varepsilon}_{\mathbf{0}} \Rightarrow \hat{\boldsymbol{\beta}}^{OLS} = (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\mathbf{y}$

• IV: residuals uncorrelated with instruments, $E\left(\mathbf{Z}_{T \times k}^{T} \boldsymbol{\varepsilon}_{T \times 1}\right) = \mathbf{0}_{k \times 1}$

- these are k "moment conditions" from which we infer the estimates
 Z^Ty = Z^TXβ + Z^T ∉ ⇒ β^{OLS} = (Z^TX)⁻¹ Z^Ty
- what if we get more than l (> k) "moment conditions", i.e. more than we actually need? Z^T_{T×l}ε_{T×1} = 0_{l×1}
 - there are *l* (> *k*) "moment conditions", so not all of them can be fulfilled by modifying β (only *k* parameters)

•
$$Z^T y = Z^T X \beta + \underbrace{Z^T \varepsilon}_{0} \Rightarrow$$
 but we won't invert $Z^T X$ (not a square matrix this time!)

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Introduction	GMM estimator	Example: Gali&Gertler (1999)
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Limitations of IV estimation

•
$$\boldsymbol{\beta}^{\prime \prime \prime} = \left(\mathbf{Z}^{T} \mathbf{X} \right)^{-1} \mathbf{Z}^{T} \mathbf{y}$$

- Z must have the same number of columns as X for this operation to be feasible
- in IV, there must be as many instrumental variables as regressors (some of which can instrumentalise themselves)
- what are Z?
 - uncorrelated with ε , correlated with X
 - no reason to assume that there is a limited number of such variables

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GMM estimator	Example: Gali&Gertler (1999)
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3 Application: Gali&Gertler's hybrid Phillips curve (1999)

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	GMM estimator	Example: Gali&Gertler (1999)
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So let's take yet another perspective...

- OLS minimises the quadratic form [X^Tε (β)]^T [X^Tε (β)]
 wrt. β_{k×1} (down to zero!)
- IV minimises the quadratic form [Z^Tε (β)]^T [Z^Tε (β)]
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- IV cannot minimise the quadratic form $(\mathbf{Z}^{T}\varepsilon)^{T}(\mathbf{Z}^{T}\varepsilon)$
 - wrt. $\beta_{k \times 1}$ with $\mathbf{Z}_{l \times T}$ down to zero, i.e. it is impossible to perfectly fulfil all the moment conditions
 - cannot solve l equations (moment conditions) for k < l unknowns

• ...so take quadratic form $(\mathbf{Z}^T \varepsilon)^T \mathbf{W} (\mathbf{Z}^T \varepsilon)$ instead!: $\mathbf{W}_{I \times I}$

• weigh the squared differences between left-hand and right-hand side and minimise the sum!

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How to obtain W?

- in general: any symmetric, positive-definite matrix sized $I \times I$
- according to *Hansen (1982)*, you should assign the lowest weights to the moment conditions with lowest precision of estimation, which implies

$$V = \frac{1}{T} Z^T \Omega Z$$

- with Ω variance-covariance matrix of the estimator
 - under white noise: $\hat{\boldsymbol{\Omega}} = \hat{\sigma}^2 \boldsymbol{I}$
 - under non-spherical disturbances White or Newey-West versions (see: serial correlation / heteroskedasticity lectures)

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	GMM estimator	Example: Gali&Gertler (1999)
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GMM as iterative procedure

- - estimate under the assumption of white noise
 - 2 calculate the residuals
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- various implementations of this algorithm in the software (i.a. as a two-step or iterative estimator cf. GLS)

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	GMM estimator	Example: Gali&Gertler (1999)
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Let us minimise the quadratic form with respect to β :

$$\begin{aligned} \varepsilon^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} \varepsilon &= (\mathsf{y} - \beta \mathsf{X})^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} (\mathsf{y} - \beta \mathsf{X}) = \\ &= \mathsf{y}^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} \mathsf{y} - \mathsf{y}^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} \mathsf{X} \beta \\ &- \beta^{\mathsf{T}} \mathsf{X}^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} \mathsf{y} + \beta^{\mathsf{T}} \mathsf{X}^{\mathsf{T}} \mathsf{Z} \mathsf{W} \mathsf{Z}^{\mathsf{T}} \mathsf{X} \beta \\ &\to \min \end{aligned}$$

From matrix calculus:

$$\frac{\partial \mathbf{x}^{\mathsf{T}} \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = \mathbf{x}^{\mathsf{T}} \left(\mathbf{A}^{\mathsf{T}} + \mathbf{A} \right)$$
$$\frac{\partial \mathbf{x}^{\mathsf{T}} \mathbf{a}}{\partial \mathbf{x}} = \mathbf{a}^{\mathsf{T}}$$

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	GMM estimator	Example: Gali&Gertler (1999)
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$$-\mathbf{y}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X} - \mathbf{y}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X} + \beta^T (\mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X} + \mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X}) = 0$$

• $\beta^T (\mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X}) = \mathbf{y}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X}$
• $(\mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X}) \beta = \mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{y}$

•
$$\hat{\boldsymbol{\beta}}^{GMM} = \left(\mathbf{X}^{\mathsf{T}} \mathbf{Z} \mathbf{W} \mathbf{Z}^{\mathsf{T}} \mathbf{X} \right)^{-1} \mathbf{X}^{\mathsf{T}} \mathbf{Z} \mathbf{W} \mathbf{Z}^{\mathsf{T}} \mathbf{y}$$

Nonlinear GMM:

- you can also (numerically) minimise the quadratic form for a nonlinear model;

- it is advisable to write the moment conditions "as close to linear as possible".

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	GMM estimator	Example: Gali&Gertler (1999)
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	GMM estimator	Example: Gali&Gertler (1999)
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GMM: special cases

•
$$\mathbf{Z}_{T \times k}, \mathbf{X}_{T \times k}$$

 $\hat{\boldsymbol{\beta}}^{GMM} = (\mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{y} =$
 $(\mathbf{Z}^T \mathbf{X})^{-1} \mathbf{W}^{-1} (\mathbf{X}^T \mathbf{Z})^{-1} \mathbf{X}^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T \mathbf{y} = (\mathbf{Z}^T \mathbf{X})^{-1} \mathbf{Z}^T \mathbf{y} = \hat{\boldsymbol{\beta}}^{IV}$
• $\mathbf{Z} = \mathbf{X}$
 $\hat{\boldsymbol{\beta}}^{GMM} = (\mathbf{X}^T \mathbf{X} \mathbf{W} \mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \mathbf{W} \mathbf{X}^T \mathbf{y} =$
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 $\hat{\boldsymbol{\beta}}^{OLS}$

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GMM: special cases

$$\begin{aligned} \mathbf{Z}_{T\times k}, \mathbf{X}_{T\times k} \\ \hat{\boldsymbol{\beta}}^{GMM} &= (\mathbf{X}^{T} \mathbf{Z} \mathbf{W} \mathbf{Z}^{T} \mathbf{X})^{-1} \mathbf{X}^{T} \mathbf{Z} \mathbf{W} \mathbf{Z}^{T} \mathbf{y} = \\ (\mathbf{Z}^{T} \mathbf{X})^{-1} \mathbf{W}^{-1} (\mathbf{X}^{T} \mathbf{Z})^{-1} \mathbf{X}^{T} \mathbf{Z} \mathbf{W} \mathbf{Z}^{T} \mathbf{y} = (\mathbf{Z}^{T} \mathbf{X})^{-1} \mathbf{Z}^{T} \mathbf{y} = \hat{\boldsymbol{\beta}}^{IV} \end{aligned}$$
$$\begin{aligned} \mathbf{Z} &= \mathbf{X} \\ \hat{\boldsymbol{\beta}}^{GMM} &= (\mathbf{X}^{T} \mathbf{X} \mathbf{W} \mathbf{X}^{T} \mathbf{X})^{-1} \mathbf{X}^{T} \mathbf{X} \mathbf{W} \mathbf{X}^{T} \mathbf{y} = \\ (\mathbf{X}^{T} \mathbf{X})^{-1} \mathbf{W}^{-1} (\mathbf{X}^{T} \mathbf{X})^{-1} \mathbf{X}^{T} \mathbf{X} \mathbf{W} \mathbf{X}^{T} \mathbf{y} = (\mathbf{X}^{T} \mathbf{X})^{-1} \mathbf{X}^{T} \mathbf{y} \end{aligned}$$

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Andrzej Torój (10) GMM

	GMM estimator	Example: Gali&Gertler (1999)
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GMM		

Variance-covariance of GMM estimator

- GMM estimator: consistent, asymptotically normally distributed
- asymptotic variance-covariance estimator in a linear model:

$$Var\left(\beta^{GMM}\right) = \frac{1}{T} \left[\frac{1}{T} \left(\mathbf{X}^{T} \mathbf{Z}\right) \mathbf{W}^{-1} \frac{1}{T} \left(\mathbf{Z}^{T} \mathbf{X}\right)\right]^{-1}$$

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	GMM estimator	Example: Gali&Gertler (1999) o ooo
GMM		

J statistic

- the non-zero value of the minimised quadratic form is interpretable
 - how far are we from fulfilling the (excessive) orthogonality conditions?
 - the lower the value, the lower the "distance" to perfect fulfilment of excessive conditions
- divided by T, it is χ^2 -distributed with degrees of freedom equal to l k (instruments in excess of parameters)

J-test of orthogonality

$$J\left(\hat{eta}, \mathbf{\hat{\Omega}}^{-1}
ight) = rac{1}{ au} arepsilon \left(\hat{eta}
ight)^{ au} \mathsf{Z} \mathbf{\hat{\Omega}}^{-1} \mathsf{Z}^{ au} arepsilon \left(\hat{eta}
ight)$$

 H_0 : all orthogonality conditions fulfilled

 H_1 : some orthogonality conditions not fulfilled

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GMM estimator	Example: Gali&Gertler (1999)
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Outline





3 Application: Gali&Gertler's hybrid Phillips curve (1999)

16/19

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	GMM estimator 0 00000000	Example: Gali&Gertler (1999) ○ ●○○
Gali&Gertler (1999)		

Hybrid Phillips curve

• Gali, Gertler (1999):

 $\pi_t = \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda mc_t + \varepsilon_t$ where π_t - inflation rate, mc_t - real marginal cost, ε_t - error term.

- Why GMM?
 - there is an unobservable variable on the right-hand side, $E_t \pi_{t+1}$
 - we can just replace it with observable $\pi_{t+1} = E_t \pi_{t+1} + v_{t+1}$, where v_t – expectations error

 $\pi_t = \gamma_b \pi_{t-1} + \gamma_f \pi_{t+1} \left(v_{t+1} \right) + \lambda m c_t + \varepsilon_t$

• but v_{t+1} clearly not independent from ε_t – inconsistency!

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	GMM estimator 0 00000000	Example: Gali&Gertler (1999) ○ ●○○
Gali&Gertler (1999)		

Hybrid Phillips curve

• Gali, Gertler (1999):

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	GMM estimator	Example: Gali&Gertler (1999)
		0 0 0 0
Gali&Gertler (1999)		

Orthogonality conditions

•
$$E_t [\varepsilon_t \mathbf{z}_t] = E_t [(\pi_t - \gamma_b \pi_{t-1} - \gamma_f E_t \pi_{t+1} - \lambda mc_t) \mathbf{z}_t] = E_t [(\pi_t - \gamma_b \pi_{t-1} - \gamma_f \pi_{t+1} - \lambda mc_t) \mathbf{z}_t] = \mathbf{0}$$

- the expected value on the orthogonality condition allows to drop the expected value on π_{t+1}

- in this context, we interpret the instrument set Z as variables that allow to forecast inflation one period ahead without systematic errors
- there should be more than 3 instruments to use GMM here

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	GMM estimator o oooooooo	Example: Gali&Gertler (1999) ○ ○○●
Gali&Gertler (1999)		

Example

And

(Imperfect) replication of Gali-Gertler results:

- Read the paper by Gali and Gertler.
- Consider the following set of variables as linear one-period-ahead predictors:
 - inflation (4 lags), log real ULC (1 lag), output gap (1 lag), short- vs long-term interest rate spread (1 lag), log-differences of wage index (1 lag), log-differences of commodity price index (1 lag)
- Define the instruments and the initial weight matrix W.
- Estimate the model. Discuss the results. Do they confirm Your intuition? If not, look for the (economic) solution in the article.

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